DYNAMICS

[For B. A. & B. Sc. *Students of All Indian Universities*]

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PREFACE TO THE EIGHTEENTH EDITION

It gives me great pleasure in bringing out the Eighteenth Edition of this book in such a short time.

This book has been thoroughly reviseed according to the *latest Syllabuses. A number of new examples selected from recent* examinations papers, have been added.

Besides giving due credit to the printers and publishers, I express my thanks to the professors and students for the appreciation and patronage of the book.

Suggestions for further improvement of the book will be highly appreciated. —*Author*

PREFACE TO THE FIRST EDITION

The present book comprising the subject "Dynamics" is meant for the students appearing in the B. A. & B. Sc. Examinations of All Indian Universities. Efforts have been made to make the treatment logical and simple.

I gratefully acknowledge my indebtedness to various authors and publishers whose books have been freely consulted during the preparation of this book.

I shall be grateful to the readers for pointing out errors and omissions that, inspite of all care, might have crept in.

I look forward to the suggestions from the readers for the improvement of the book. —*Author*

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Note :—[illegible] are marked as* and [illegible] ook.

Mathematics For Compatitions

A BOON TO STUDENTS PREPARING FOR ENGINEERING & OTHER COMPETITIONS

(TWO VOLUMES)

By :

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Velocity and Acceleration

§ 1. **Definitions :**

Dynamics. It is that branch of Mathematics which treats the bodies in motion (a body is said to be in motion if it changes position with respect to the surrounding objects).

Space. The region in which different events occur is called space.

A body. A portion of matter occupying finite space *i.e.* limited in every direction is called a body.

A particle. A particle is defined as a body which is so small that for the purpose of reasoning its position at any instant coincides with a geometrical point.

Path of a particle. A particle occupies different positions in space while in motion and the curve joining such positions of a particle is known as the path of the particle.

Rigid Body. A rigid body is defined as the agglomeration of innumerable particles which does not change its size or shape and the distance between any two particles of it remains canstant throghout the motion.

According to the above definition of particle and rigid body Dynamics has further been divided into two parts *viz.* **Dynamics of a particle and Dynamics of a rigid body.**

In order to describe the motion of particle or of a rigid body two things are necessary, (i) a frame of reference and (ii) time.

Frame of Reference. Motion relative to surrounding objects can only be described and motion implies change of position in space. To locate the position of an object an origin O and a set of axes through O are needed. In general a system of two or three rectangular axes through O are chosen as a frame of reference. In the case of bodies moving on or near the surface of the earth, O is generally chosen on the surface of the earth and the axes through O fixed with reference to the earth. [illegible] of the planets generally a frame of [illegible] origin at the centre of the sun.

Time. It is the measure of succession of events. Its unit is generally taken as second, though it is sometimes taken as minute or hour also.

§ 2. **Speed** The speed of a moving point or particle is the rate at which the point or the particle is describing its path which may be a straight line or a curve.

Average Speed.

So if a particle moves a distance s in a total time t, then the ratio $\frac{s}{t}$ is defined as the *average speed* of the particle during this interval of time t i.e. Average speed $= \frac{\text{Total distance travelled}}{\text{Total time taken}}$

Here it must be noted that actual speed may be sometimes greater than this average speed and less than at other times and is a scalar quantity.

Instantaneous Speed.



(Fig. 1)

Let a particle be moving along a curve AB as shown in the adjoining figure. Let the particle be at P and Q at times t and $t+\delta t$ respectively such that arc $AP=s$ and arc $AQ=s+\delta s$. Then the particle has described a distance δs (=arc PQ) in time δt, so its average speed is $\frac{\delta s}{\delta t}$.

$\therefore$ Limit of $\frac{\delta s}{\delta t}$ as $\delta t \to 0$ i.e. $\lim\limits_{\delta t \to 0} \frac{\delta s}{\delta t}$ of the particle at time t i.e. at P is defined as the instantaneous speed and is denoted by $\frac{ds}{dt}$.

Generally we use the word 'speed' for instantaneous speed.

Variable and Uniform Speed.

Let the distance s moved by a particle be given by the relation $s=a \cos \mu t+b \sin \mu t$, where a, b and μ are constants. Then speed of the particle $=ds/dt=-a\mu \sin \mu t+b\mu \cos \mu t$, which evidently varies as t varies.

So in this case we say that the speed is *variable.*

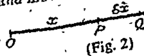
Again if s and t are related by $s=\lambda t$, then $ds/dt=\lambda$, a constant for all values of time t.

In this case we say that the speed is *constant or uniform.*

Displacement. Let a particle move from P to Q in a certain interval of time along any path. Then displacement of the particle in this interval is defined as the change in the positions of the particle. The displacement is a *vector* quantity, its magnitude is the length PQ and direction along PQ.

§ 3. Velocity at a point or at an instant.

Suppose a particle starts from O and moves along the straight line OA. Let the particle be at P after an interval of time t, such that $OP=x$ and let it be at Q after time $(t+\delta t)$. Let $PQ=\delta x$,



(Fig. 2)

then the particle has traversed the distance PQ $(=\delta x)$ in time $(t+\delta t)-t$ *i.e.* δt.

∴ The displacement of the particle in time $\delta t=\delta x$.

∴ Average velocity of the moving point during the interval of time $\delta t=\dfrac{\text{displacement in the time } \delta t}{\text{time } \delta t}=\dfrac{\delta x}{\delta t}$.

∴ **Velocity at P at time t** $=\lim_{\delta t\to 0}\dfrac{\delta x}{\delta t}=\dfrac{dx}{dt}$,

and is in the sense of x increasing. This velocity dx/dt is also denoted by $\dot{x}$, where dot denotes differentiation with respect to t.

In vector notations :—

If **i** be the unit vector along OA, then if **v** be the velocity vector of the particle at P, we have

$\mathbf{v}=(dx/dt)\,\mathbf{i}$ and it is in the sense of x increasing.

Note : Velocity is a vector quantity whereas the speed is a scalar one.

Units of Velocity.

The most common units of velocity are *cm./sec.*, *km./hr.* and *ft./sec.*

If L and T denote the units of length and time respectively then the unit of velocity is LT^{-1}.

§ 4. Uniform and Variable Velocity.

[illegible]

moving with uniform velocity.

If on the other hand either the magnitude or the direction or both change from time to time *i.e.* from point to point then the particle is said to be moving with variable velocity and this may happen when the particle is moving in a straight line or a curve.

Acceleration at a point.

Definition. The rate of change of velocity with respect to time is defined as the acceleration.

∴ If the particle has a velocity v at P at time t and has a velocity $v+\delta v$ at Q at time $t+\delta t$, *i.e.* after an interval of time δt, then in time δt, the change in velocity $=(v+\delta v)-v=\delta v$

∴ Average rate of change of velocity $=\delta v/\delta t$

∴ Rate of change of velocity $=\lim_{\delta t\to 0}\dfrac{\delta v}{\delta t}=\dfrac{dv}{dt}$

or **acceleration at time t or at P** $=\dfrac{dv}{dt}$ ...(i)

But as proved in § 3 Page 2 of this chapter, we have the velocity v of the particle at $P=dx/dt$

$\therefore$ From (i), we get acceleration at time $t = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

Also we can write $\frac{dv}{dt}$ as $\frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx} v$.

$\therefore$ Acceleration at time $t = v\frac{dv}{dx}$.

Hence the acceleration of a particle moving along a straight line can be expressed in three ways as

$$\frac{dv}{dt}, \frac{d^2x}{dt^2} \text{ and } v\frac{dv}{dx} \text{ or } \dot{v}, \ddot{x} \text{ and } v\frac{dv}{dx}. \qquad \text{(Remember)}$$

In vector notations :—

We know from § 3 Page 2 that if $\mathbf{v}$ be the velocity vector of the particle at P then $\mathbf{v}=(dx/dt)\,\mathbf{i}$, where $\mathbf{i}$ is the unit vector in the direction in which x increases.

If $\mathbf{a}$ is the acceleration vector in the direction of $\mathbf{i}$ *i.e.* in the direction in which x increases, then

$$\mathbf{a} = \frac{d}{dt}(\mathbf{v}) = \frac{d}{dt}\left(\frac{dx}{dt}\mathbf{i}\right) = \left(\frac{d^2x}{dt^2}\right)\mathbf{i} \qquad \text{...(ii)}$$

Also $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{dx}\cdot\frac{dx}{dt} = \frac{d\mathbf{v}}{dx}(v)$, where $v = |\mathbf{v}|$

$$= \left(v\frac{dv}{dx}\right)\mathbf{i}, \quad \because \quad \mathbf{v} = v\mathbf{i} \qquad \text{(Note)} \quad \text{...(iii)}$$

Also $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\,\mathbf{i}) = \left(\frac{dv}{dt}\right)\mathbf{i}$ ...(iv)

Here from (ii), (iii) and (iv), we conclude that the magnitude of acceleration of a particle moving along a straight line can be expressed as before, in three ways as $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ and $\frac{dv}{dt}$.

Note 1. Magnitude of acceleration vector is denoted by f or a. Here also the direction of acceleration is the direction in which x increases.

Note 2. Acceleration is also a vector quantity.

Units of Acceleration.

The most common units of acceleration are cm./sec^2, km/.hr^2, and ft./sec^2.

If L and T denote the units of length and time respectively then the unit of acceleration is LT^{-2}.

Solved Examples on § 3 to § 5.

Ex. 1. A particle is moving in a straight line and its velocity at a distance x from the origin is $k\sqrt{(a^2-x^2)}$. Find the acceleration and nature of motion.

Sol. Given $v=k\sqrt{(a^2-x^2)}$ ...(i)

$\therefore$ $dv/dx=k.\frac{1}{2}(a^2-x^2)^{-1/2}(-2x)=-kx/\sqrt{(a^2-x^2)}$ ...(ii)

$\therefore$ Required acceleration$=v\,(dv/dx)$

$=k\sqrt{(a^2-x^2)}\,[-kx/\sqrt{(a^2-x^2)}]$, from (i) and (ii)

$=-k^2x$,

which varies as its distance from the origin and being negative is towards it.

Also from (i) writing v as dx/dt, we can get

$k\,dt=[1/\sqrt{(a^2-x^2)}]\,dx$.

Integrating, $kt+c=\sin^{-1}(x/a)$, which gives the distance of the particle at a particular instant.

Ex. 2. A particle moves along a straight line, so that after seconds its distance x from a fixed point O on the line is given by $x=t^2(t-1)$. Find the velocity and acceleration on each occassion when it passes through O.

Sol. Given $x=t^2(t-1)=t^3-t^2$ ...(i)

$\therefore$ $dx/dt=3t^2-2t$ ...(ii) and $d^2x/dt^2=6t-2$ ...(iii)

From (i) when the particle passes through O *i.e.* when $x=0$ we get $0=t^3-t^2$ or $t^2(t-1)=0$ or $t=0, 1$.

i.e. the particle passes through O, at two instants viz. when $t=0$ and $t=1$.

$\therefore$ From (ii) at $t=0$, the velocity of the particle$=0$ and at $t=1$, the velocity of the particle$=3(1)^2-2(1)=3-2=1$ unit of velocity.

And from (iii) at $t=0$, the acceleration of the particle

$=6(0)-2=-2$ units/sec^2.

And at $t=1$, the acceleration of the particle

$=6(1)-2=4$ units/sec^2. Ans.

Ex. 3. A particle moves along a straight line such that its displacement x from a fixed point on the line at time t, is given by

$$x=t^3-9t^2+24t+6$$

Determine, (i) the instant when the acceleration becomes zero, (ii) the position of the particle at that instant and (iii) the velocity of the particle at that instant.

Sol. Given $x=t^3-9t^2+24t+6$.

$\therefore$ $dx/dt=3t^2-18t+24$ and $d^2x/dt^2=6t-18$.

(i) When the acceleration is zero, we get

$\dfrac{d^2x}{dt^2}=0$ or $6t-18=0$ or $t=3$ units. Ans.

(ii) At $t=3$, from $x=t^3-9t^2+24t+6$ we have

$x=(3)^3-9(3)^2+24(3)+6=24$ units Ans.

(iii) At $t=3$, from $dx/dt=3t^2-18t+24$ we get

the required velocity$=3(3)^2-18(3)+24=-3$ units. Ans.

Ex. 4 (a). **If at time t the displacement s of a particle moving away from the origin is given by $s=a \sin t+b \cos t$, find the velocity and acceleration of the particle.**

Sol. We are given that $s=a \sin t+b \cos t$.

Differentiating with respect to t, we have the velocity

$$=ds/dt=a \cos t-b \sin t.$$

Differentiating again with respect to t, the acceleration

$$=\frac{d^2s}{dt^2}=-a \sin t-b \cos t=-s.$$ **Ans.**

Ex. 4 (b). **A point moves in a fixed straight path so that $s=\sqrt{(t)}$; show that the acceleration is negative and proportional to the cube of the velocity.**

Sol. Given $s=\sqrt{t}=t^{1/2}$.

Differentiating w. r. to t, velocity $=ds/dt=\frac{1}{2}t^{-1/2}$.

Again differentiating, $\frac{d^2s}{dt^2}=-\frac{1}{4}t^{-3/2}$

or acceleration $=-\frac{1}{4}t^{-3/2}=-2\,[\frac{1}{2}t^{-1/2}]^3=-2$ (velocity)^3

or $\frac{\text{acceleration}}{(\text{velocity})^3}=-2=$ constant. Hence proved.

Ex. 5. **If a particle moves along a straight line according to the law $v^2=ax-bx^3$, prove the $27\,bv^4=4\,(a-2f)\,(a+f)^2$, where a and b are constants and f is the acceleration.**

Sol. We are given $v^2=ax-bx^3$ ...(i)

Differentiating with respect to x, we get

$$2v\,(dv/dx)=a-3bx^2$$

$\therefore$ Acceleration $=v\,(dv/dx)=\frac{1}{2}\,[a-3bx^2]=f$ (given) ...(ii)

Now $4\,(a-2f)\,(a+f)^2$

[illegible] from (ii)

[illegible]

[illegible], from (i)

$=27bv^4$ Hence proved.

Ex. 6. **Prove that if a particle moves so that the space described is proportional to the square of the time of description, the velocity will be proportional to time and the rate of increase of velocity will be constant.**

Sol. Given $x=kt^2$, where k is a constant of proportionality.

Differentiating w.r. to t, we get $dx/dt=2kt$ ...(i)

i.e. the velocity dx/dt is proportional to time t.

Differentiating (i) again, acceleration $=\frac{d^2x}{dt^2}=2k$, which is constant and positive.

Hence the rate of increase of velocity is constant.

Ex. 7. A particle moves along a straight line, the law of motion being $s=A\cos(nt+k)$, show that the acceleration is directed to the origin and varies as the distance.

Sol. Given $s=A\cos(nt+k)$.

Differentiating w. r. to t we get $ds/dt=-An\sin(nt+k)$.

Again differentiating we get $\dfrac{d^2s}{dt^2}=-An^2\cos(nt+k)$.

or the acceleration $=-n^2s$.

The negative sign shows that the direction of the acceleration is in the sense of s decreasing, *i.e.* towards the origin.

Also acceleration $=-n^2s$, *i.e.* varies as the distance s.

Ex. 8. A point moves in a straight line so that its distance from a fixed point in that line is the square root of the quadratic function of the time; prove that its acceleration varies inversely as the cube of the distance from the fixed point.

Sol. Let the distance of the particle from a fixed point on the line be x at time t.

Then according to the problem, we have

$$x=\sqrt{(at^2+bt+c)}, \quad \ldots(i)$$

where a, b and c are constants.

Differentiating (i) with respect to t, we have

$$\frac{dx}{dt}=\frac{1}{2\sqrt{(at^2+bt+c)}}\cdot(2at+b)=\frac{2at+b}{2x}, \text{ from (i)}$$

Again differentiating w. r. to t, we get the acceleration

$$\frac{d^2x}{dt^2}=\frac{2x(2a)-(2at+b)\,2\,(dx/dt)}{4x^2}$$

$$=\frac{4ax-(2at+b).\dfrac{2(2at+b)}{2x}}{4x^2}, \text{ substituting the values of } \frac{dx}{dt}$$

$$=\frac{8ax^2-2(2at+b)^2}{8x^3}=\frac{4a(at^2+bt+c)-(4a^2t^2+4abt+b^2)}{4x^3},$$

putting the value of x^2 from (i)

$$=\frac{4ac-b^2}{4x^3}=\frac{k}{x^3}, \text{ where } k=\frac{4ac-b^2}{4}$$

$\therefore$ acceleration varies as $1/x^3$ or inversely as x^3.

Ex. 9. [illegible] straight line is given by th[illegible] he acceleration varies as [illegible]

Sol. Given $v^2=ax^2+2bx+c$.

Differentiating w. r. to x, we have $2v\,(dv/dx)=2ax+2b$

or acceleration $=v\dfrac{dv}{dx}=ax+b=a\left(x+\dfrac{b}{a}\right)$

i.e. the acceleration varies as the distance from a fixed point $x=-(b/a)$. Hence proved.

Ex. 10 (a). If a point moves in a straight line in such a manner that its retardation is proportional to its speed prove that the space described in any time is proportional to the speed destroyed in that time.

Sol. Given that the retardation$=k\times$speed,
where k is constant

or $$-\frac{d^2x}{dt^2}=k\frac{dx}{dt}$$ (Note the $-$ve sign is for retardation)

or $$\frac{d^2x}{dt^2}+k\frac{dx}{dt}=0.$$

Integrating w.r. to t, we have $(dx/dt)+kx=C$,
where C is constant of integration.

If the particle moves such that $dx/dt=0$ when $x=0$ then $C=0$ and we have $\frac{dx}{dt}+kx=0$ or $x=-\frac{1}{k}\frac{dx}{dt}$

or x varies as $-(dx/dt)$

or space described is proportional to the speed destroyed.

Ex. 10 (b). A particle moves in a straight line from a fixed point O with velocity V under force which produces an acceleration ax, where x is the distance from O. Find the time taken for the velocity to be increased to $2V$.

Sol. Given acceleration $v\frac{dv}{dx}=ax$

or $$2v\,dv=2ax\,dx$$

Integrating, $v^2=ax^2+C$ ...(i)

Initially at O, $v=V$, $x=0$

$\therefore$ $V^2=0+C$ or $C=V^2$.

$\therefore$ From (i) we get $v^2=ax^2+V^2$

or $(dx/dt)^2=ax^2+V^2$ or $dx/dt=\sqrt{a}\sqrt{[x^2+(V^2/a)]}$

or $$\sqrt{a}\,dt=\frac{dx}{\sqrt{[x^2+(V^2/a)]}}$$ ...(iii)

Also when $v=2V$, let $x=x_1$ then from (ii), we get

$(2V)^2=ax_1^2+V^2$ or $ax_1^2=3V^2$ or $x_1=V\sqrt{3}/\sqrt{a}$.

Now we are to find time of moving from $x=0$ to $x=x_1$

i.e. $V\sqrt{3}/\sqrt{a}$, and if this time be T, then from (iii), we get

$$\sqrt{a}\,T=\int_0^{x_1}\frac{dx}{\sqrt{(x^2+(V^2/a))}}=\Big(\log[x+\sqrt{(x^2+(V^2/a))}]\Big)_0^{x_1}$$

or $$T=\frac{1}{\sqrt{a}}\Big(\log[x_1+\sqrt{(x_1^2+(V^2/a))}]-\log[\sqrt{(V^2/a)}]\Big)$$

or $\quad T=\frac{1}{\sqrt{a}}\left[\log\left\{\frac{V\sqrt{3}}{\sqrt{a}}+\frac{2V}{\sqrt{a}}\right\}-\log\left(\frac{V}{\sqrt{a}}\right)\right]$

$$=\frac{1}{\sqrt{a}}\log\left[\frac{(2+\sqrt{3})\,V}{\sqrt{a}}\times\frac{\sqrt{a}}{V}\right]=\frac{1}{\sqrt{a}}\log(2+\sqrt{3}).\ \text{Ans.}$$

Ex. 11. A particle initially at rest moves from a fixed point in a straight line so that at the end of t seconds its acceleration is $(\sin t)+[1/(t+1)^2]$.

Show that its distance from the fixed point at the end of π seconds is $2\pi-\log(1+\pi)$.

Sol. Given $\frac{dv}{dt}=\sin t+\frac{1}{(t+1)^2}$

Integrating with respect to t, we get

$$v=-\cos t-\frac{1}{(t+1)}+C, \quad \ldots(i)$$

where C is constant of integration.

Initially velocity $v=0$ and $t=0$, so from (i) we get

$$0=-1-(1/1)+C \quad \text{or} \quad C=2$$

$\therefore$ From (i) we have $v=-\cos t-[1/(t+1)]+2$

or $\quad \frac{dx}{dt}=-\cos t-\frac{1}{(t+1)}+2, \quad \because \ v=\frac{dx}{dt}$

Again integrating with respect to t, we have

$$x=-\sin t-\log(t+1)+2t+k, \quad \ldots(ii)$$

where k is constant of integration.

Let $x=0$ at $t=0$, then from (ii) we get $k=0$

$\therefore$ From (ii) we have

$$x=-\sin t-\log(t+1)+2t \quad \ldots(iii)$$

If x_1 be the required distance (measured from the fixed point $x=0$) at the end of π seconds, then $x=x_1$ when $t=\pi$.

$\therefore$ From (iii) we get $x_1=-\sin\pi-\log(\pi+1)+2\pi$

or $\quad x_1=2\pi-\log(1+\pi)$. *Hence proved.*

***Ex. 12. The law of motion in a straight line is being given by $s=\frac{1}{2}vt$, prove that the acceleration is constant.**

Sol. Given that $\quad s=\frac{1}{2}vt$.

Differentiating w. r. to t, we get $ds/dt=\frac{1}{2}v+\frac{1}{2}t\,(dv/dt)$

or $\quad v=\frac{1}{2}v+\frac{1}{2}t\,(dv/dt)$, since $v=ds/dt$

or $\quad \frac{1}{2}v=\frac{1}{2}t\,(dv/dt) \quad$ or $\quad v=t\,(dv/dt)$

Again differentiating we get $\frac{dv}{dt}=t\frac{d^2v}{dt^2}+\frac{dv}{dt}$ or $\frac{d^2v}{dt^2}=0$.

Integrating w. r. to t, we get $dv/dt=$ constant.

or the acceleration is constant. Hence proved.

**Ex. 13. If time t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by $f^3 \frac{d^2t}{dv^2}$, f being the acceleration.

Sol. Suppose $t=F(v)$ .. (i)

Differentiating w. r. to t, we have $1=F'(v)\frac{dv}{dt}$

or acceleration $f=\frac{dv}{dt}=\frac{1}{F'(v)}$ .. (ii)

Differentiating both sides of (ii) w. r. to t, we have

$$\frac{d^2v}{dt^2}=-\frac{F''(v)}{[F'(v)]^2}\frac{dv}{dt}=-\frac{F''(v)}{[F'(v)]^2}\times\frac{1}{F'(v)}, \text{ from (ii)}$$

$$=-\frac{F''(v)}{[F'(v)]^3}=-f^3.F''(v), \text{ since } f=\frac{1}{F'(v)} \quad \text{.. (iii)}.$$

Again from (i), we have $\frac{dt}{dv}=F'(v)$ and $\frac{d^2t}{dv^2}=F''(v)$.

$\therefore$ From (iii) we get $\frac{d^2v}{dt^2}=-f^3.\frac{d^2t}{dv^2}$ or $\frac{d}{dt}\left(\frac{dv}{dt}\right)=-f^3\frac{d^2t}{dv^2}$, which is negative.

Hence rate of decrease of acceleration $=f^3.\frac{d^2t}{dv^2}$.

Ex. 14. A point moves in a straight line so that its distance s from a fixed point at any time t is proportional to t^n. If v be the velocity and f the acceleration at any time t, show that

$$v^2=nfs/(n-1).$$

Sol. Given $s=kt^n$, where k is constant of proportionality.

$\therefore$ velocity $v=ds/dt=knt^{n-1}$

and acceleration $f=dv/dt=kn(n-1)t^{n-2}$.

$$\therefore \frac{nfs}{(n-1)}=\frac{nkn(n-1)t^{n-2}kt^n}{n-1}$$

$$=n^2k^2t^{2n-2}=(knt^{n-1})^2=v^2. \qquad \text{Hence proved.}$$

Ex. 15. Prove that if a point moves with a velocity varying as any power (not less than unity) of its distance from a fixed point which it is approaching, it will never reach that point.

Sol. Let O be the fixed point. Let the particle after time t be at P, such that $OP=x$. Also the particle is approaching O, i.e. the particle is moving towards O, i.e. in the sense of x decreasing.



(Fig. 3)

$\therefore$ According to the problem $dx/dt = -kx^n$, where k is constant and n is any power of the distance x and $n \geqslant 1$.

or $$dt = -\frac{dx}{kx^n} = -\frac{1}{k} x^{-n}\, dx.$$

Integrating we get $t = \frac{1}{k} \cdot \frac{x^{-n+1}}{(-n+1)} + C$,

where C is constant of integration.

or $$t = \frac{1}{k(n-1)x^{n-1}} + C.$$

If $n > 1$, putting $x=0$, we get $t = \infty$.

i.e. the particle will take infinite time to reach O where $x=0$.

If $n=1$, the given condition becomes

$$\frac{dx}{dt} = -kx \quad \text{or} \quad dt = -\frac{1}{k}\frac{dx}{x}$$

Integrating, $t = -(1/k) \log x + B$, where B is constant.

If we put $x=0$, the above result gives $t = \infty$.

Hence if $n \geqslant 1$, the particle will never reach the point O.

Exercises on § 1 – § 5

Ex. 1. A particle moves in a straight line so that after t seconds its distance from a given point O on the line is given by

$x = (t-2)^2 (t-5)$. Find the motion.

Ans. At time t, velocity $= 3t^2 - 18t + 24$; acceleration $= 6t - 18$.

Ex. 2. The velocity v of a particle is given by $v = 6 \cos 3x$, where x is the distance from a fixed point. Find its acceleration.

Ans. $-54 \sin 6x$.

***Ex. 3.** A particle starts from a fixed point O such that its acceleration after t secs. is $1/(t+2)^2$. Find the distance described in 9 seconds and its velocity then.

Ex. 4. A body moves from rest from a point O so that its acceleration after t seconds from O is $1/(t+10)^3$. Find the distance described in 5 seconds and its velocity then.

Ex. 5 - Fill up the blank :—

Displacement about a point is always a .....

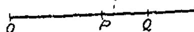
Ans. vector quantity.

Ex. 6. Distinguish between 'speed' and 'velocity', and give on example of each.

§ 6. Rectilinear Motion. It means the motion which is taking place in a straight line.

Velocity at time t.

Let us consider the motion of a particle along a straight line Ox on which O is a fixed point.



(Fig. 4)

Let P and Q be the positions of the particles at time t and t' respectively such that $OP=x$ and $OQ=x'$.

Then the distance moved in time $(t'-t)$ is PQ *i.e.* $(x'-x)$.

$\therefore$ The average rate of displacement of the particle is $(x'-x)/(t'-t)$ or the average velocity during the interval $(t'-t)$ is $(x'-x)/(t'-t)$.

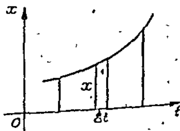
If this ratio has the same value for all intervals of time, then the velocity of the particle is said to be *uniform* otherwise *variable*. In the case of uniform velocity the particle travels equal distance in equal times.

The limiting value of the ratio $(x'-x)/(t'-t)$ as t' tends to t (whether this ratio is constant or not) is defined as the velocity of the partice at the instant t. Also from our knowledge of Differential Calculus we know that this limiting value is the differential coefficient of x with respect to t *i.e.* if v be this velocity of the particle at time t, then $v=dx/dt$ or $\dot{x}$

Space-time curve.

Definition. If a curve is plotted such that abscissae represent times taken and ordinates represent the distances travelled by the particle, then the curve is called the space-time curve.

This curve represents graphically the relation between the distance moved and the time taken. Also



(Fig. 5)

the tangent of angle which the tangent at any point makes with the time-axis *i.e.* gradient of the curve gives the value of dx/dt *i.e.* the velocity.

Acceleration at time t.

If v and v' be the velocities of the particle moving along line Ox, where O is the fixed point (See Fig. 4 above), at P and Q respectively, then $v'-v$ is the change is velocity in time $t'-t$.

$\therefore$ The average rate of change of velocity or the average acceleration during the interval $t'-t$ is $(v'-v)/(t'-t)$ and if this ratio is independent of the interval $t'-t$ *i.e.* remains unchanged during the interval $t'-t$ then this average acceleration is *uniform* or constant otherwise variable.

The limiting value of the ratio $(v'-v)/(t'-t)$ as t' tends to t (whether this ratio is constant or not) is defined as the acceleration of the particle at the instant t and this limiting values is dv/dt, *i.e.* the differential coefficient of the velocity v with respect to time t.

$$\therefore \text{ acceleration at time } t=\frac{dv}{dt}=\frac{d^2x}{dt^2}, \quad \because \quad v=\frac{dx}{dt}$$

$$=\dot{x} \text{ or } \frac{dv}{dx}\cdot\frac{dx}{dt}=v\frac{dv}{dx}. \qquad \text{(Note)}$$

Velocity-time curve.

Definition. If a curve is plotted such that the abscissae represent times taken and ordinates represent velocities, then the curve is called the velocity-time curve (In Fig. 5 on Page 12 replace x by v).

The gradient of this curve gives the acceleration dv/dt of this particle at any instant.

And the area under this velocity-time curve.

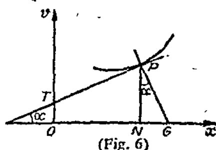
$$=\int v\,dt \qquad \text{(See Figure 5 on Page 12)}$$

$$=\int\frac{dx}{dt}\,dt=\int dx=(x), \text{ taken between proper limits}$$

$=$ the distance covered in the corresponding time.

Velocity-space Curve.

Definition. If a curve is plotted such that the abscissae represent the distances moved and the ordinates represent the velocities, then the curve is called the velocity-space curve.



(Fig. 6)

The gradient of the curve is *differential coefficient of v with respect to x i.e. dv/dx* and in the adjoining figure we find that $\tan\alpha=dv/dx$.

But $\alpha=\angle NPG$, where PN is the ordinate v and PG is the normal at P.

i.e., $\tan\angle NPG=dv/dx$.

Also from $\triangle PNG$, we get $NG/PN=\tan\angle NPG$

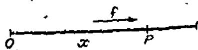
or the subnormal $NG=PN.\tan\angle NPG=v\,(dv/dx)$.

Also we know that the only curve for which the subnormal is constant is parabola, hence if the acceleration is constant i.e. $v\, dv/dx$ is constant then the subnormal is constant and then the velocity-space curve is a parabola.

§ 7. Rectilinear Motion with Uniform Acceleration.

To investigate the motion of a particle moving in a straight line with uniform (i.e. constant throughout the motion) acceleration 'f', staarting from some point on the line with a given velocity 'u'.

Let O be the fixed point on the line, from which the distances are measured and the particle starts. (It is not necessary to take O as the point af start, we may choose another point as the point of start).



(Fig. 7)

Let the particle be at P after time t, such that $OP=x$. Then at P the equation of motion is

$$d^2x/dt^2=f \quad ...(i)$$

Integrating with respect to t, we get $dx/dt=ft+C_1$, where C_1 is constant of integration

At O, $t=0$ and $dx/dt=u$ (given),

$$\therefore \quad u=f.0+C_1 \quad \text{or} \quad C_1=u.$$

$$\therefore \quad dx/dt=ft+u \quad \text{or} \quad v=u+ft \quad ...(ii)$$

where v is the velocity of the particle at P.

Again integrating with respect to t, we get

$x=\frac{1}{2}ft^2+ut+c_2$, where c_2 is constant of integration.

At O, $t=0$, $x=0$, we have $0=0+0+c_2$ or $c_2=0$.

$$\therefore \quad x=ut+\tfrac{1}{2}ft^2. \quad ...(iii)$$

The results (i), (ii) and (iii) give the acceleration, velocity and position of the particle at time t.

Also from (i), we have $v\dfrac{dv}{dx}=f$, $\because$ $\dfrac{d^2x}{dt^2}=v\dfrac{dv}{dx}$ (§ 5 Page 3)

Integrating with respect to x, we get

$\frac{1}{2}v^2=fx+c_3$, where c_3 is a constant of integration.

At O, velocity $v=u$, $x=0$, so we have

$$\tfrac{1}{2}u^2=0+c_3 \quad \text{or} \quad c_3=\tfrac{1}{2}u^2$$

$$\therefore \quad \tfrac{1}{2}v^2=fx+\tfrac{1}{2}u^2 \quad \text{or} \quad v^2=u^2+2fx. \quad ...(iv)$$

This result gives the velocity of the particle at any point.

If however the particle starts from rest then $u=0$ and from (iii) and (iv), we get $v=ft$, $x=\frac{1}{2}ft^2$ and $v^2=2fx$.

Note. The formulae (ii), (iii) and (iv) are valid only if the acceleration of the particle is uniform *i.e.* constant throughout the motion.

§ 8. Distance travelled in the nth second.

Distance travelled in the nth second by a particle moving in a straight line with a constant acceleration f

$=$distance travelled in n seconds$-$distance travelled in $(n-1)$ seconds

$=(un+\frac{1}{2}fn^2)-[u(n-1)+\frac{1}{2}f(n-1)^2]$, see § 7 (iii), Page 14

$=u+\frac{1}{2}f(2n-1)$.

Solved Examples on § 6 – § 8

Ex. 1 (a). A particle moves along a straight line in such a way that its distance x from a fixed point O on the line at time t from the start, is given by. $x=3t^2+4t+5$.

Find the velocity and the distance from O at start, the motion being with uniform acceleration. Find also the distance covered in 5th second.

Sol. Given $x=3t^2+4t+5$.

$\therefore$ $dx/dt=5t+4$

$\therefore$ At start, *i.e.* $t=0$ we get $x=5$ and $dx/dt=4$.

$\therefore$ At start, velocity$=4$ units

and its distance from $O=5$ units.

Also the distance covered in 5th second

$=$distance covered in 5 seconds$-$distance covered in 4 seconds

$=(3.5^2+4.5+5)-(3.4^2+4.4+5)=100-69=31$ units.

Ex. 1 (b). If the distance s is given by $s=at^2+bt+c$, where t is the time and a, b, c are constants, prove that $4a(s-c)=v^2-b^2$, where v is the velocity. Ans.

Sol. Given that $s=at^2+bt+c$ ...(i)

$\therefore$ $v=ds/dt=2at+b$

$\therefore$ $v^2-b^2=(2at+b)^2-b^2=4a^2t^2+4abt+b^2-b^2$

$=4a^2t^2+4abt=4a(at^2+bt)$

$=4a[(at^2+bt+c)-c]=4a(s-c)$, from (i)

i.e. $v^2-b^2=4a(s-c)$. Hence proved.

Ex. 1 (c). A particle moving in a straight line with uniform acceleration describes 25m. in the 5th second and 33m. in the 7th second. Find its initial velocity and acceleration.

Sol. Let u m/sec and f m/sec². be the initial velocity and acceleration of the particle. Then from

$s=u+\frac{1}{2}f(2n-1)$ ...See § 8 above

we have $25 = u + \frac{1}{2} f (2 \times 5 - 1) = u + (9/2) f$ ...(i)

and $35 = u + \frac{1}{2} f (2 \times 7 - 1) = u + (13/2) f$ ...(ii)

Solving (i) and (ii) we get $f = 4$ *m/sec²*, $u = 7$ *m/sec.* **Ans.**

Ex. 2. A car starts from rest and accelerates for 13 seconds. Its speed at the end of this period is 19 km./hour.

Find (i) the acceleration of the car ; (ii) the distance travelled in 6 seconds ; (iii) the distance travelled in 7th second and (iv) the distance from start at which its speed is 50 km/hour.

Sol. Let f be the acceleration of the car and v its velocity at distance x from start at time. Then (See § 7 Page 14)

(i) From '$v = ft$', we have

Here $t = 10$ seconds and

$$v = 10 \text{ km./hr} = \frac{10 \times 1000 \times 100}{60 \times 60} \text{ cms./sec.} = \frac{2500}{9} \text{ cms./sec.}$$

$$\therefore \frac{2500}{9} = f.10 \quad \text{or} \quad f = \frac{250}{9} = 27\frac{7}{9} \text{ cms./sec}^2.$$

(ii) From "$x = \frac{1}{2} f t^2$", we have

the distance travelled in 6 seconds $= \frac{1}{2} \cdot \frac{250}{9} \cdot (6)^2$ cms. $= 500$ *cms.*

(iii) Distance travelled in 6th second

$$= \text{"}u + \frac{1}{2} f (2n - 1)\text{"} \qquad \text{...§ 8 Page 15}$$

$$= 0 + \frac{1}{2} \cdot \frac{250}{9} (2 \times 6 - 1) = \frac{1375}{9} = 152\frac{7}{9} \text{ cms.}$$

(iv) From "$v^2 = u^2 + 2fx$", we have

$$\left(\frac{50 \times 1000 + 10}{60 \times 60}\right)^2 = 0 + 2 \times \frac{250}{9} x, \text{ where } x \text{ is the required distance}$$

or $$x = \frac{312500}{9} = 34722\frac{2}{9} \text{ cms.}$$

Ex. 5. A bullet fired into a target loses half its ***velocity after*** **penetrating 3 centimeters. How much further will it penetrate ?**

Sol. Let u cms /sec be the *velocity* with which the bullet strikes the target, then after penetrating 3 centimeters in the target its velocity becomes $\frac{1}{2}u$ cm./sec. Let f cm./sec² be the acceleration of the bullet in the target then *form* "$v^2 = u^2 + 2fx$" we get

$$(\tfrac{1}{2}u)^2 = u^2 + 2f(3) \quad \text{or} \quad f = (1/6)(\tfrac{1}{4}u^2 - u^2) = -\tfrac{1}{8}u^2.$$

The negative sign of f shows that there is retardation of the bullet in the target.

Let us suppose that bullet penetrates a distance of s centimeters further in the target before coming to rest. Then for this motion we have $'u'=\frac{1}{2}u$, $'f'=-\frac{3}{8}u^2$, $'v'=0$, $'x'=s$

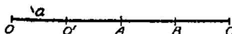
$\therefore$ From "$v^2=u^2+2fx$" we get

$0=(\frac{1}{2}u)^2+2(-\frac{3}{8}u^2)s$ or $s=1$ cm. Ans.

Ex. 4. **A particle moves in a straight line with constant acceleration and its distance from an origin O on the line (not necessarily the position at time $t=0$) at times t_1, t_2, t_3 are x_1, x_2, x_3 respectively. Show that if t_1, t_2, t_3 form an AP. whose common difference is d and x_1, x_2, x_3 are in G. P. then the acceleration is**

$$\sqrt{(x_1-\sqrt{x_3})^2/d^2}.$$

Sol. O is the given origin and let O' be the point of start. Let $OO'=a$ (time is being measured from O').



(Fig. 8)

Let u be the initial velocity and f the acceleration of the particle, then we have

$$O'A=x_1-a=ut_1+\tfrac{1}{2}ft_1^2, \quad \text{...(i)}$$
$$O'B=x_2-a=ut_2+\tfrac{1}{2}ft_2^2, \quad \text{...(ii)}$$
$$O'C=x_3-a=ut_3+\tfrac{1}{2}ft_3^2. \quad \text{...(iii)}$$

Adding (i) and (iii), we get $x_1+x_3-2a=u(t_1+t_3)+\frac{1}{2}f(t_1^2+t_3^2)$. Subtracting two times (ii) from this sum, we get

$$x_1+x_3-2x_2=u(t_1+t_3-2t_2)+\tfrac{1}{2}f(t_1^2+t_3^2-2t_2^2) \quad \text{...(iv)}$$

$\because$ x_1, x_2 and x_3 are in G.P., we get $x_2=\sqrt{(x_1 x_3)}$ ...(v)

As t_1, t_2, t_3 are in A.P., so we get $2t_2=t_1+t_3$. ...(vi)

Also d being the common difference of A.P., we get

$$t_2-t_1=t_3-t_2=d. \quad \text{...(vii)}$$

$\therefore$ from (iv), (v) and (vi), we get

$$x_1+x_3-2\sqrt{(x_1 x_3)}=u(2t_2-2t_2)+\tfrac{1}{2}f(t_1^2+t_3^2-2t_2^2)$$

or $(\sqrt{x_1}-x_3)^2=\frac{1}{2}f[t_1^2+t_3^2-2\{\frac{1}{2}(t_1+t_3)\}^2]$, from (vi)

$=\frac{1}{4}f(t_3-t_1)^2$, on simplifying

$=\frac{1}{4}f\{(t_3-t_2)+(t_2-t_1)\}^2$

$=\frac{1}{4}f\{d+d\}^2$, from (vii) (Note)

$=fd^2$

or $f=(\sqrt{x_1}-\sqrt{x_3})^2/d^2$. Hence proved.

Ex. 5 (a). **If the distances described by any particle during the pth, qth and rth seconds of its motion are a, b, c respectively, prove that $a(q-r)+b(r-p)+c(p-q)=0$.**

Sol. According to the given problem, we have

$a = u + \frac{1}{2} f(2p-1)$ [see § 8 Page 15] ...(i)

$b = u + \frac{1}{2} f(2q-1)$ ...(ii) $c = u + \frac{1}{2} f(2r-1)$, ...(iii)

where u and f are initial velocity and acceleration of the particle respectively.

Multiplying (i), (ii) and (iii) by $(q-r)$, $(r-p)$ and $(p-q)$ respectively and adding we get $a(q-r)+b(r-p)+c(p-q)$

$$= u\,[(q-r)+(r-p)+(p-q)] + \tfrac{1}{2} f\,[(2p-1)(q-r) + (2q-1)(r-p) + (2r-1)(p-q)]$$

$$= u\,[0] + \tfrac{1}{2} f\,[2\{p(q-r)+q(r-p)+r(p-q)\} - \{(q-r)+(r-p)+(p-q)\}]$$

$$= \tfrac{1}{2} f\,[2(0)-(0)] = 0.$$ Hence proved.

***Ex. 5 (b). If the co-ordinates of a point moving with the constant acceleration be x_1, x_2, x_3 at the instants t_1, t_2, t_3 respectively, prove that the acceleration is**

$$2\left[\frac{(x_2-x_3)\,t_1+(x_3-x_1)\,t_2+(x_1-x_2)\,t_3}{(t_2-t_3)\,(t_3-t_1)\,(t_1-t_2)}\right]$$

Sol. Let u be the initial velocity and f the acceleration of the point, then we have

$$x_1 = ut_1 + \tfrac{1}{2} f t_1^2, \quad ...(i)$$

$$x_2 = ut_2 + \tfrac{1}{2} f t_2^2, \quad ...(ii)$$

$$x_3 = ut_3 + \tfrac{1}{2} f t_3^2. \quad ...(iii)$$

From (i) and (ii), we have $x_1 - x_2 = u(t_1-t_2) + \frac{1}{2} f (t_1^2 - t_2^2)$

or $(x_1-x_2)\,t_3 = u\,(t_1t_3 - t_2t_3) + \frac{1}{2} f\,(t_1^2 t_3 - t_2^2\,t_3)$. ...(iv)

Similarly $(x_2-x_3)\,t_1 = u\,(t_2t_1 - t_3t_1) + \frac{1}{2} f\,(t_2^2t_1 - t_3^2t_1)$ ...(v)

and $(x_3-x_1)\,t_2 = u\,(t_3t_2 - t_1t_2) + \frac{1}{2} f\,(t_3^2t_2 - t_1^2t_2)$. ...(vi)

Adding (iv), (v) and (vi), we get

$$\Sigma\,[(x_2-x_3)\,t_1] = \tfrac{1}{2} f\,(t_1^2t_3 - t_2^2t_3 + t_2^2t_1 - t_3^2t_1 + t_3^2t_2 - t_1^2t_2)$$

or $$f = \frac{2\,[(x_2-x_3)\,t_1+(x_3-x_1)\,t_2+(x_1-x_2)\,t_3]}{t_1^2t_3 - t_2^2t_3 + t_2^2t_1 - t_3^2t_1 + t_3^2t_2 - t_1^2t_2} \quad ...(vii)$$

Now $t_1^2t_3 - t_2^2t_3 + t_2^2t_1 - t_3^2t_1 + t_3^2t_2 - t_1^2t_2$

$= t_1^2\,(t_3-t_2) + t_1\,(t_2^2-t_3^2) + (t_3^2t_2 - t_2^2t_3)$,

arranging in descending order of t_1

$= t_1^2\,(t_3-t_2) + t_1\,(t_2^2-t_3^2) + t_2t_3\,(t_3-t_2)$

$= (t_3-t_2)\,[t_1^2 - t_1\,(t_2+t_3) + t_2t_3]$

$= (t_3-t_2)\,[(t_1^2 - t_1t_2) + t_2t_3 - t_1t_3]$

$= (t_3-t_2)\,[t_1\,(t_1-t_2) - t_3\,(t_1-t_2)]$

$= (t_3-t_2)\,(t_1-t_2)\,(t_1-t_3) = (t_1-t_2)\,(t_2-t_3)\,(t_3-t_1)$

$\therefore$ From (vii), $f=\dfrac{2\,[(x_2-x_3)\,t_1+(x_3-x_1)\,t_2+(x_1-x_2)\,t_3]}{(t_1-t_2)\,(t_2-t_3)\,(t_3-t_1)}$

****Ex. 5 (c). If a point moving in a straight line with constant acceleration describes distances S_1 and S_2 in two successive intervals of time t_1, t_2 prove that acceleration is $2\,(t_1S_2-t_2S_1)/[t_1t_2\,(t_1+t_2)]$.**

Sol. Let u be the initial velocity and f the acceleration of the point, then from "$x=ut+\frac{1}{2}ft^2$" we have

$$S_1=ut_1+\tfrac{1}{2}ft_1^2 \quad \text{...(i)}$$

and $$S_1+S_2=u\,(t_1+t_2)+\tfrac{1}{2}f\,(t_1+t_2)^2. \quad \text{...(ii)}$$

Subtracting (i) from (ii), we get

$$S_2=ut_2+\tfrac{1}{2}f\,(2t_1t_2+t_2^2) \quad \text{...(iii)}$$

Multiplying (i) by t_2 and (iii) by t_1 and subtracting, we get

$$S_2t_1-S_1t_2=[ut_2+\tfrac{1}{2}f\,(2t_1t_2+t_2^2)]\,t_1-[ut_1+\tfrac{1}{2}ft_1^2]\,t_2$$

$$=ft_1^2t_2+\tfrac{1}{2}ft_2^2t_1-\tfrac{1}{2}ft_1^2t_2=\tfrac{1}{2}ft_2t_1\,(t_1+t_2)$$

or $$f=2\,(S_2t_1-S_1t_2)/[t_1t_2\,(t_1+t_2)].$$ Hence proved.

[or $$f=2\left(\frac{S_2t_1-S_1t_2}{t_1t_2}\right)\cdot\frac{1}{(t_1+t_2)}$$

$$=2\left(\frac{S_2}{t_2}-\frac{S_1}{t_1}\right)\Big/(t_1+t_2)].$$

***Ex. 6. Two cars start off to race with velocities u, u' and move with acceleration f, f'; the result being a dead heat. Prove that the length of the course is $2\,(u-u')\,(uf'-u'f)/(f-f')^2$.**

Sol. The result of the race is dead heat that means the cars have travelled equal distances in equal time.

Let the length of the course be s.

Then for the first car, we get $s=ut+\frac{1}{2}ft^2$ ...(i)

and for the second car, we get $s=u't+\frac{1}{2}f't^2$. ...(ii)

(i) and (ii) can be rewritten as

$$ft^2+2ut-2s=0 \text{ and } f't^2+2u't-2s=0.$$

Solving these equations simultaneously, we get

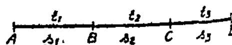
$$\frac{t^2}{-4us+4u's}=\frac{t}{-2f's+2fs}=\frac{1}{2u'f+2uf'}$$

or $$\frac{t^2}{2s\,(u'-u)}=\frac{t}{s\,(f-f')}=\frac{1}{(u'f-uf')}.$$

Eliminating t, we get $s^2\,(f-f')^2=2s\,(u'-u)\,(u'f-uf')$

or $$s=2\,(u'-u)\,(u'f-uf')/(f-f')^2, \quad \because s\neq 0.$$

*Ex. 7. A point moves with uniform acceleration and v_1, v_2, v_3 denote the average velocities in three successive intervals t_1, t_2, t_3; prove that $(v_1-v_2)/(v_2-v_3)=(t_1+t_2)/(t_2+t_3)$.



(Fig. 9)

Sol. Let u be the initial velocity and f the acceleration. Let the successive intervals be $AB=s_1$, $BC=s_2$ and $CD=s_3$, described in times t_1, t_2 and t_3 respectively.

Then $AB=s_1=ut_1+\frac{1}{2}ft_1^2$, ...(i)

$AC=s_1+s_2=u(t_1+t_2)+\frac{1}{2}f(t_1+t_2)^2$ ...(ii)

$AD=s_1+s_2+s_3=u(t_1+t_2+t_3)+\frac{1}{2}f(t_1+t_2+t_3)^2$. ...(iii)

Subtracting (i) from (ii), $s_2=ut_2+\frac{1}{2}f(t_2^2+2t_1t_2)$ ...(iv)

Subtracting (ii) from (iii), $s_3=ut_3+\frac{1}{2}f(t_3^2+2t_1t_3+2t_2t_3)$ ...(v)

∴ If v_1, v_2 and v_3 be the average velocities in these intervals, then $v_1=s_1/t_1=u+\frac{1}{2}ft_1$, from (i)

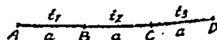
$v_2=s_2/t_2=u+\frac{1}{2}f(t_2+2t_1)$, from (iv)

$v_3=s_3/t_3=u+\frac{1}{2}f(t_3+2t_1+2t_2)$, from (v).

$$\therefore \frac{v_1-v_2}{v_2-v_3}=\frac{\{u+\frac{1}{2}ft_1\}-\{u+\frac{1}{2}f(t_2+2t_1)\}}{\{u+\frac{1}{2}f(t_2+2t_1)\}-\{u+\frac{1}{2}f(t_3+2t_1+2t_2)\}}$$

$$=\frac{-\frac{1}{2}f(t_2+t_1)}{-\frac{1}{2}f(t_3+t_2)}=\frac{t_2+t_1}{t_3+t_2}$$ Hence proved.

**Ex. 8 (a). A point moving in a straight line with uniform acceleration describes successive equal distances in times t_1, t_2, t_3, then show that $\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}=\frac{3}{t_1+t_2+t_3}$.



(Fig. 10)

Sol. Let the successive equal distances be $AB=BC=CD=a$ (say). Let u be the initial velocity and f the acceleration. Then from

"$x=ut+\frac{1}{2}ft^2$", we get ...(i)

$AB=a=ut_1+\frac{1}{2}ft_1^2$ ...(ii)

$AC=2a=u(t_1+t_2)+\frac{1}{2}f(t_1+t_2)^2$ ...(iii)

and $AD=3a=u(t_1+t_2+t_3)+\frac{1}{2}f(t_1+t_2+t_3)^2$ ...(iv)

From (i), we have $a/t_1=u+\frac{1}{2}ft_1$.

Subtracting (i) from (ii) and dividing by t_2, we get

$a/t_2=u+ft_1+\frac{1}{2}ft_2$. ...(v)

Subtracting (ii) from (iii) and dividing by t_2, we get

$$a/t_3 = u + ft_1 + ft_2 + \tfrac{1}{2} ft_3. \quad \text{...(vi)}$$

From (iv), (v) and (vi), we get

$$\frac{a}{t_1} - \frac{a}{t_2} + \frac{a}{t_3} = (u + \tfrac{1}{2} ft_1) - (u + ft_1 + \tfrac{1}{2} ft_2) + (u + ft_1 + ft_2 + \tfrac{1}{2} ft_3)$$

$$= u + \tfrac{1}{2} f(t_1 + t_2 + t_3) = 3a/(t_1 + t_2 + t_3), \text{ from (iii)}$$

or $$\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}.$$ Hence proved.

Ex. 8 (b). Prove that when a particle moves with uniform acceleration, the distances described in consecutive equal intervals of time are in A. P.

Sol. Let f be the uniform acceleration, u the velocity at the start of the first interval and t be the equal interval of time.

Now distance travelled in the nth interval

=[distance described during n intervals (*i.e.*, in time nt)] − [distance described during $(n-1)$ intervals {*i.e.* in time $(n-1)t$}]. **(Note)**

$$= [u(nt) + \tfrac{1}{2} f(nt)^2] - [u(n-1)t + \tfrac{1}{2} f\{(n-1)t\}^2]$$

$$= ut\{n-(n-1)\} + \tfrac{1}{2} ft^2\{n^2 - (n-1)^2\}$$

$$= ut + \tfrac{1}{2} ft^2(2n-1).$$

Putting $n = 1, 2, 3, \ldots$ in (i), the distances described in 1st, 2nd, 3rd,...intervals are respectively

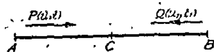
$$ut + \tfrac{1}{2} ft^2(1),\ ut + \tfrac{1}{2} ft^2(3),\ ut + \tfrac{1}{2} ft^2(5), \ldots,$$

Evidently these distances are in A.P. whose common difference

$= [ut + \tfrac{1}{2} ft^2(3)] - [ut + \tfrac{1}{2} ft^2(1)] = ft^2.$ Hence proved.

Ex. 9. Two points P and Q move in a straight line AB. The point P starts from A in the direction of AB with velocity u and acceleration f, and at the same time Q starts from B in the direction of BA with velocity u_1 and acceleration f_1; if they pass one another at the middle point of AB and arrive at the other ends of AB with equal velocities, prove that $(u+u_1)(f-f_1) = 8(fu_1 - f_1 u)$.

Sol. Let C be the middle point of AB. Let $AB = 2s$, then $AC = BC = s$.



(Fig. 11)

Let t be the time taken by each particle to reach C.

∴ From "$x = ut + \tfrac{1}{2} ft^2$, we have

for the point P, $s=ut+\frac{1}{2}ft^2$ ...(i)

and for the point Q, $s=u_1t+\frac{1}{2}f_1t^2$. ...(ii)

$\therefore$ From (i) and (ii), we get $ut+\frac{1}{2}ft^2=u_1t+\frac{1}{2}f_1t^2$

or $(u-u_1)=\frac{1}{2}t(f_1-f)$ or $t=\dfrac{2(u-u_1)}{(f_1-f)}$ ...(iii)

Let v be the velocity with which each particle reaches the other end.

Then from "$v^2=u^2+2fx$", we have

for the point P, $v^2=u^2+2f(2s)$,

and for the point Q, $v^2=u_1^2+2f_1(2s)$.

Equating these two results, we get $u^2+4fs=u_1^2+4f_1s$

or $4s(f-f_1)=u_1^2-u^2$ or $s=(u_1^2-u^2)/[4(f-f_1)]$ ...(iv)

Substituting the values of t_1 and s from (iii) and (iv) in (i),

we get $\dfrac{(u_1^2-u^2)}{4(f-f_1)}=u\left\{\dfrac{2(u-u_1)}{f_1-f}\right\}+\frac{1}{2}f\left\{\dfrac{2(u-u_1)}{(f_1-f)}\right\}^2$

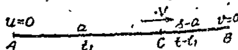
or $\dfrac{(u_1-u)(u_1+u)}{4(f_1-f)}=\dfrac{2(u_1-u)}{(f-f_1)}\left[u+f\left(\dfrac{u_1-u}{f-f_1}\right)\right]$

or $(u_1+u)=8[u(f-f_1)+f(u_1-u)]/(f-f_1)$

or $(u_1+u)(f-f_1)=8[u_1f-uf_1]$. Hence proved.

***Ex. 10. A train travels a distance s in t seconds. It starts from rest and ends at rest. In the first part of the journey it moves with constant acceleration f and in the second part with constant retardation f'. Show that $\sqrt{[2s.\{(1/f)+(1/f')\}]}$.**

Sol. A and B are two stations. The train moves from A to C with acceleration f and from C to B with retardation f'.



(Fig. 12)

Let $AC=a$, then $CB=s-a$. Let t_1 be the time taken in moving from A to C, then time from C to B is $t-t_1$. Let V be the velocity at C.

$\therefore$ For the motion from A to C, from "$v=u+ft$" and "$v^2=u^2+2fx$", we have $V=0+ft_1$ ...(i)

and $V^2=0+2fa$. ...(ii)

Also for the motion from C to B, from

"$v=u+ft$" and "$v^2=u^2+2fx$",

we have $0=V-f'(t-t_1)$ ...(iii)

and $0=V^2-2f'(s-a)$. ...(iv)

From (ii) and (iv) we get $2a=\frac{V^2}{f}$ and $\frac{V^2}{f'}=2(s-a)$

Adding these, we get $V^2\left(\frac{1}{f}+\frac{1}{f'}\right)=2s.$

Now total time taken

$=t=t_1+(t-t_1)=\frac{V}{f}+\frac{V}{f'}$, from (i) and (iii)

$=\left(\frac{1}{f}+\frac{1}{f'}\right)V=\left(\frac{1}{f}+\frac{1}{f'}\right)\frac{\sqrt{(2s)}}{\sqrt{[1/f+1/f']}}$, from (v)

$=\sqrt{\left[2s\left(\frac{1}{f}+\frac{1}{f'}\right)\right]}.$ Hence proved.

***Ex. 11. Two points move in the same straight line starting at the same moment from the same point in the same direction. The first moves with constant velocity u and the second with constant acceleration f (its initial velocity being zero). Show that the greatest distance between the points before the second catches first is $(u^2/2f)$ at the end of the time u/f from the start.**

Sol. The distance moved by the first particle in time $t=ut$.

And the distance moved by the second particle in time t

$$=0.t+\tfrac{1}{2}ft^2=\tfrac{1}{2}ft^2.$$

Let s be the distance between the particle at time t.

Then $\quad s=ut-\frac{1}{2}ft^2.$ (Note) ...(i)

Differentiating (i) with respect to t, we get

$$(ds/dt)=u-ft. \quad ...(ii)$$

Equating (ds/dt) to zero, we get $u-ft=0$ or $t=u/f$.

Also from (ii) differentiating again, we get

$$(d^2s/dt^2)=-f=\text{negative}.$$

$\therefore$ s is maximum for $t=u/f$.

And maximum value of $s=u(u/f)-\frac{1}{2}f(u/f)^2$, from (i)

$$=\frac{u^2}{f}-\frac{u^2}{2f}=\frac{u^2}{2f}.$$ Hence proved.

Ex. 12. A body projected vertically upwards with a velocity u, after time t another body is projected vertically upwards from the same point with a velocity v, where $v<u$. If they meet as soon as possible, prove that $t=\{u-v+\sqrt{(u^2-v^2)}\}/g$.

Sol. Let the two bodies meet each other at a hight h after time T of the projection of second body. Then before meeting,

the first body was in motion for time $(t+T)$ whereas the second body was in motion for time T.

The distance moved by the second body in time $(t+T)$

$$=u\,(t+T)-\tfrac{1}{2}g\,(t+T)^2.$$

And the distance moved by the second body in time T

$$=vT-\tfrac{1}{2}gt^2=h \text{ (supposed above).} \quad ...(i)$$

$\because$ The two bodies meet each other.

$\therefore$ They are equidistant from the point of projection.

Hence $\quad u\,(t+T)-\tfrac{1}{2}g\,(t+T)^2=vT-\tfrac{1}{2}gT^2$

or $\quad u\,(t+T)-\tfrac{1}{2}g\,(t^2+2tT)=vT$

or $\quad gt^2+2t\,(gT-u)+2\,(v-u)\,T=0. \quad ...(ii)$

Also from (i) we get, $\quad h=vT-\tfrac{1}{2}gT^2$

$$\therefore \quad (dh/dT)=v-gT$$

$\because$ h increases as T increases, $\therefore$ T is minimum when h is minimum, *i.e.* when $dh/dT=0$, *i.e.* when $v-gT=0$ or $T=v/g$.

Substituting this value of T in (ii), we get

$$gt^2+2t\,(v-u)+2\,(v-u)\,(v/g)=0$$

or $\quad g^2t^2-2gt\,(u-v)-2v\,(u-v)=0$

or $\quad t=\dfrac{2g\,(u-v)+\sqrt{[4g^2\,(u-v)^2+8vg^2\,(u-v)]}}{2g^2}$

or $\quad t=[(u-v)+\sqrt{(u^2-v^2)}]/g,$

neglecting the negative sign which gives negative value of t.

****Ex. 13. If a point moves with the constant acceleration, the space average of the velocity over any distance is $\tfrac{2}{3}\cdot\dfrac{u_1^2+u_1u_2+u_2^2}{u_1+u_2}$ and the time average of the velocity is $\tfrac{1}{2}\,(u_1+u_2)$, where u_1 and u_2 are the initial and final velocities.**

Sol. Let v be the velocity and s be the distance travelled from the starting point at time t. Let a be the whole distance moved and T be the whole time taken, then we have

$$\text{space average of velocity}=\frac{1}{a}\int_{s=0}^{a} v\,ds$$

and time average of velocity $=\int_{t=0}^{T} v\,dt.$ (Note)

Let f be the acceleration of the point. Then from "$v^2=u^2+2fs$", we get

$$v^2=u_1^2+2fs \quad ...(i)$$

and

$$u_2^2=u_1^2+2fa \quad ...(ii)$$

Also from "$v=u+ft$" we get $v=u_1+ft$ ...(iii)

and $u_2=u_1+fT.$ ...(iv)

$\therefore$ Space average of velocity

$$=\frac{1}{a}\int_{s=0}^{a} v\,ds=\frac{1}{a}\int_{s=0}^{a}(u_1^2+2fs)^{1/2}\,ds, \text{ from (i)}$$

$$=\frac{1}{a}\left[\frac{2}{3}\frac{(u_1^2+2fs)^{3/2}}{2f}\right]_0^a=\frac{(u_1^2+2fa)^{3/2}-(u_1^2)^{3/2}}{3fa}$$

$$=\frac{(u_2^2)^{3/2}-(u_1^2)^{3/2}}{\frac{3}{2}(u_2^2-u_1^2)},\ \because fa=\frac{u_2^2-u_1^2}{2} \text{ from (ii)}$$

$$=\frac{2(u_2^3-u_1^3)}{3(u_2^2-u_1^2)}=\frac{2(u_2^2+u_1u_2+u_1^2)}{3(u_2+u_1)}.$$

And time average of velocity $=\frac{1}{T}\int_{t=0}^{T} v\,dt=\frac{1}{T}\int_0^T (u_1+ft)\,dt,$ from (ii)

$$=\frac{1}{T}\left[\frac{(u_1+ft)^2}{2f}\right]_0^T=\frac{(u_1+fT)^2-u_1^2}{2fT}$$

$$=\frac{u_2^2-u_1^2}{2(u_2-u_1)}, \text{ from (iv) } fT=u_2-u_1$$

$$=\tfrac{1}{2}(u_1+u_2).$$ Hence proved.

****Ex. 14. Prove that the average velocity is the mean of the initial and final velocities and is equal to the velocity at the middle of the interval.**

Sol. Let u be the initial velocity of the particle and let it travel a distance s in time t. If v be the final velocity and f be the acceleration, then we have

$$v=u+ft \quad \text{and} \quad s=ut+\tfrac{1}{2}ft^2. \qquad \text{...(i)}$$

$\therefore$ Average velocity $=\dfrac{\text{total distance travelled}}{\text{total time taken}}$

$$=\frac{s}{t}=\frac{ut+\frac{1}{2}ft^2}{t}=\frac{2u+ft}{2}$$

$$=\tfrac{1}{2}[u+(u+ft)] \qquad \text{(Note)}$$

$$=\tfrac{1}{2}(u+v), \text{ from (i)}$$

= mean of initial and final velocities.

Also average velocity $=\dfrac{2u+ft}{2}$, as before

$$=u+f(\tfrac{1}{2}t)$$

= velocity at time $\frac{1}{2}t$, from $v=u+ft$

$=$ velocity at the middle of interval.

Hence proved.

Ex. 15. Prove that the mean kinetic energy of a particle of mass m moving under a constant acceleration, in any interval of time is $\frac{1}{6}m(u_1^2+u_1u_2+u_2^2)$, where u_1 and u_2 are the initial and final velocities.

Sol. Let after time t, the velocity of the particle be v. Let f be the constant acceleration. Let T be the total time taken.

Then from "$v=u+ft$", we have

$$v=u_1+ft \quad \text{...(i)} \qquad \text{and} \qquad u_2=u_1+fT \quad \text{...(ii)}$$

Now mean K.E. $= \frac{1}{T}\int_{t=0}^{T} (\frac{1}{2}mv^2)\, dt$ (Note)

$$= \frac{m}{2T}\int_0^T (u_1+ft)^2\, dt, \text{ from (i)}$$

$$= \frac{m}{2T}\left[\frac{(u_1+ft)^3}{3f}\right]_0^T = \frac{m}{6fT}\left[(u_1+fT)^3-u_1^3\right]$$

$$= \frac{m}{6(u_2-u_1)}\left[u_2^3-u_1^3\right], \text{ from (ii)},$$

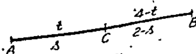
putting $fT=u_2-u_1$

$$=(1/6)m(u_2^2+u_1u_2+u_1^2).$$

Hence proved.

***Ex. 16.** A train stopping at two stations 2 kms. apart on a straight line takes 4 minutes for the journey. Assuming that its motion is first uniformly accelerated and then uniformly retarded. Prove that $1/x+1/y=4$, where x and y are the magnitude of the acceleration and retardation respectively, a km. and a minute being the units.

Sol Let A and B be the stations. Let the train accelerate from A to C then retard from C to B. Let $AC=s$ kms.



(Fig. 13)

$\therefore$ $CB=(2-s)$ kms. Let time taken from A to C be t minutes then the time from C to $B=(4-t)$ minutes. Let v kms. per minute be the velocity at C.

Then for the motion from A to C, from "$v=u+ft$" and "$v^2=u^2+2fs$" we get $v=xt$ ...(i) and $v^2=2xs$. ...(ii)

And for the motion from ... have

$$0=v-y(4-t) \quad \text{...(iii)} \qquad v^2=2y(2-s)$$

From (i) and (iii), we get v/x ... $=4-t$,

Adding these we have $(v/x)+(v/y)=4.$ ...(v)

From (ii) and (iv), we get $v^2/x=2s$ and $v^2/y=4-2s.$

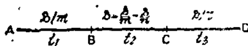
Adding these, we have $(v^2/x)+(v^2/y)=4.$ ...(vi)

Dividing (vi) by (v), we get $v=1$

and therefore from (v), we get $(1/x)+(1/y)=4.$ Hence proved

***Ex. 17.** For $1/m$ of the distance between two stations a train is uniformly accelerated and $1/n$ of the distance it is uniformly retarded, it starts from rest at one station and comes to rest at the other. Prove that the ratio of its greatest velocity to its average velocity is $\left(1+\frac{1}{m}+\frac{1}{n}\right):1.$

Sol. A and B are the stations. Let $AB=s$, from A to B the train accelerates, from B to C it moves with uniform velocity v (say) and from C to D it retards. Let f and f' be the acceleration and retardation respectively. Let t_1, t_2 and t_3 be the times taken in moving from A to B, B to C and C to D respectively.



(Fig. 14)

Then the greatest velocity $=v$ and

average velocity $=\frac{\text{total distance}}{\text{total time}}=\frac{s}{(t_1+t_2+t_3)}.$

$\therefore$ Required ratio $=\frac{\text{greatest velocity}}{\text{average velocity}}=\frac{v}{s/(t+t_2+t_3)}$

$=\frac{vt_1+vt_2+vt_3}{s}$ ...(i)

Now $AB=\frac{s}{m}$ and $CD=\frac{s}{n}$ $\therefore$ $BC=s-\frac{s}{m}-\frac{s}{n}.$

Also for the motion from A to B,

from "$v=u+ft$" and "$x=ut+\frac{1}{2}ft^2$",

we get $v=ft_1$ and $s/m=\frac{1}{2}ft_1^2.$ ...(ii)

$\therefore$ $vt_1=ft_1^2=2s/m,$ from (ii)

or $vt_1=2s/m.$ ...(iii)

For the motion from B to C, we have

$BC=vt_2$ or $vt_2=s-(s/m)-(s/n).$ ...(iv)

For the motion from C to D, from

"$x=ut+\frac{1}{2}ft^2$" and "$v=u+ft$"

we have $s/n=vt_3-\frac{1}{2}f't_3^2,$ ...(v) and $0=v-f't_3$ ...(vi)

$\therefore$ from (v), $s/n = f'\,t_3^2 - \frac{1}{2}f'\,t_3^2$, from (vi) $v = f'\,t_3$ ...(vii)

or $2s/n = f'\,t_3^2 = vt_3$, from (vi)

Hence from (i), required ratio $= \dfrac{\left(\dfrac{2s}{m}\right) + \left(s - \dfrac{s}{m} - \dfrac{s}{n}\right) + \left(\dfrac{2s}{n}\right)}{s}$,

substituting the values of vt_1, vt_2 and vt_3 from (iii), (iv) and (vii)

$= \dfrac{s\,[1 + (1/m) + (1/n)]}{s} = \dfrac{[1 + (1/m) + (1/n)]}{1}$. Hence proved.

*Ex. 18. The speed of a train increases at a constant rate α from 0 to v, then remains constant for an interval and finally decreases to zero at a constant rate β. If l be the total distance described, prove that the total time occupied is $l/v + \frac{1}{2}v\,(1/\alpha + 1/\beta)$. And find the least value of time when $\beta = \alpha$.

Sol. Let a, b and c be distances of the first, second and third parts of the journey. Then $l = a + b + c$. ...(i)

Let t_1, t_2 and t_3 be the times taken to cover these distances.

Then for first part of the motion $v = \alpha t_1$ ...(ii)

and $v^2 = 2\alpha\, a$. ...(iii)

And for the third part of the motion, $v = \beta t_3$ ...(iv)

and $v^2 = 2\beta c$ ...(v)

From (ii) and (iii), we get $a = \frac{1}{2}vt_1$ ...(vi)

And from (iv) and (v), we get $c = \frac{1}{2}vt_3$. ...(vii)

For the second part of the motion, $vt_2 = b$.

$\therefore$ From (i) and (vii) we get $a + b + c = \frac{1}{2}vt_1 + vt_2 + \frac{1}{2}vt_3$

or $l = \frac{1}{2}\,v\,(t_1 + 2t_2 + t_3)$, from (i) (Note)

$= v\,(t_1 + t_2 + t_3) - \frac{1}{2}v\,(t_1 + t_3)$

or $\dfrac{l}{v} = (t_1 + t_2 + t_3) - \frac{1}{2}\left(\dfrac{v}{\alpha} + \dfrac{v}{\beta}\right)$, from (iii) and (iv)

or $(t_1 + t_2 + t_3) = \dfrac{l}{v} + \dfrac{v}{2}\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)$ Hence proved.

Let t be the total time taken, when $\alpha = \beta$.

Then $t = \dfrac{l}{v} + \dfrac{v}{2}\left(\dfrac{1}{\alpha} + \dfrac{1}{\alpha}\right) = \dfrac{l}{v} + \dfrac{v}{\alpha}$. ...(viii)

$\therefore$ $\dfrac{dt}{dv} = -\dfrac{l}{v^2} + \dfrac{1}{\alpha}$ and $\dfrac{d^2t}{dv^2} = \dfrac{2l}{v^3} =$ positive

For minimum t, $\dfrac{dt}{dv} = 0$ or $-\dfrac{l}{v^2} + \dfrac{1}{\alpha} = 0$ or $\dfrac{l}{v} = \dfrac{v}{\alpha}$

Also d^2t/dt^2 being positive, t is minimum.

$\therefore$ from (viii), least value of $t = v/\alpha + v/\alpha$, $\because l/v = v/\alpha$

$= 2v/\alpha$. **Ans.**

Ex. 19. **A dog, seeing a hare at a distance d starts with velocity u and moves with acceleration α in order to catch it, while the hare with acceleration β starts from rest. Show that the dog will overtake the hare if $\alpha < \beta < \alpha + (u^2/2d)$.**

Sol. Let the dog overtake the hare after time t when the hare has gone through a distance x from rest.

So for the hare, we have

'u'=initial velocity$=0$, 'f'=acc.$=\beta$, 's'$=x$ and 't'$=t$.

So from $s = ut + \frac{1}{2} ft^2$, we have

$$x = 0.t + \tfrac{1}{2}\beta t^2 \quad \text{or} \quad x = \tfrac{1}{2}\beta t^2 \qquad \text{...(i)}$$

For the dog, we have

'u'=initial velocity$=u$; 'f'=acceleration$=\alpha$,

's'=distance$=d+x$, 't'$=t$

$\therefore$ From "$s = ut + \frac{1}{2} ft^2$" we have

$$d + x = ut + \tfrac{1}{2}\alpha t^2 \qquad \text{...(ii)}$$

Subtracting (i) from (ii) we get

$$d = ut + \tfrac{1}{2}(\alpha - \beta) t^2 \quad \text{or} \quad (\alpha - \beta) t^2 + 2ut - 2d = 0.$$

If the dog overtakes the hare, then t given by the above quadratic equation must be real and the condition for the same is

$$b^2 - 4ac \geqslant 0$$

i.e. $(2u)^2 - 4(\alpha - \beta)(-2d) \geqslant 0$

i.e. $u^2 + 2d(\alpha - \beta) \geqslant 0$ i.e. $u^2 - 2d(\beta - \alpha) \geqslant 0$ ...(iii)

$\therefore$ L.H.S. of (iii) is greater than or equal to zero

if $\alpha < \beta$ but $u^2 > 2d(\beta - \alpha)$

i.e. if $\alpha < \beta$ but $(u^2/2d) + \alpha > \beta$

i.e. if $\alpha < \beta < (u^2/2d) + \alpha$. Hence proved.

****Ex. 20.** **A lift ascends with constant acceleration f, then with constant velocity and finally stops under constant retardation f. If the total distance ascended is s and total time occupied is t, show that the time during which the lift is ascended with constant velocity is $\sqrt{(t^2 - 4s/f)}$.**

Sol. The whole journey consists of three parts. Let v be the maximum velocity acquired in 1st part, which will remain uniform in 2nd part and will gradually reduce to zero in 3rd part due to retardation. Since acceleration in the first part and retardation in 3rd part are equal, therefore the time taken and distance covered

in acquiring velocity v in first part from start will be equal to the corresponding time and distance in 3rd part in destroying the velocity v. Let distance travelled be x_1 and time taken be t_1 in each of these first and third parts.

$\therefore$ In the first and last parts of motion, we have

$$v = ft_1 \quad \text{...(i)} \quad \text{and} \quad v^2 = 2fx_1. \quad \text{...(ii)}$$

In the second part of motion, distance moved $= s - 2x_1$ and time taken $= t - 2t_1$.

$$\therefore \quad (s - 2x_1) = v\,(t - 2t_1). \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get $s - v^2/f = ft_1\,(t - 2t_1)$

or $\quad s - ft_1^2 = ft_1\,(t - 2t_1)$, from (i)

or $\quad ft_1^2 - ftt_1 + s = 0 \quad$ or $\quad t_1 = \dfrac{ft \pm \sqrt{\{f^2t^2 - 4fs\}}}{2f}$

or $\quad 2t_1 = t \pm \sqrt{\{t^2 - (4s/f)\}} \quad$ or $\quad t - 2t_1 = \sqrt{\{t^2 - (4s/f)\}}$,

which gives the required time.

Exercises on § 6—§ 8

Ex. 1. Find the acceleration of the car which accelerates from 10 km./hr. to 40 km /hr. in 5 seconds. Ans. : 6 km./hr^2.

Ex 2. A car is brought to rest from a speed of 60 km./h. in 10 seconds. Find the average retardation. Ans. 125/9 m./sec^2.

Ex. 3. The greatest possible acceleration is 1 m./sec^2 and the greatest possible retardation is $\frac{4}{3}$ m /sec^2. Find the least time taken to run between two stations 12 km. apart if the maximum speed is 22 m./sec.

Ex. 4. An electric train starts from a station and where its speed is v, its acceleration is $k\,(v_1 - v)$. Find the time it takes to attain half of its maximum speed v_1 and show that it has then travelled a distance $(\log 2 - \frac{1}{2}) \times (v_1/k)$.

Ex. 5. The speed of a train is reduced from 40 m. p. h. to 20 m.p.h. whilst it traveles a distance of 150 yards. If the retardation be uniform, find how much further will it travel before coming to rest. [1 mile = 1760 yards ; 1 yard = 3 feet]

§ 9. Bodies falling under gravity.

If a body is freely falling vertically downwards with constant acceleration, then this acceleration is due to gravity and is generally denoted by 'g'. If it is projected vertically upwards, its retardation is constant and equal to g.

The value of 'g' in F. P. S. system is 32 ft/sec^2, in C. G. S. system 981 cm/sec^2 and in M.K.S. system is 98 m/sec^2.

In the case of bodies falling under gravity, we can find results similar to those in § 7 and § 8 by using 'g' in place of the constant acceleration 'f'. And if the body is projected vertically upwards, we are to replace 'f' by '$-g$'.

Solved Examples on § 9.

Ex. 1. A particle falls freely from the top of a tower and during the last two seconds it falls through (3/4)th of the height of the tower. Find the height of the tower.

Sol. Let h be the height of the tower and n seconds be the total time taken by the particle in falling the distance h.

Then distance fallen in $(n-2)$ seconds$=h-(3/4)\ h=h/4$ and distance fallen in n seconds$=h$.

$\therefore$ From '$s=ut+\frac{1}{2}ft^2$' we have

$$h/4=0+\tfrac{1}{2}g\,(n-2)^2 \qquad ...(i)$$

and $$h=0+\tfrac{1}{2}g\,(n)^2 \qquad ...(ii)$$

Subtracting (i) from (ii) we get $3h/4=\frac{1}{2}g\,(4n-4)$

or $3h=8g\,(n-1)$ or $n=(3h/8g)+1=(3h+8g)/g$

$\therefore$ From (ii), $h=\frac{1}{2}g\,[(3h+8g)/8g]^2$

or $128gh=(3h+8g)^2$ or $9h^2-80gh+64g^2=0$

or $h=8g$ or $(8g/9)$. **Ans.**

Ex. 2. A particle is projected vertically upwards with velocity u. Show that it cannot go above the height of $u^2/(2g)$ and will return to ground after an interval of time $2(u/g)$.

Sol. Let h be the maximum height reached by the particle in its upwards motion. Then for the upward motion we have,

'u'$=u$, 'f'$=-g$, 'x'$=h$, 'v'$=0$

$\therefore$ From "$v^2=u^2+2fx$", we get

$0=u^2+2\,(-g)\,h$ or $h=u^2/(2g)$. Hence proved.

Let t_1 be the time taken by the particle in covering this height h in its upward motion.

$\therefore$ From "$v=u+ft$", we get $0=u+(-g)\,t_1$

or $$t_1=u/g \qquad ...(i)$$

Again let t_2 be the time taken by the particle in falling this height h from rest, then for the downwards motion of the particle we have 'u'$=0$, 'f'$=g$, 'x'$=h=u^2/(2g)$, 't'$=t_2$

∴ From "$x=ut+\frac{1}{2}ft^2$", we get

$$\frac{u^2}{2g}=0+\tfrac{1}{2}g\,t_2^2 \quad\text{or}\quad t_2^2=\frac{u^2}{g^2} \quad\text{or}\quad t_2=\frac{u}{g}, \qquad ...(ii)$$

taking the positive value.

∴ From (i) and (ii), total time taken by the particle to return to ground $=t_1+t_2=\frac{u}{g}+\frac{u}{g}=\frac{2u}{g}$. Hence proved.

***Ex. 3. Choose the correct answer :**

A particle is projected vertically upwards with a velocity u from a point O under gravity only. Then maximum height h and total time of flight t, are

(i) $h=u^2/(2g)$; $t=u/(2g)$; (ii) $h=2u^2/g$, $t=2u/g$;

(iii) $h=u^2/g$, $t=u/g$; (iv) $h=u^2/(2g)$, $t=2u/g$.

Hint. See Ex. 1 above. **Ans.** (iv)

Ex. 4. A particle is let fall under gravity only from rest at a point at height h above the ground. Find (i) the time required to strike the ground, (ii) the velocity with which it will strike the ground and (iii) the distance travelled during the nth second from start.

Sol. Let the particle strike the ground after time t with velocity v.

(i) Then from "$x=ut+\frac{1}{2}ft^2$", we get

$$h=0.t+\tfrac{1}{2}gt^2 \quad\text{or}\quad t=\sqrt{(h/g)}\text{ sec.}$$

(ii) From "$v^2=u^2+2fx$", we get

$$v^2=0+2gh \quad\text{or}\quad v=\sqrt{(2gh)}\text{ units.}$$

(iii) Distance travelled during the nth second from start

$=$"$u+\frac{1}{2}f(2n-1)$" ...See § 8 Page 15

$=0+\frac{1}{2}g(2n-1)=\frac{1}{2}g(2n-1)$.

Ex. 5. A balloon ascends with a uniform acceleration f m/sec². At the end of t seconds a body is released from it. Find the time that elapses before the body reaches the ground.

Sol. For the motion of the balloon :

'u'$=0$, 'f'$=f$ m/sec², 't'$=t$ sec.

Let v be its velocity and h be the height after time t seconds.

Then from $v=u+ft$ and $s=ut+\frac{1}{2}ft^2$, we get

$$v=ft \text{ and } h=\tfrac{1}{2}ft^2 \qquad ...(i)$$

For the motion of body : At the time of release of body from the balloon, the velocity of the body will be the same as that of the balloon both in magnitude and direction *i.e.* the velocity of the body will be v, given by (i), in vertically upward direction.

∴ For the body, we have

'u'$=-v$ m/sec., 'f'$=g$ m/sec^2, 's'$=h$ metres and $t=T$ sec. (say).

∴ From '$s=ut+\frac{1}{2}ft^2$', we have $h=-vT+\frac{1}{2}g\,T^2$

or $\frac{1}{2}ft^2=-ft\,T+\frac{1}{2}g\,T^2$, from (i)

or $gT^2-2ft\,T-ft^2=0$

or $T=[2ft\pm\sqrt{\{4f^2t^2+4fgt^2\}}]/2g$

$=t[f\pm\sqrt{(f^2+fg)}]/g$

or $T=t\,[f+\sqrt{(f^2+fg)}]/g$, as $T>0$. **Ans.**

Ex. 6. A particle is projected upwards with velocity u and t seconds afterwards, another particle is similarly projected with the same velocity. Find when and where they will meet ?

Sol. Let the second

T of its

$(t+T)$ b

by the f

second particle in time T.

or $u\,(t+T)+\frac{1}{2}(-g)(t+T)^2=uT+\frac{1}{2}(-g)\,T^2$

or $ut-\frac{1}{2}g\,(t^2+2tT)=0$ or $u-\frac{1}{2}gt-gT=0$

or $T=(u/g)-\frac{1}{2}t$. **Ans.**

Also the height at which the second particle meets the first $=uT-\frac{1}{2}gT^2$, where $T=(u/g)-\frac{1}{2}t$

$$=u\left(\frac{u}{g}-\frac{t}{2}\right)-\frac{1}{2}g\left(\frac{u}{g}-\frac{t}{2}\right)^2$$

$$=\frac{u^2}{g}-\frac{ut}{2}-\frac{g}{2}\left(\frac{u^2}{g^2}-\frac{ut}{g}+\frac{t^2}{4}\right)=\frac{u^2}{2g}-\frac{gt^2}{8}.$$ **Ans.**

****Ex. 7. A load W is to be raised by a rope, from rest to rest through a height a ; the greatest tension which the rope can safely bear is nW. Show that the least time in which the ascent can be made is $[2nh/(n-1)\,g]^{1/2}$.**

Sol. Let m be the mass of the load W, then $m=W/g$ ...(i)

In the first part of the motion, let the load move upwards with an acceleration f and greatest tension nW.

Then the equation of motion (from Newton's 2nd law) is

$mf=nW-W$ or $(W/g)f=(n-1)\,W$, from (i)

or $$f=(n-1)\,g. \qquad \text{...(ii)}$$

In the second part of the motion tension ceases to act and the load is moving under gravity only.

Let x_1 and x_2 be the distances moved and t_1, t_2 be the times taken in these two parts of motion. Let v be the max. velocity acquired at the end of first part, *i e.* at the time when tension ceases to act.

Then for the first part of the motion, we have

$$v=ft_1 \text{ and } v^2=2fx_1. \qquad \text{...(iii)}$$

And for the second part of the motion, we have

$$v=gt_2 \text{ and } v^2=2gx_2. \qquad \text{...(iv)}$$

Also $x_1+x_2=h$ (given). ...(v)

From (iii) and (iv), we get $x_1+x_2=(v^2/2f)+(v^2/2g)$

or $$h=\tfrac{1}{2}v^2\,[(1/f)+(1/g)], \text{ from (v)} \qquad \text{...(vi)}$$

$\therefore$ the required time

$=t_1+t_2=(v/f)+(v/g)$, from (iii) and (iv)

$=v\left[\dfrac{1}{f}+\dfrac{1}{g}\right]=\sqrt{\left[\dfrac{2h}{1/f+1/g}\right]}\left(\dfrac{1}{f}+\dfrac{1}{g}\right)$, from (vi)

$=\sqrt{\left[2h\left(\dfrac{1}{f}+\dfrac{1}{g}\right)\right]}=\sqrt{\left[2h\left\{\dfrac{1}{(n-1)\,g}+\dfrac{1}{g}\right\}\right]}$, from (ii)

$=\sqrt{[2nh/\{(n-1)\,g\}]}$. Hence proved.

***Ex. 8. A man, in a lift ascending with an acceleration f, throws a ball vertically upwards with a velocity v and catches it after a time t_1. Afterwards when the lift is descending with the same acceleration the man again throws a ball vertically upwards with the same velocity and catches it after a time t_2. Determine the velocity v and acceleration f in terms of t_1, t_2 and g the acceleration due to gravity.**

Sol. The man in the lift throws the ball upwards and catches the ball. Therefore relative to the lift, the velocity of projection of the ball is v and the distance moved is zero in both the cases when the lift is ascending and descending.

When the lift is ascending, the acceleration of the ball relative to the lift is $(f+g)$ vertically downwards or $[-(f+g)]$ vertically upwards.

$\therefore$ From $s=ut+\frac{1}{2}ft^2$, we get

$$0=vt_1+\tfrac{1}{2}\{-(f+g)\}\,t_1^2 \quad \text{or} \quad v=\tfrac{1}{2}(f+g)\,t_1 \qquad \text{...(i)}$$

When the lift is descending, the acceleration of the ball relative to the lift is $(g-f)$ vertically downwards or $[-(g-f)]$ vertically upwards. (Note)

$\therefore$ As before from $s=ut+\frac{1}{2}ft^2$, we have

$$0=vt_2-\frac{1}{2}(g-f)t_2^2 \quad \text{or} \quad v=\frac{1}{2}(g-f)t_2 \quad \text{...(ii)}$$

From (i) and (ii), equating values of v, we get

$$\frac{1}{2}(g+f)t_1=\frac{1}{2}(g-f)t_2$$

or $$f=(t_2-t_1)g/(t_1+t_2)$$ Ans.

Substituting this value in (i), we get

$$v=gt_1t_2/(t_1+t_2).$$ Ans.

Ex. 9. A stone is dropped from an aeroplane which is rising with acceleration f and t seconds after this another stone is dropped. Prove that the distance between the stones at time T after the second stone is dropped is $\frac{1}{2}(g+f)t(t+2T)$.

Hint. Acc. of each stone when released relative to the aeroplane$=f+g$. The initial velocity of each stone relative to the aeroplane$=0$.

$\because$ The first stone has been in motion for $(t+T)$ seconds and the second for T seconds.

$\therefore$ Distance covered by the first stone relative to the aeroplane$=\frac{1}{2}(f+g)(t+T)^2$ and distance covered by the second stone relative to the aeroplane$=\frac{1}{2}(f+g)T^2$.

$\therefore$ Required distance $=\frac{1}{2}(f+g)(t+T)^2-\frac{1}{2}(f+g)T^2$

$=\frac{1}{2}(f+g)[(t+T)^2-T^2]$

$=\frac{1}{2}(f+g)(t+2T)t.$ Hence proved.

Exercises on Velocity and Acceleration

Ex. 1. A particle falling from the top of a vertical tower has descended x metres when another is let fall from a point y metres below the top. If they reach the ground together show that the height of the tower is $\frac{1}{4}(x+y)^2/x$ metres.

Ex. 2. A ball is dropped from the top of a tower h metres high; and at the same moment another ball is projected upwards from the bottom. They meet when the upper one has described $1/n$ of the distance. Show that the velocities when they meet are in the ratio $2:(n-2)$ and the initial velocity of the ball projected from the bottom of the tower is $\sqrt{(\frac{1}{2}ngh)}$.

Ex. 3. Prove that the shortest time from rest to rest in which a steady load P can lift a weight w through a vertical distance h is $\sqrt{[(2h/g)\{P/(P-w)\}]}$ seconds.

[Hint. See Ex. 7 Page 33].

Ex. 4. Are the following statements true or false :—

(i) A particle moving with an uniform acceleration describes equal distances in equal intervals of time.

(ii) The average speed of a particle moving in a straight line with uniform acceleration for an interval of time is equal to half the sum of the velocities of the particle at the initial and the final instants.

(iii) Bodies of different weights falling freely from a fixed height reach the surface of the earth with different velocities.

Ans. (i) False, (ii) True ; (ii) False

Newton's Laws of Motion and Rectilinear Motion with Variable Acceleration

§ 1. Newton's Laws of Motion.

(Bundelkhand 87; Rajasthan 86)

The Newtonian Mechanics is based on three laws (or axioms) known as Newton's Laws of Motion and are most commonly used in Mechanics. They are stated as follows :—

Law I. *Every body (or particle) continues in its state of rest or of uniform motion in a straight line unless it is compelled by some external force (or forces) to change that state.*

Law II *The rate of change of momentum is directly proportional to the impressed force and takes place in the direction in which the force acts.*

Law III. *To every action there is equal and opposite reaction.*

§ 2. Newton's Law I defines a force to be the agent which changes the state of body i.e. in the absence of a f

was so previously. Hence we conclude that if a body is not moving in a straight line or is moving with a variable speed, it is under the action of some external force.

Force.

Definition. Force is a cause which tends to change or changes the state of rest or of uniform motion of a body in a straight line.

Laws of Inertia.

This law I of Newton also indicates the property of a body due to which it continues in its state of uniform motion if no external force is applied and this property is called **Inertia** and hence Law I of Newton is also known as **Law of Inertia.**

***Issac Newton (1642–1727)** was born in Woolsthrope, England. He was the father of Differential Calculus but his works remained unpublished until many years after their discovery. His main works were in Optics, Algebra, Geometry, Calculus, Laws of gravitation and celestial mechanics.

Mass.

Definition. The measure of the inertia of a body is known as its mass.

§ 3. Newton's Law II gives us a measure for the force and thus makes the concept of force more precise. This Law is also called Law of Motion and gives a relation between the force and the acceleration produced by it.

Momentum.

Definition 1. If m be the mass of a particle and v be its velocity then the momentum p is defined by the relation $p=mv$.

Definition 2. (*By vectors*). If $\mathbf{v}$ be the velocity vector and m the mass of a particle, then the linear momentum vector of the particle is $m\mathbf{v}$.

Equation of motion.

By Newton's Law II, we find that the rate of change of momentum, *viz.* $\frac{d}{dt}(p)$ is directly proportional to the impressed force F(say) and takes place in the same direction in which F acts

i.e. $\frac{d}{dt}(p)=kF$, where k is a scalar constant

or $\frac{d}{dt}(mv)=kF$, since $p=mv$

or $m(dv/dt)=kF$, if m is constant

or $mf=kF$, where f is the acceleration.

Now if we choose the unit of force in such a way that an unit force acting on a particle of unit mass produces an unit acceleration, then we have $k=1$ and then the above equation reduces to

$$mf=F. \quad \text{...(i)}$$

This equation is known as **equation of motion** or **fundamental equation of dynamics** and gives a relation between a force and the acceleration it produces on a given particle of mass m.

The equation (i) can be written as $F-mf=0$ ...(ii)

Here mf is known as **effective force** on the particle of mass m. Thus we can interpret (ii) as :—The external force acting on a particle and the reversed effective force taken together keep a particle in equilibrium. This is known as D' Alembert's Principle for the motion of a particle.

Equation of motion (vector form).

From Newton's Law II, the rate of change of momentum *viz.* $\frac{d}{dt}(m\mathbf{v})$ is proportional to the impressed force $\mathbf{F}$ (say) and takes place in the direction in which the force $\mathbf{F}$ acts.

∴ We can write $\frac{d}{dt}(m\mathbf{v})=k\mathbf{F}$,

where k is the constant of proportionality

or $m\frac{d}{dt}(\mathbf{v})=k\mathbf{F}$, if m does not depend on time

or $m\mathbf{a}=k\mathbf{F}$, where $\mathbf{a}$ is the acceleration vector of the body.

Now if we suppose that an unit force acting on an unit mass produces an unit acceleration, then we have $k=1$ and the above equation takes the form $m\mathbf{a}=\mathbf{F}$,
which is known as the equation of motion of the body.

Exercises on § 3

Ex. Is the statement 'The relation "force=mass×acceleration" is only true for uniform acceleration' true or false?

§ 4. Newton's Law III describes that forces always occur in pairs, the two forces being such that they are equal in magnitude but opposite in direction *i.e.* it describes the equal and opposite force interactions of two bodies.

§ 5. Principle of Physical Independence of Forces.

Let m be the mass of a particle. Let a force F_1 acting on this particle produce an acceleration f_1, then from Newton's Law II, we have $mf_1=F_1$. ...(i)

Similarly if F_2 be another force acting on this particle which produces an acceleration f_2, then $mf_2=F_2$. ...(ii)

Adding (i) and (ii), we get $m(f_1+f_2)=F_1+F_2$. ...(iii)

From result (iii) above we conclude that a force F_1+F_2 acting on the same particle of mass m produces an acceleration f_1+f_2 in it *i.e.* the additional force F_2 produces an additional acceleration f_2 alone on the particle of mass m [see result (iii)] as it would do if acting alone on the particle of mass m [see result (ii)].

This result is known as Principle of Physical Independence of forces.

§ 6. Deduction of Law I from Law II.

From Newton's Law II, we have $\frac{d}{dt}(mv)=kF$, ...(i)

where m is the mass of a particle, v is the velocity and F is the external force acting on it.

Now if no force is acting on the particle, then $F=0$, and so from (i), we get $\frac{d}{dt}(mv)=0$.

Integrating with respect to t, we have $mv=c$, where c is an arbitrary constant

or $$v=c/m=u, \text{ a constant.} \quad ...(ii)$$

i.e. velocity of the particle is constant which shows that the particle continues to move in a straight line with constant speed.

If however, $u=0$ then from (ii) we find that particle continues to be at rest.

This is nothing but Newton's Law I of motion and hence the deduction of Law I from Law II.

§ 7. Units of Force.

(i) In C.G.S. system, the unit of force is a dyne and we define a dyne as that force which when acts on a particle of mass one gram produces in it an acceleration of 1 cm/sec².

(ii) In M.K.S. system, the unit of force is a **newton** and we define a newton as that force which when acts on a particle of mass one kilogram produces in it an acceleration of 1 m./sec².

(iii) In F.P.S. system, the unit of force is a **poundal** and we define a poundal as that force which when acts on a particle of mass one pound produces in it an acceleration of 1 ft./sec².

Note 1. From definitions of dyne and newton it is evident that

one newton = 100,000 dynes.

Note 2. Dyne in C.G.S. system and poundal in F.P.S. system are called the **absolute units of force** as these are free from g, the acceleration due to gravity and remain the same at all places.

A unit of force based on the weight of a unit mass is known as gravitational unit of force.

In C.G.S. system the gravitational unit of force is **gram weight** or gm. wt.

In M.K.S. system this unit of force is **kilogram weight** or kg. wt. and in F.P.S. system it is **pound weight** or lb. wt.

We know g, the acceleration due to gravity, is 980 cm./sec² in C.G.S. system; 9·8 m./sec² in M.K.S. system and 32 ft./sec² in F.P.S. system.

Hence we have **One gram weight = 980 dynes.**

One kilogram weight = 9·8 newtons,

One pound weight = 32 poundals.

§ 8. Dimensions of Force and Momentum.

If M, L and T denote the units of mass, length and time respectively then (i) the momentum being the product of mass and velocity its unit is denoted by MLT^{-1}.

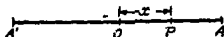
(ii) The force being the product of mass and acceleration its unit is denoted by MLT^{-2}.

§ 9. Motion under Inverse Square Law.

A particle moves in a straight line under an attraction towards a fixed point on the line, varying inversely as the square of the distance from the fixed point; to investigate the motion.

(Gorakhpur 87; Meerut 86)

O is the fixed point on the line. Let P be the position of the particle at time t, where $OP=x$.



(Fig. 1)

∴ The equation of motion is

$$m\frac{d^2x}{dt^2}=-\frac{m\mu}{x^2}, \qquad \text{...(i)}$$

where μ is constant.

or $$v\frac{dv}{dx}=-\frac{\mu}{x^2}, \quad \because \frac{d^2x}{dt^2}=v\frac{dv}{dx}.$$

Integrating with respect to x, we get $\frac{1}{2}v^2=(\mu/x)+c_1$, ...(ii)
where c_1 constant of integration.

Let the particle start from rest from a point A, where $OA=a$.
Then at A, $x=a$ and $v=0$.

∴ From (ii), $0=(\mu/a)+c_1$ or $c_1=-(\mu/a)$.

∴ From (ii), we get $v^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right)$

or $$\left(\frac{dx}{dt}\right)^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right), \quad \because \frac{dx}{dt}=v.$$

or $$\frac{dx}{dt}=-\sqrt{(2\mu)}\sqrt{\left(\frac{1}{x}-\frac{1}{a}\right)}, \qquad \text{...(ii)}$$

the negative sign is due to the fact that as t increases, x decreases

or $\sqrt{\left(\frac{ax}{a-x}\right)}dx=-\sqrt{(2\mu)}\,dt$ or $\sqrt{\left(\frac{ax}{a-x}\right)}dx=-(2\mu)\int dt+c_2$,

where c_2 is constant of integration.

Putting $x=a\cos^2\theta$ or $dx=-2a\cos\theta\sin\theta\,d\theta$, we get

$$\int\sqrt{a}.\frac{\cos\theta}{\sin\theta}.(-2a\cos\theta\sin\theta)\,d\theta=-\sqrt{(2\mu)}\int dt+c_2$$

or $$a\sqrt{a}\int 2\cos^2\theta\,d\theta=\sqrt{(2\mu)}\,t-c_2$$

or $$a\sqrt{a}\int(1+\cos 2\theta)\,d\theta=\sqrt{(2\mu)}\,t-c_2$$

or $$a\sqrt{a}\,(\theta+\tfrac{1}{2}\sin 2\theta)=\sqrt{(2\mu)}\,t-c_2$$

or $$a\sqrt{a}\,(\theta+\sin\theta\cos\theta)=\sqrt{(2\mu)}\,t-c_2$$

or $$a\sqrt{a}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left(1-\frac{x}{a}\right)}\cdot\sqrt{\left(\frac{x}{a}\right)}\right]=\sqrt{(2\mu)}\,t-c_2,$$

$\because \cos\theta=\sqrt{(x/a)}$.

At A, we have $x=a; t=0$, $\therefore$ $c^2=0$ and we get

$$a\sqrt{a}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left(1-\frac{x}{a}\right)}\sqrt{\left(\frac{x}{a}\right)}\right]=\sqrt{(2\mu)}.t$$

or
$$t=\frac{a\sqrt{a}}{\sqrt{(2\mu)}}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left(1-\frac{x}{a}\right)}\sqrt{\left(\frac{x}{a}\right)}\right] \quad \text{...(iv)}$$

Putting $x=0$ in (iv) we get the time taken by the particle in reaching O from $A=\frac{a\sqrt{a}}{\sqrt{(2\mu)}}[\cos^{-1}(0)]=\frac{a\sqrt{a}}{\sqrt{(2\mu)}}\cdot\frac{\pi}{2}$. ...(v)

Also from (i) and (iii) we find at O, *i.e.* at $x=0$, the acceleration is infinite and the velocity is (—infinity). Therefore the particle will dash through O and the negative sign of the velocity shows that it will move towards the left of O. Since the acceleration is always directed towards O therefore retardation will begin and the velocity would go on decreasing and ultimately the particle would come to momentary rest at A' such that $OA'=a=OA$.

It will then retrace its ... and will again come to momentary rest a... O and its period=time tak...

$$=4.(\text{time taken in moving from } A \text{ to } O)$$
$$=4.\frac{\pi a\sqrt{a}}{2\sqrt{(2\mu)}}, \text{ from (v)}$$
$$=\pi a\sqrt{(2a/\mu)}.$$ *(Gorakhpur 87)*

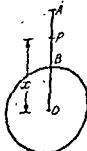
Note. The law of attraction towards the centre of the earth for a body moving outside the surface of the earth is inverse square whereas inside the surface of the earth attraction is proportional to the distance from the centre.

Solved Examples on Inverse Square Law.

Ex. 1. A particle is shot upwards from the earth's surface with a velocity of 1 km./sec. Considering variation in gravity, find roughly in kilometres the greatest height attained.

Sol. O is the centre of the earth and B is the point of projection on its surface. Let P be the position of the particle at time t, such that $OP=x$.

Since outside the surface ... the earth, the ...verse square of ... the equation of ...n, at P is



$$m\frac{d^2x}{dt^2}=-\frac{m\mu}{x^2} \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{\mu}{x^2} \quad \text{...(i)}$$

(Fig. 2)

On the surface of the earth acceleration is g and x=radius of the earth =6400 kms.=6400×1000 metres.

$\therefore$ From (i), $g=\dfrac{\mu}{(6400\times 1000)^2}$

or $\mu=(6400\times 1000)^2 g.$ ...(ii)

Again from (i), we have

$$v\frac{dv}{dx}=-\frac{\mu}{x^2} \quad \because \frac{d^2x}{dt^2}=v\frac{dv}{dx}.$$

Integrating with respect to x, we get $\frac{1}{2}v^2=(\mu/x)+c_1$, ...(iii)
where c_1 is constant of integration.

At B (*i.e.* on the surface of the earth) $v=1$ km/sec$=1000$ metres per sec. and $x=6400\times 1000$ metres.

$\therefore$ From (iii) we get $\frac{1}{2}(1000)^2=\dfrac{\mu}{6400\times 1000}+c_1$

or $\frac{1}{2}\times(1000)^2=\dfrac{(6400\times 1000)^2 g}{6400\times 1000}+c_1$, from (ii)

or $c_1=\frac{1}{2}(1000)^2-(6400\times 1000)\,g$.

$\therefore$ From (iii), $v^2=(2\mu/x)+(1000)^2-2\times 6400\times 1000\times g$.

At the highest point $v=0$ and $x=h$ (say).

$\therefore$ $0=(2\mu/h)+(1000)^2-2\times 6400\times 1000\times g$.

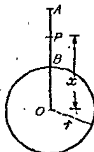
or
$$h=\frac{2\mu}{2\,(6400\times 1000)\,g-(1000)^2}$$
$$=\frac{2\times(6400\times 1000)^2 g}{2g\,(6400\times 1000)-(1000)^2}$$
$$=\frac{2\times 6400\times 6400\times 1000\times 9\cdot 8}{2\times 9\cdot 8\times 6400-1000}\text{ metres.}$$
$$=\frac{6400\times 6400\times 1960}{(196\times 64-100)\times 1000}\text{ kms.}=6451\cdot 43\text{ kms.}$$

$\therefore$ Required height$=6451\cdot 43-6400=51\cdot 43$ kms. **Ans.**

****Ex. 2. If h be the height due to the velocity v at the earth's surface, supposing its attraction constant and H the corresponding height when the variation of gravity is taken into account, prove that**

$$\frac{1}{h}-\frac{1}{H}=\frac{1}{r},$$

where r is the radius of the earth. (*Avadh 89*)



(Fig. 3)

Sol. Supposing the acceleration due to gravity to be constant and equal to g according to the problem from "$v^2=u^2+2fx$", we have

$v^2=0+2gh$ or $v^2=2gh$ .. (i)

When the variation of gravity is taken into account, let P be the position of the perticle at time t, such that $OP=x$.

$\therefore$ The equation of motion is

$$m\frac{d^2x}{dt^2}=-\frac{m\mu}{x^2} \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{\mu}{x^2} \quad \text{...(ii)}$$

On the surface of the earth $x=r$ (given),

$\therefore$ $g=\mu/r^2$ or $\mu=gr^2$. ...(iii)

From (ii) and (iii), we have

$$v\frac{dv}{dx}=-\frac{gr^2}{x^2}, \quad \because \quad \frac{d^2x}{dt^2}=v\frac{dv}{dx}$$

Integrating with respect to x, we get $\frac{1}{2}v^2=\frac{gr^2}{x}+C_1$, ...(iv)

where C_1 is constant of integration.

At the highest point A, $v=0$ and $x=OB+BA=r+H$

$\therefore$ From (iv), we get $0=\frac{gr^2}{H+r}+C_1$ or $C_1=-\frac{gr^2}{H+r}$.

$\therefore$ From (iv), $v^2=2gr^2\left[\frac{1}{x}-\frac{1}{H+r}\right]$.

On the surface of the earth, *i.e.* at B, $v=v$ (given) and $x=r$.

$$\therefore \quad v^2=2gr^2\left[\frac{1}{r}-\frac{1}{H+r}\right]$$

or $$2gh=2gr^2\left[\frac{(H+r)-r}{r(H+r)}\right], \quad \because \text{ from (i) } v^2=2gh$$

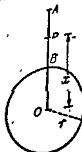
or $$h=\frac{rH}{H+r} \quad \text{or} \quad \frac{1}{h}=\frac{H+r}{rH}=\frac{1}{r}+\frac{1}{H} \quad \text{or} \quad \frac{1}{h}-\frac{1}{H}=\frac{1}{r}.$$

***Ex. 3. A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is**

$$\tfrac{1}{3}\sqrt{(2a/g)}\,[(1+h/a)^{3/2}-1],$$

where a is the radius of the earth. *(Kanpur 87, Rohilkhand 88)*

Sol. O is the centre of the earth and B is the point of projection on its surface. Let P be the position of the particle at time t, such that $OP=x$.



(Fig. 4)

We know that outside the surface of the earth, the law of acceleration is that of inverse square of the distance from the centre, hence the equation of motion at P is

$$d^2x/dt^2=-\mu/x^2. \quad \text{...(i)}$$

On the surface of the earth, this acceleration is g and $x=a$ (given).

$\therefore$ From (i), $g=\mu/a^2$ or $\mu=a^2g$.

Hence from (i), $\dfrac{d^2x}{dt^2}=-\dfrac{a^2g}{x^2}$ ...(ii)

Multiplying both sides by 2 (dx/dt) and integrating, we get

$\left(\dfrac{dx}{dt}\right)^2=\dfrac{2a^2g}{x}+C$, where C is constant.

Given that $dx/dt=0$ when $x=\infty$, $C=0$.

Hence $\left(\dfrac{dx}{dt}\right)^2=\dfrac{2a^2g}{x}$ or $\dfrac{dx}{dt}=\dfrac{a\sqrt{(2g)}}{\sqrt{x}}$

(positive sign is due to the fact that x increases as t increases)

or $\sqrt{x}\,dx=a\sqrt{(2g)}\,dt$. ...(iii)

$\therefore$ Required time from the surface of earth to a height h, *i.e.* from $x=a$ to $x=a+h$ is obtained from (iii) by integration as

$$t=\frac{1}{a\sqrt{(2g)}}\int_{x=a}^{a+h}\sqrt{x}\,dx=\frac{1}{a\sqrt{(2g)}}\left[\tfrac{2}{3}x^{3/2}\right]_a^{a+h}$$

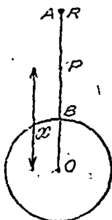
$$=\frac{2}{3a\sqrt{(2g)}}[(a+h)^{3/2}-a^{3/2}]=\tfrac{1}{3}\,\frac{\sqrt{2a^{3/2}}}{a\sqrt{g}}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right]$$

$=\tfrac{1}{3}\sqrt{(2a/g)}\,[\{1+(h/a)\}^{3/2}-1]$. Hence proved.

Ex. 4 (a). **A particle falls towards the earth form infinity; show that its velocity on reaching the surface of the earth is the same as that which it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius.**

(Agra 87)

Sol. Let O be the centre and a the radius of the earth. Let P be the position of
law of acceleration h-
O, the equation of m



Multiplying both sides by 2 (dx/dt) and integrating, we have $(dx/dt)^2=2\mu/x+C$, where C is constant of integration.

At $x=\infty$, $dx/dt=0$ (given). $\therefore$ $C=0$.

Hence

$\left(\dfrac{dx}{dt}\right)^2=\dfrac{2\mu}{x}$ or $\dfrac{dx}{dt}=-\dfrac{\sqrt{(2\mu)}}{\sqrt{x}}$. ...(ii)

(The negative sign shows that x decreases as t increases).

Also on the surface of the earth acceleration due to gravity is g.

$\therefore$ From (i), $g=\mu/a^2$ or $\mu=a^2g$.

$\therefore$ From (ii),

$(dx/dt)=-a\sqrt{(2g/x)}$, .. (iii) (Fig 5)

$\therefore$ On the surface of the earth, *i.e.* at $x=a$, the velocity of the particle $=a\sqrt{(2g/a)}=\sqrt{(2ag)}$.

=velocity which the particle would have acquired in falling through a distance a under constant acceleration g.

Ex. 4 (b). A particle falls freely from infinity on the earth. Show that its velocity on the earth's surface is $\sqrt{(2ga)}$, where a is the radius of the earth. *(Agra 87)*

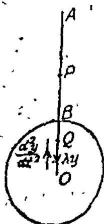
What shall be its velocity on reaching the centre of the earth?

Sol. First part of the problem is the same as Ex. 4 (a) above.

Now inside the surface of the earth attraction towards the centre of the earth is proportional to the distance from the centre. Let Q be the position of the particle at time t, such that $OQ=y$. Then the equation of motion of the particle of mass m (say) at Q is

$$m\frac{d^2y}{dt^2}=-\lambda y \text{ or } \frac{d^2y}{dt^2}=-ky. \quad \ldots(i)$$

where $k=\lambda/m$.



(Fig. 6)

On the surface of the earth, the acceleration is g and $y=a$ (given).

∴ From (i) we get $g=ka$

or $k=g/a$

Multiplying both sides of (i) by $2(dy/dt)$ and integrating, we have

$$\left(\frac{dy}{dt}\right)^2=-ky^2+C, \text{ where } C \text{ is constant} \quad \ldots(iii)$$

At B, $(\text{velocity})^2=(dy/dt)^2=[\sqrt{(2ga)}]^2=2ga$ and $y=a$

∴ $2ga=-ka^2+C$ or $2ga=-ga+C$, from (ii)

or $C=3ga$.

∴ From (iii) we get $(dy/dt)^2=-(g/a)\,y^2+3ga$ ...(iv)

At the centre O, $y=0$ and so from (iv) we get

$(\text{vel.})^2=0+3ga$ or velocity at the centre $=\sqrt{(3ga)}$. **Ans.**

****Ex. 5.** Calculate in kilometers per second the least velocity which will carry a body from the earth's surface to infinity.

Or

Find the velocity of a particle with which if projected from the surface of the earth it may never return to it.

Sol. Let O be the centre of the earth, whose radius=6400

kilometres or 6400×1000 metres. Let P be the position of the particle at time t, such that $OP = x$, when the particle is projected from the point B on the surface of the earth with a velocity v m./sec.

$\therefore$ The equation of the motion at P is $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$ ...(i)

On the surface of the earth, acceleration due to gravity is g.

$\therefore$ From (i), $g = \mu/(6400 \times 1000)^2$

or $\mu = (9 \cdot 8) \times (6400 \times 1000)^2$ ...(ii)

Multiplying both sides of (i) by $2dx/dt$ and integrating, we get $(dx/dt)^2 = (2\mu/x) + C$, ...(iii)

where C is constant of integration.

As $x = \infty$, $dx/dt = 0$ (given), $\therefore$ $C = 0$.

$\therefore$ From (iii) we get $(dx/dt)^2 = (2\mu/x)$.

$\therefore$ On the surface of the earth, *i.e.* at B,

$dx/dt = v$ and $x = 6400 \times 1000$

$$\therefore v^2 = \frac{2 \times (9 \cdot 8) \times (6400 \times 1000)^2}{6400 \times 1000} = 196 \times 64 \times 10000$$

or $v = 14 \times 8 \times 100 = 11200$ metres/sec.

$= 11 \cdot 2$ kms/sec. **Ans.**

*Ex. 6 *If the earth's attraction vary inversely as the square of the distance from its centre, and g be its magnitude at the surface the time of falling from a height h above the surface to the surface is*

$$\sqrt{\left(\frac{a+h}{2g}\right)}\left[\sqrt{\left(\frac{h}{a}\right)} + \frac{a+h}{a}\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}\right],$$

where a is the radius of the earth.

Sol. Let the particle fall from A to B, where $AB = h$.

Let P be the position of the partible at time t, such that $OP = x$, where O is the earth.

$\because$ Outside the surface of the earth, law of attraction is inverse square of the distance from the centre of the earth.

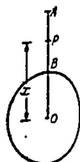
$\therefore$ The equation of motion is

$$d^2x/dt^2 = -\mu/x^2 \quad ...(i)$$

On the surface of the earth acceleration is g and $x = a$,

$\therefore$ $g = \mu/a^2$ or $\mu = a^2 g$.

$\therefore$ from (i we get $\dfrac{d^2x}{dt^2}=-\dfrac{a^2g}{x^2}$ ...(ii)



(Fig. 7)

Multiplying both sides by $2\,(dx/dt)$, and integrating, we get

$\left(\dfrac{dx}{dt}\right)^2=\dfrac{2a^2g}{x}+C$, where C is constant.

At A, $x=h+a$ and $dx/dt=0$.

$\therefore\quad 0=\dfrac{2a^2g}{(a+h)}+C$ or $C=-\dfrac{2a^2g}{(a+h)}$

$\therefore\quad \left(\dfrac{dx}{dt}\right)^2=2a^2g\left(\dfrac{1}{x}-\dfrac{1}{a+h}\right)$

or $\quad \dfrac{dx}{dt}=a\sqrt{(2g)}\sqrt{\left(\dfrac{1}{x}-\dfrac{1}{a+h}\right)}$, .. (iii)

negative sign is due to the fact that x decreases as t increases.

Putting $a+h=b$, we find from (iii)

$$\frac{dx}{dt}=-a\sqrt{(2g)}\sqrt{\left(\frac{1}{x}-\frac{1}{b}\right)}=-a\sqrt{(2g)}\sqrt{\left(\frac{b-x}{bx}\right)}$$

or

$$dt=-\frac{1}{a\sqrt{(2g)}}\sqrt{\left(\frac{bx}{b-x}\right)}\,dx.$$

$\therefore$ Integrating between the limits $x=a+h$ to $x=a$ or $x=b$ to $x=a$, we have the required time

$$=-\frac{1}{a\sqrt{(2g)}}\int_{x=b}^{a}\sqrt{\left(\frac{bx}{b-x}\right)}dx$$

$$=-\frac{1}{a\sqrt{(2g)}}\int_{\pi/2}^{\sin^{-1}\sqrt{(a/b)}}\sqrt{\left(\frac{b^2\sin^2\theta}{b\cos^2\theta}\right)}.2b\sin\theta\cos\theta\,d\theta,$$

putting $x=b\sin^2\theta$ or $dx=2b\sin\theta\cos\theta\,d\theta$

$$=-\frac{2b\sqrt{b}}{a\sqrt{(2g)}}\int_{\pi/2}^{\sin^{-1}\sqrt{(a/b)}}\sin^2\theta\,d\theta$$

$$=-\frac{b\sqrt{b}}{a\sqrt{(2g)}}\int_{\pi/2}^{\sin^{-1}\sqrt{(a/b)}}(1-\cos 2\theta)\,d\theta$$

$$=-\frac{b\sqrt{b}}{a\sqrt{(2g)}}\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\sin^{-1}\sqrt{(a/b)}}$$

$$=-\frac{b\sqrt{b}}{a\sqrt{(2g)}}\left[\sin^{-1}\sqrt{(a/b)}-\sqrt{(a/b)}\sqrt{\left(1-\frac{a}{b}\right)}-\frac{\pi}{2}\right]$$

$$=\frac{b\sqrt{b}}{a\sqrt{(2g)}}\left[\{\tfrac{1}{2}\pi-\sin^{-1}\sqrt{(a/b)}\}+\sqrt{\left(\frac{a}{b}\right)}\sqrt{\left(1-\frac{a}{b}\right)}\right] \quad \text{(Note)}$$

$$=\frac{b\sqrt{b}}{a\sqrt{(2g)}}\left[\cos^{-1}\sqrt{\left(\frac{a}{b}\right)}+\sqrt{\left(\frac{a}{b}\right)}\sqrt{\left(1-\frac{a}{b}\right)}\right].$$

$\because\ \tfrac{1}{2}\pi-\sin^{-1}\theta=\cos^{-1}\theta$

$$=\frac{b\sqrt{b}}{a\sqrt{(2g)}}\left[\sin^{-1}\sqrt{\left(1-\frac{a}{b}\right)}+\sqrt{\left(\frac{a}{b}\right)}\sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$\ldots\because\ \cos^{-1}\theta=\sin^{-1}\sqrt{(1-\theta^2)}$

$$=\frac{(a+h)^{3/2}}{a\sqrt{(2g)}}\left[\sin^{-1}\sqrt{\left(1-\frac{a}{a+h}\right)}+\sqrt{\left(\frac{a}{a+h}\right)}\sqrt{\left(1-\frac{a}{a+h}\right)}\right]$$

$\ldots\because\ b=a+h$

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\left(\frac{a+h}{a}\right)\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}+\sqrt{\left(\frac{h}{a}\right)}\right].$$ Hence proved.

Ex. 7. Show that the time occupied by a body, under the acceleration kx^3 towards the origin, to fall from rest at distance a to distance x from the attraction centre can be put in the form

$$\sqrt{(a^3/2k)}\ [\cos^{-1}\sqrt{(x/a)}+\sqrt{(x/a)}.\sqrt{[1-(x/a)]}].$$

Prove also that the time occupied from $x=3a/4$ to $a/4$ is one third of the whole time of descent from a to 0.

Sol. The equation of motion of the body when it is at distance x from the attracting centre is

$$d^2x/dt^2=-k/x^2 \qquad \ldots(i)$$

Multiplying both sides by $2\,dx/dt$ and integrating we get

$$(dx/dt)^2=-(2k/x)+C, \text{ where } C \text{ is constant.}$$

At $x=a$, $dx/dt=0$ (given)

$$\therefore\ 0=(2k/a)+C \quad \text{or} \quad C=-(2k/a).$$

$$\therefore\ \left(\frac{dx}{dt}\right)^2=2k\left(\frac{1}{x}-\frac{1}{a}\right) \text{ or } \frac{dx}{dt}=-\sqrt{(2k)}\sqrt{\left(\frac{1}{x}-\frac{1}{a}\right)}, \qquad \ldots(iii)$$

(negative sign shows that x decreases as t increases)

or
$$dt=-\frac{1}{\sqrt{(2k)}}\sqrt{\left(\frac{ax}{a-x}\right)}\,dx.$$

$\therefore$ Required time (from $x=a$ to $x=x$)

$$=-\frac{1}{\sqrt{(2k)}}\int_{x=a}^{x}\sqrt{\left(\frac{ax}{a-x}\right)}\,dx=\frac{2a\sqrt{a}}{\sqrt{(2k)}}\int_{\theta=0}^{\theta}\cos^2\theta\,d\theta,$$

putting $x=a\cos^2\theta$

or $dx=-2a\cos\theta\sin\theta\,d\theta$

$$=\frac{a\sqrt{a}}{\sqrt{(2k)}}\int_{\theta=0}^{\theta}(1+\cos 2\theta)\,d\theta=\frac{a\sqrt{a}}{\sqrt{(2k)}}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta}$$

$$=[a\sqrt{a}/\sqrt{(2k)}]\ [\theta+\sin\theta\cos\theta]$$

$$=\sqrt{(a^3/2k)}\ [\cos^{-1}\sqrt{(x/a)}+\sqrt{(x/a)}\sqrt{\{1-(x/a)\}}].$$ Hence proved.

Do the second part yourself.

***Ex. 8.** *Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance; show that a particle falling from rest at a distance b ($b>a$) from the centre of the earth would on reaching the centre acquire a velocity $\sqrt{[ga(3b-2a)/b]}$ and the time to travel from the surface to the centre of the*

earth is $\sqrt{(a/g)} \sin^{-1} \sqrt{[b/(3b-2a)]}$, where a is the radius of the earth and g is acceleration due to gravity. (Agra 86)

Sol Refer fig 7 Page 12.

Let O be the centre of the earth and A be the point from which the particle falls, $OA=b$ (given). Let P be the position of the particle at time t, such that $OP=x$.

Since the law of acceleration outside the surfrce of the earth is that of inverse square of the distance from the centre of the earth, so at P, the equation of motion is

$$mv\frac{dv}{dx}=-\frac{m\mu}{x^2} \text{ or } v\frac{dv}{dx}=-\frac{\mu}{x^2} \quad ...(i)$$

On the surface of the earth $x=a$ and acceleration $=g$

$$\therefore \ g=\mu/a^2 \text{ or } \mu=a^2g.$$

$\therefore$ From (i), we get $v\dfrac{dv}{dx}=-\dfrac{a^2g}{x^2}$.

Integrating with respect to x, we get $\frac{1}{2}v^2=(a^2g/x)+C_1$, ...(ii)

where C_1 is constant of integration.

At A, $x=b$ and $v=0$ (given), so from (ii), we have

$$0=(a^2g/b)+C_1 \text{ or } C_1=-(a^2g/b)$$

$$\therefore \text{ From (ii), } v^2=2a^2g\left(\frac{1}{x}-\frac{1}{b}\right)=2a^2g\left(\frac{b-x}{bx}\right). \quad ...(iii)$$

$\therefore$ If V be the velocity of the particle on the surface of the earth, *i.e.* at B, *i.e.* at $x=a$, then from (iii), we have

$$V^2=2a^2g\left(\frac{b-a}{ab}\right)=2ag\left(1-\frac{a}{b}\right). \quad ...(iv)$$

When the particle penetrates into earth, the law of acceleration changes and the equation of motion becomes

$$m.v\frac{dv}{dx}=-m.\mu'x \text{ or } v\frac{dv}{dx}=-\mu'x \quad ...(v)$$

Also on the surface of the earth, $x=a$ and acceleration is g,

$$\therefore \ g=\mu'a \text{ or } \mu'=g/a.$$

$\therefore$ From (v), we get $v\dfrac{dv}{dx}=-\left(\dfrac{g}{a}\right)x$.

Integrating with respect to x, we get

$$\tfrac{1}{2}v^2=-(g/a).\tfrac{1}{2}x^2+C_2, \quad ...(vi)$$

where C_2 is constant of integration.

At B, *i.e.* at $x=a$, we have $v=V$ and therefore from (vi), we get $\frac{1}{2}V^2=-(g/a).\frac{1}{2}a^2+C_2$ or $C_2=\frac{1}{2}(V^2+ga)$.

$\therefore$ From (vi), we have $v^2=-(g/a)\,x^2+(V^2+ga)$

or $\quad v^2=-(g/a)\,x^2+ag\,[3-(2a/b)]$, from (iv)
$=(g/a)\,[c^2-x^2]$, where $c^2=[3a^2-(2a^2/b)]$

or $\quad dx/dt=-\sqrt{(g/a)}\,\sqrt{(c^2-x^2)}$, ...(vii)

$-$ve sign shows that x decreases as t increases

or $\quad dt=-\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{dx}{\sqrt{(c^2-x^2)}}$

Integrating between the limits $x=a$ to $x=0$, we have the required time from B to O

$$=-\sqrt{\left(\frac{a}{g}\right)}\int_{x=a}^{0}\frac{dx}{\sqrt{(c^2-x^2)}}=-\sqrt{\left(\frac{a}{g}\right)}\left[\sin^{-1}\left(\frac{x}{c}\right)\right]_a^0$$

$$=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left(\frac{a}{c}\right)=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left[\frac{a}{\sqrt{(3a^2-2a^2/b)}}\right]$$

$$=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left[\sqrt{\left(\frac{b}{3b-2a}\right)}\right]$$

And from (vii) the velocity at the centre, *i.e.* at $x=0$

$$=-\sqrt{\left(\frac{g}{a}\right)}\,c=-\sqrt{\left(\frac{g}{a}\right)}\cdot\sqrt{\left(3a^2-\frac{2a^2}{b}\right)}$$

$$=-\sqrt{\left[ga\left(\frac{3b-2a}{b}\right)\right]},$$

the negative sign gives the direction of motion.

**Ex. 9. Show that the time of descent to the centre of the force varying inversely as the square of the distance from the centre through first half of its initial distance is to that through the last half as $(\pi+2):(\pi-2)$. (*Avadh 90; Gorakhpur 90; Rohilkhand 87*)

Sol. Let O be the centre of the force and A the point of start, such that $OA=a$ (say). Let B be the mid-point of OA. Let P be the position of the particle at time t, such that $OP=x$.

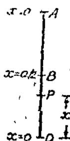
Then at P, equation of motion is $\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}$. ...(i)

Multiplying both sides by $2\,(dx/dt)$ and integrating, we get
$(dx/dt)^2=(2\mu/x)+C$, where C is constant.

At A, $dx/dt=0$ and $x=a$.

$\therefore\quad 0=(2\mu/a)+C$ or $C=-(2\mu/a)$

$\therefore\quad \left(\frac{dx}{dt}\right)^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right)=2\mu\left(\frac{a-x}{ax}\right)$



or $\quad \frac{dx}{dt}=-\sqrt{\left(\frac{2\mu}{a}\right)}\sqrt{\left(\frac{a-x}{x}\right)}$

(negative sign shows that x decreases as t increases)

or $\quad dt=-\sqrt{\left(\frac{a}{2\mu}\right)}\sqrt{\left(\frac{x}{a-x}\right)}\,dx.$

Let t_1 and t_2 be the times taken in moving from A to B and A to O respectively.

Then $t_2=$ time taken in moving from $x=a$ to $x=0$ (Fig. 8)

$$=-\sqrt{\left(\frac{a}{2\mu}\right)}\int_{x=a}^{0}\sqrt{\left(\frac{x}{a-x}\right)}dx$$

$$=\sqrt{\left(\frac{a}{2\mu}\right)}\int_{x=0}^{a}\sqrt{\left(\frac{x}{a-x}\right)}dx=\sqrt{\left(\frac{a}{2\mu}\right)}\int_{\theta=0}^{\pi/2}2\sin^2\theta\, d\theta,$$

putting $x=a\sin^2\theta$;

$$=\sqrt{(a/2\mu)}\,(2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\pi)=\tfrac{1}{2}\pi\sqrt{(a/2\mu)}.$$

And t_1=time taken in moving from $x=a$ to $x=\frac{1}{2}a$

$$=-\sqrt{\left(\frac{a}{2\mu}\right)}\int_{x=a}^{a/2}\sqrt{\left(\frac{x}{a-x}\right)}dx$$

$$=-\sqrt{\left(\frac{a}{2\mu}\right)}\int_{\pi/2}^{\pi/4}2\sin^2\theta\, d\theta,\text{ putting } x=a\sin^2\theta$$

$$=\sqrt{\left(\frac{a}{2\mu}\right)}\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2},\text{ integrating } 2\sin^2\theta=1-\cos\theta$$

$$=\sqrt{(a/2\mu)}\,[\tfrac{1}{2}\pi-(\tfrac{1}{4}\pi-\tfrac{1}{2})]=\tfrac{1}{4}(\pi+2)\sqrt{(a/2\mu)};$$

$\therefore$ Required ratio $=\dfrac{\text{time from } A \text{ to } B}{\text{time from } B \text{ to } O}=\dfrac{t_1}{t_2-t_1}$

$$=\frac{\frac{1}{4}(\pi+2)\sqrt{(a/2\mu)}}{\frac{1}{2}\pi\sqrt{(a/2\mu)}-\frac{1}{4}(\pi+2)\sqrt{(a/2\mu)}}=\frac{\pi+2}{2\pi-(\pi+2)}=\frac{\pi+2}{\pi-2}$$

Exercises on § 9

Ex. 1. Assuming that the gravity inside the earth varies as its distance from its centre, show that if a smooth straight tunnel is cut between Delhi to London, a train starting from rest from Delhi under the action of gravity would reach London in about 42 minutes. Assume the earth's radius to be 6400 kilometers.

Ex. 2. A particle is let fall from a height h from the earth's centre, a being the radius of the earth. Find its velocity on reaching the centre.

Ex. 3. Find the time of falling a height h above earth's surface to the surface.

[Hint : See Ex. 6 Page 11 of this chapter].

****Ex. 4.** Complete the following :

The law of force upon a particle above the earth's surface is ....and inside the surface is ....

[Hint : See note at the end of § 9 on Page 6 of this chapter].

Ex 5. A particle is falling under attraction due to earth from a height h above the Earth's surface. Find the velocity of the particle on reaching the Earth's surface. Also find the time in reaching the surface.

[Hint : See Ex. 6 Page 11 of this chapter]

Ex. 6. If $\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}$, find the expression of time.

(Gorakhpur 89)

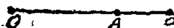
[Hint : See § 9 result (iv) Page 6 of this chapter]

****§ 10. Motion under Miscellaneous Laws.**

A particle moves under acceleration varying as the distance, and directed away from a fixed point ; to investigate the motion.

(Kanpur 87)

Let O be the fixed point and A the point of start, where $OA=a$. Let P be the position of the particle at time t, such that $OP=x$.


(Fig. 9)

According to the problem, the equation of motion at P of the particle of mass m (say) is

$$mv\frac{dv}{dx}=m\mu x \quad \text{or} \quad v\frac{dv}{dx}=\mu x, \qquad \text{...(i)}$$

here the acceleration is in the sense of x increasing.

Integrating both sides with respect to x, we get

$$\tfrac{1}{2}v^2=\tfrac{1}{2}\mu x^2+C_1, \qquad \text{...(ii)}$$

where C_1 is constant of integration.

At A, *i.e.* at $x=a$, we find that $v=0$.

$\therefore$ From (ii), we get $0=\frac{1}{2}\mu a^2+C_1$ or $C_1=-\frac{1}{2}\mu a^2$.

$\therefore$ From (ii), $v^2=\mu(x^2-a^2)$ or $v=\sqrt{\mu}\sqrt{(x^2-a^2)}$ ...(iii)

or $\frac{dx}{dt}=\sqrt{\mu}\sqrt{(x^2-a^2)}$ or $dt=\frac{dx}{\sqrt{\mu}\sqrt{(x^2-a^2)}}$. ...(A)

Integrating, $t=\frac{1}{\sqrt{\mu}}\cosh^{-1}\left(\frac{x}{a}\right)+C_2$, where C_2 is constant of integration.

At A, $x=a$, $t=0$, $\therefore$ $C_2=0$.

Hence $t=(1/\sqrt{\mu})\cosh^{-1}(x/a)$ or $x=a\cosh(\sqrt{\mu}t)$ ...(iv)

$\therefore$ (i), (iii) and (iv) give acceleration, velocity and position of the particle at time t.

[Again from (A), on integrating we can have

$\sqrt{\mu}t=\log\{x+\sqrt{(x^2-a^2)}\}+C_3$, where C_3 is constant of integration.

At A, $x=a$, $t=0$

$\therefore$ $0=\log\{a\}+C_3$ or $C_3=-\log a$

Hence $\sqrt{\mu}t=\log[\{x+\sqrt{(x^2-a^2)}\}-\log a$

or $t\sqrt{\mu}=\log[\{x+\sqrt{(x^2-a^2)}/a]$ *(Gorakhpur 89)*

Solved Examples on Motion under Miscellaneous Laws.

Ex. 1 A particle moves towards a centre of attraction starting from rest at a distance 'a' from the centre. If its velocity when at a distance 'x' from the centre varies as $\sqrt{\{(a^2-x^2)/x^2\}}$, find the law of force.

Sol. Since the particle moves towards the centre of attraction and its velocity when at distance x from the centre varies as $\sqrt{\{(a^2-x^2)/x^2\}}$.

$$\therefore \qquad \frac{dx}{dt}=-\mu\sqrt{\left\{\frac{a^2-x^2}{x^2}\right\}}$$

or $$\left(\frac{dx}{dt}\right)^2=\mu^2\left(\frac{a^2-x^2}{x^2}\right)=\mu^2\left(\frac{a^2}{x^2}-1\right).$$

Differentiating both sides with respect to t, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=\mu^2\left(-\frac{2a^2}{x^3}\right)\frac{dx}{dt} \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{a^2\mu^2}{x^3}.$$

Hence the law of force is inverse cube of the distance from the centre and is directed towards the centre.

Ex. 2. Assuming that at a distance x from a centre of force the speed v of a particle, moving in a straight line is given by the equation $x=ae^{bv^2}$, where a and b are constants. Find the law and nature of the force.

Sol. Given that $x=ae^{bv^2}$.

Differentiating both sides with respect to x, we get

$$1=ae^{bv^2}.2bv\frac{dv}{dx}=2bvx\frac{dv}{dx}, \text{ from (i)}$$

or $$v\frac{dv}{dx}=\frac{1}{2bx} \quad \text{or} \quad \frac{d^2x}{dt^2}=\frac{1}{2bx}, \quad \because v\frac{dv}{dx}=\frac{d^2x}{dt^2}$$

Hence the law of force is that of inverse distance and is directed away from the centre as this acceleration is positive.

****Ex. 3.** A particle moves such that its acceleration varies inversely as cube of the distance from a fixed point and is directed towards the fixed point; discuss the motion.**

Or

A particle moves in a straight line from a distance a, towards the centre of force varying inversely as the cube of distance. Find the time of its descent to the centre. *(Kanpur 87; Purvanchal 89)*

Sol. Let the particle be at time t at a distance x from the centre, then the equation of motion according to the problem is

$$v\frac{dv}{dx}=-\frac{\mu}{x^3}. \qquad \text{...(i)}$$

Integrating both sides with respect to x, we get

$$\tfrac{1}{2}v^2=(\mu/2x^2)+c_1, \qquad \text{.. (ii)}$$

where c_1 is constant of integration.

Let the particle start from rest at $x=a$. Then from (ii) we have $0=(\mu/2a^2)+c_1$ or $c_1=-(\mu/2a^2)$

$\therefore$ From (ii) we get $v^2=\mu\left(\frac{1}{x^2}-\frac{1}{a^2}\right)=\frac{\mu}{a^2}\left(\frac{a^2-x^2}{x^2}\right)$

or $v=-\sqrt{(\mu/a^2)}\sqrt{[(a^2-x^2)/x^2]}$, ...(iii)

the negative sign show that x decreases as t increases.

From (iii) we get $dt=-\frac{a}{\sqrt{\mu}}\cdot\frac{x\,dx}{\sqrt{(a^2-x^2)}}$, $\because$ $v=\frac{dx}{dt}$

Integrating, $t=(a/\sqrt{\mu}).\sqrt{(a^2-x^2)}+c_2$,
where c_2 is constant of integration.

At $x=a$, $t=0$ and therefore $c_2=0$.

Hence $t=(a/\sqrt{\mu})\sqrt{(a^2-x^2)}$. ...(iv)

Results (i), (iii) and (iv) give acceleration, velocity and position of the particle at time t.

Putting $x=0$ in (iv), we find that the time taken by the particle in reaching the centre of attraction from $x=a$ is

$$(a/\sqrt{\mu})\sqrt{(a^2-0)} \text{ i.e. } a^2/\sqrt{\mu}$$

*Ex. 4. A particle starts from rest at a distance a from the centre of force which attracts inversely as the distance. Prove that the time of arriving at the centre is $a\sqrt{(\pi/2\mu)}$.

Sol. The equation of motion is $\frac{d^2x}{dt^2}=-\frac{\mu}{x}$. ...(i)

Integrating, $(dx/dt)^2=-2\mu\log x+C$, where C is constant.

Initially at $x=a$, $\frac{dx}{dt}=0$ (given)

$\therefore$ $0=-2\mu\log a+C$. or $C=2\mu\log a$

$\therefore$ $(dx/dt)^2=2\mu(\log a-\log x)$

or $dx/dt=-\sqrt{(2\mu)}\sqrt{\{\log(a/x)\}}$, ...(ii)

the negative sign is due to the fact that the particle is moving towards the centre.

$\therefore$ from (ii) we get, $dt=-\frac{1}{\sqrt{(2\mu)}}\cdot\frac{dx}{\sqrt{\{(\log a/x)\}}}$

$\therefore$ Required time from $x=a$ to $x=0$

$$=-\frac{1}{\sqrt{(2\mu)}}\int_{x=a}^{0}\frac{dx}{\sqrt{\{\log(a/x)\}}}$$

$$=-\frac{1}{\sqrt{(2\mu)}}\int_{0}^{\infty}\frac{-2aze^{-z^2}}{z}, \text{ putting } \log(a/x)=z^2$$

or $x=ae^{-z^2}$ or $dx=-2aze^{-z^2}dz$

$$=\frac{2a}{\sqrt{(2\mu)}}\int_0^\infty e^{-z^2}dz=\frac{2a}{\sqrt{(2\mu)}}\cdot\frac{\sqrt{\pi}}{2}, \quad \because \int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$$

$=a\sqrt{(\pi/2\mu)}$ Hence proved.

Ex. 5 (a). A particle moves in a straight line under a force to a point in it varying as $(\text{distance})^{-4/3}$; show that the velocity acquired in falling from rest at infinity to a distance a is equal to that in falling from rest at a distance a to distance $a/8$.

Sol. Given that $d^2x/dt^2 = -\mu x^{-4/3}$. ...(i)

Multiplying both sides by $(2dx/dt)$ and integrating we get

$$\left(\frac{dx}{dt}\right)^2 = -2\mu \cdot \frac{x^{(-4/3+1)}}{(-\frac{4}{3}+1)} + C, \text{ where } C \text{ is constant.}$$

or $$\left(\frac{dx}{dt}\right)^2 = \frac{6\mu}{x^{1/3}} + C. \quad ...(ii)$$

If the particle falls from rest at infinity, then at

$$x = \infty \; ; dx/dt = 0, \quad \therefore \quad C = 0$$

$\therefore$ From (ii) we get $\left(\dfrac{dx}{dt}\right)^2 = \dfrac{6\mu}{x^{1/3}}$

$\therefore$ If v be the velocity of the particle at $x = a$, then

$$v^2 = 6\mu/a^{1/3} \quad ...(iii)$$

Again if the particle starts from rest at $x = a$, then from (ii)

$$0 = \frac{6\mu}{a^{1/3}} + C \quad \text{or} \quad C = -\frac{6\mu}{a^{1/3}}$$

$\therefore$ From (ii) we get, $\left(\dfrac{dx}{dt}\right)^2 = 6\mu\left[\dfrac{1}{x^{1/3}} - \dfrac{1}{a^{1/3}}\right]$

$\therefore$ If v_1 be the velocity at $x = a/8$, then

$$v_1^2 = 6\mu\left[\left(\frac{8}{a}\right)^{1/3} - \frac{1}{a^{1/3}}\right] = 6\mu\left[\frac{2}{a^{1/3}} - \frac{1}{a^{1/3}}\right]$$

or $$v_1^2 = 6\mu/a^{1/3}. \quad ...(iv)$$

$\therefore$ From (iii) and (iv), we get $v = v_1$. Hence proved.

***Ex. 5 (b).** A [illegible] force to a fixed point [illegible] falling [illegible] equal [illegible] to's [illegible]

Sol. Given that $d^2x/dt^2 = -\mu/x^n = -\mu x^{-n}$, ...(i)

where x is the distance of the particle from the fixed point at time t.

Multiplying both sides by $2\,(dx/dt)$ and integrating we get

$(dx/dt)^2 = -2\mu\,[x^{-n+1}/(-n+1)] + C$, where C is constant

or $$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{(n-1)\,x^{n-1}} + C. \quad ...(ii)$$

If the particle falls from rest at infinity, then at $x = \infty$, $dx/dt = 0$, $\therefore$ $C = 0$.

$\therefore$ From (ii), we get $(dx/dt)^2 = 2\mu/[(n-1)\,x^{n-1}]$.

$\therefore$ If v be the velocity of the particle at $x = a$, then

$$v^2 = 2\mu/[(n-1)\,a^{n-1}]. \quad ...(iii)$$

Again if the particle starts from rest from $x=a$, then at $x=a$, $dx/dt=0$ and from (ii) we have

$$0=\frac{2\mu}{(n-1)\,a^{n-1}}+C \quad \text{or} \quad C=-\frac{2\mu}{(n-1)\,a^{n-1}}$$

$\therefore$ From (ii), we get $\left(\frac{dx}{dt}\right)^2=\frac{2\mu}{(n-1)}\left[\frac{1}{x^{n-1}}-\frac{1}{a^{n-1}}\right]$.

$\therefore$ If v_1 be the velocity at $x=\frac{1}{4}a$, we have

$$v_1^2=\frac{2\mu}{(n-1)}\left[\frac{1}{(\frac{1}{4}a)^{n-1}}-\frac{1}{a^{n-1}}\right] \quad ...(iv)$$

Now if $v=v_1$ from (iii) and (iv), we get

$$\frac{2\mu}{(n-1)\,a^{n-1}}=\frac{2\mu}{(n-1)}\left[\frac{4^{n-1}}{a^{n-1}}-\frac{1}{a^{n-1}}\right]$$

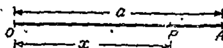
or $1=4^{n-1}-1$ or $2=4^{n-1}=(2^2)^{n-1}=2^{2n-2}$

or $2^3=2^{2n}$ or $2n=3$ or $n=\frac{3}{2}$. Hence proved.

****Ex. 6. If a particle is projected towards the centre of repulsion, the repulsion varying as the distance from the centre, from a distance a from it with a velocity $a\sqrt{\mu}$, prove that the particle will approach the centre but never reach it.**

Sol. Let the particle be at P at a distance x from the centre O after time t. Also $OA=a$ (given).

The force of repulsion means that the acceleration is away from O in the direction OP. Also the particle is projected from A towards O.



(Fig. 10)

$\therefore$ The equation of motion is $d^2x/dt^2=\mu x$. ...(i)

(Note)

(the acceleration being away from O, we have +ve sign)

Integrating (i) we have $(dx/dt)^2=\mu x^2+C$, where C is constant.

Given that at $x=a$, $dx/dt=a\sqrt{\mu}$

$\therefore$ $a^2\mu=\mu a^2+C$ or $C=0$

$\therefore$ $(dx/dt)^2=\mu x^2$ or $dx/dt=-\sqrt{\mu}\,x$ ...(ii)

(negative sign has been taken as the particle moves towards O)

or $dt=-(1/\sqrt{\mu})\,(1/x)\,dx$

$\therefore$ Time to reach the centre O from A (*i.e.* from $x=a$ to $x=0$)

$$=-\frac{1}{\sqrt{\mu}}\int_{x=a}^{0}\frac{dx}{x}=\frac{1}{\sqrt{\mu}}\left[\log x\right]_0^a=\frac{1}{\sqrt{\mu}}\,[\log a-\log 0]$$

$=(1/\sqrt{\mu})\,[\log a-(-\infty)]$, $\because$ $\log 0=-\infty$. (Note)

$=\infty$.

i.e. the particle will reach the centre after infinite time *i.e.* the particle will never reach the centre O.

Ex. 7 (a). A particle, whose mass is 'm', is acted upon by a force $m\mu\,[x+(a^4/x^3)]$ towards origin, if it starts from rest at a distance 'a' show that it will arrive at the origin in time $\pi/4\sqrt{\mu}$.
(Avadh 90; Bundelkhand 87; Rohilkhand 90)

Sol. The equation of motion is

$$m.v\frac{dv}{dx}=-m\mu\left[x+\frac{a^4}{x^3}\right] \quad \text{or} \quad v\frac{dv}{dx}=-\mu\left[x+\frac{a^4}{x^3}\right] \quad ...(i)$$

Integrating both sides with respect to x, we get

$$\tfrac{1}{2}v^2=-\mu\left[\tfrac{1}{2}x^2-\frac{a^4}{2x^2}\right]+c_1, \quad ...(ii)$$

where c_1 is constant of integration.

Initially $x=a$ and $v=0$ (given)

∴ From (ii) we get $0=-\mu\,[\tfrac{1}{2}a^2-\tfrac{1}{2}a^2]+c_1$ or $c_1=0$

∴ From (ii) we get $v^2=\mu\left(\frac{a^4}{x^2}-x^2\right)=\mu\left(\frac{a^4-x^4}{x^2}\right)$ ...(iii)

or $\quad dx/dt=-\sqrt{\mu}\sqrt{[(a^4-x^4)/x^2]}$,

the negative sign has been taken as the particle moves towards origin.

or $$dt=-\frac{1}{\sqrt{\mu}}\frac{x\,dx}{\sqrt{(a^4-x^4)}}$$

∴ Required time of travelling from $x=a$ at $x=0$

$$=-\frac{1}{\sqrt{\mu}}\int_{x=a}^{0}\frac{x\,dx}{\sqrt{(a^4-x^4)}}=-\frac{1}{2\sqrt{\mu}}\int_{z=a^2}^{0}\frac{dz}{\sqrt{(a^4-z^2)}},$$
putting $x^2=z$

$$=\frac{1}{2\sqrt{\mu}}\left[\sin^{-1}\left(\frac{z}{a^2}\right)\right]_0^{a^2}=\frac{1}{2\sqrt{\mu}}\left[\sin^{-1}(1)\right]=\frac{\pi}{4\sqrt{\mu}}$$

Hence proved.

Ex. 7 (b). A particle moves with an acceleration $\mu\,[x+(a^4/x^3)]$ towards the origin. If it starts from rest at a distance a from the origin, determine its velocity when its distance from origin is $a/2$.
(Allahabad 86)

Sol. Proceed as in Ex. 7 (a) above and get the result (iii),

viz. $$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{\left(\frac{a^4-x^4}{x^2}\right)}$$

Let the required velocity be V when $x=a/2$.

Then $\quad V=-\sqrt{\mu}\sqrt{\left[\frac{a^4-(a/2)^4}{(a/2)^2}\right]}=-\tfrac{1}{2}a\sqrt{(15\mu)}$. **Ans.**

The negative sign shows that direction of V is towards the origin.

*Ex 7 (c). A particle whose mass is 'm' is acted upon by a force $m\mu\,(x^{-5/3})$ towards the centre. If it starts from rest at a distance 'a' from this centre, then show that it will arrive at the centre after time $2\,a^{4/3}/\sqrt{(3\mu)}$.

Sol. The equation of motion is

$$m.v\frac{dv}{dx}=-m\mu\,x^{-5/3} \quad \text{or} \quad v\,dv=-\mu\,x^{-5/3}\,dx$$

Integrating both sides, we get $\frac{1}{2}v^2=\frac{3}{2}\mu\,x^{-2/3}+c$, ...(i)
where c is constant of integration.

$\therefore$ From (i), we get $0=\frac{3}{2}\mu\,a^{-2/3}+c$ or $c=-\frac{3}{2}\mu\,a^{-2/3}$

$\therefore$ From (i) we get $\frac{1}{2}v^2=\frac{3}{2}\mu\,(x^{-2/3}-a^{-2/3})$

or $\left(\dfrac{dx}{dt}\right)^2=3\mu\,(x^{-2/3}-a^{-2/3})$ or $\dfrac{dx}{dt}=-\sqrt{(3\mu)}\sqrt{(x^{-2/3}-a^{-2/3})}$, ...(ii)

the negative sign has been taken as the particle moves towards the origin i.e. x decreases as t increases.

or $$dt=-\frac{1}{\sqrt{(3\mu)}}\,\frac{dx}{\sqrt{(x^{-2/3}-a^{-2/3})}}$$

$\therefore$ Required time of moving from $x=a$ to $x=0$

$$=-\frac{1}{\sqrt{(3\mu)}}\int_{x=a}^{0}\frac{dx}{\sqrt{(x^{-2/3}-a^{-2/3})}}$$

$$=-\frac{1}{\sqrt{(3\mu)}}\int_{x=a}^{0}\frac{x^{1/3}\,a^{1/3}\,dx}{\sqrt{(a^{2/3}-x^{2/3})}} \qquad \text{(Note)}$$

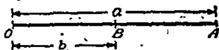
$$=-\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_{t=\pi/2}^{0}\frac{a^{1/3}\sin\theta.3a\sin^2\theta\cos\theta\,d\theta}{a^{1/3}\sqrt{(1-\sin^2\theta)}},$$

putting $x=a\sin^3\theta$

$$=\frac{3\,a^{4/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2}\sin^3\theta\,d\theta=\frac{3\,a^{4/3}}{\sqrt{(3\mu)}}\cdot\frac{2}{3}=\frac{2\,a^{4/3}}{\sqrt{(3\mu)}}.$$ Hence proved.

*Ex 8 (a) A ,
rest at $x=a$. For
the acceleration is
is μx, and at the end of this interval the particle is at the origin; prove that $\tan(\sqrt{\mu}t_1)\tanh(\sqrt{\mu}t_2)=1$. (Avadh 88)

Sol. Let OA be the axis and O be the origin.



Let the particle start from A such that $OA=a$.

Let from A to B, the acce- (Fig. 11)
leration be $-\mu x$ (i.e. towards O) and from B to O, the acceleration be μx (i.e. away from O or in the sense of x increasing). Let $OB=b$.

For the motion from A to B, the equation of motion is

$$\frac{d^2x}{dt^2}=-\mu x. \qquad \text{...(i)}$$

Integrating, we get $(dx/dt)^2=-\mu x^2+C$, where C is constant.

At A, $x=a$ and $dx/dt=0$, $\therefore$ $C=\mu a^2$.

So we get $(dx/dt)^2=\mu(a^2-x^2)$ .. (ii)

At B let velocity be v; then from (ii), $v^2=\mu(a^2-b^2)$ .. (iii)

From (ii) we get $dx/dt=-\sqrt{\mu}\sqrt{(a^2-x^2)}$, since particle is moving towards O ,

or $$dt=-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}$$

Integrating, we get $t=\dfrac{1}{\sqrt{\mu}}\cos^{-1}\left(\dfrac{x}{a}\right)+k$, where k is constant.

At A, $x=a$, $t=0$, $\therefore$ $k=0$

$\therefore$ $t=(1/\sqrt{\mu})\cos^{-1}(x/a)$

$\therefore$ If t_1 be the time taken in moving from A to B, then

$t_1=(1/\sqrt{\mu})\cos^{-1}(b/a)$ or $\cos(\sqrt{\mu}t_1)=b/a$

or $\tan(\sqrt{\mu}t_1)=[\sqrt{(a^2-b^2)}]/b$ ...(iv)

Now for the motion from B to O, the equation is $\dfrac{d^2x}{dt^2}=\mu x$.

Integrating we get $(dx/dt)^2=\mu x^2+C_1$, where C_1 is constant.

At V, $x=b$ and $dx/dt=v$, where $v^2=\mu(a^2-b^2)$, from (iii)

$\therefore$ $\mu(a^2-b^2)=\mu b^2+C_1$ or $C_1=\mu(a^2-2b^2)$

$\therefore$ $(dx/dt)^2=\mu[(a^2-2b^2)+x^2]$

or $dx/dt=-\sqrt{\mu}\sqrt{[(a^2-2b^2)+x^2]}$

(negative sign shows that x decreases as t increases)

or $$dt=-\frac{1}{\sqrt{\mu}}\cdot\frac{dx}{\sqrt{[(a^2-2b^2)+x^2]}}$$

Integrating we get, $t=-\dfrac{1}{\sqrt{\mu}}\sinh^{-1}\left[\dfrac{x}{\sqrt{(a^2-2b^2)}}\right]+C_2$

At B, $t=0$, $x=b$, $\therefore$ $C_2=\dfrac{1}{\sqrt{\mu}}\sinh^{-1}\left[\dfrac{b}{\sqrt{(a^2-2b^2)}}\right]$.

$$\therefore\quad t=\frac{1}{\sqrt{\mu}}\left[\sinh^{-1}\left\{\frac{b}{\sqrt{(a^2-2b^2)}}\right\}-\sinh^{-1}\left\{\frac{x}{\sqrt{(a^2-2b^2)}}\right\}\right] \quad ...(v)$$

$\therefore$ If t_2 be the time taken in moving from B to O, then putting $x=0$ in (v) we have

$t_2=\dfrac{1}{\sqrt{\mu}}\sinh^{-1}\left\{\dfrac{b}{\sqrt{(a^2-2b^2)}}\right\}$ or $\sinh(\sqrt{\mu}t_2)=\dfrac{b}{\sqrt{(a^2-2b^2)}}$

$\therefore$ $\cosh(\sqrt{\mu}t_2)=\sqrt{\{1+\sinh^2(\sqrt{\mu}t_2)\}}$

$$=\sqrt{\left\{1+\frac{b^2}{(a^2-2b^2)}\right\}}=\sqrt{\left\{\frac{(a^2-b^2)}{(a^2-2b^2)}\right\}}$$

$\therefore$ $\tanh(\sqrt{\mu}t_2)=b/\sqrt{(a^2-b^2)}$ ...(vi)

Hence from (iv) and (vi) we get

$\tan(\sqrt{\mu}t_1)\times\tanh(\sqrt{\mu}t_2)=1.$ Hence proved.

***Ex. 8 (b). A particle starts with a given velocity V and moves under a retardation K times the space described. Show that the distance traversed before it comes to rest is $V/\sqrt{K}$.**

Sol. Refer Fig. 10 Page 21 of this chapter.

Let the particle start from O with the given velocity V under the given retardation. Let the particle be at P at a distance x from O after time t. Then equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-Kx \qquad \text{...(i)}$$

Multiplying both sides by 2 (dx/dt) and integrating we have

$$(dx/dt)^2=-Kx^2+C \qquad \text{... (ii)}$$

Initially *i.e.* at O, dx/dt=velocity$=V$ (given) and $x=0$, so from (ii) we get $V^2=C$.

∴ From (ii) we get $(dx/dt)^2=-Kx^2+V^2$ .. (iii)

Let the particle come to rest at A, such that $x=x_1$ (say) at A. Then from (iii) we get $0=-Kx_1^2+V^2$ or $x_1^2=V^2/K$.

or $x_1=V/\sqrt{K}$. Hence proved.

****Ex. 9. A particle moves in a straight line towards a centre of force $\mu/(\text{ditance})^2$ starting from rest at a distance a from the centre of force ; show that the time of reaching a point distant b from the centre of force is $a\sqrt{[(a^2-b^2)/\mu]}$, and that its velocity then is $\sqrt{[\mu(a^2-b^2)]}/ba$.** *(Avadh 86)*

Sol Refer Fig. 11 Page 23 of this chapter.

Let the particle move along the line OA, starting from rest at A towards the centre of force O, such that $OA=a$.

Let B be a point on OA, such that $OB=b$. To find the velocity at B and the time of reaching B from A.

The equation of motion is $\dfrac{d^2x}{dt^2}=-\dfrac{\mu}{x^2}$. ...(i)

Multiplying both sides by 2 (dx/dt) and integrating we have $(dx/dt)^2=(\mu/x^2)+C$, where C is constant of integration.

At A, $x=a$, $\dfrac{dx}{dt}=0$, ∴ $C=-\dfrac{\mu}{a^2}$

Hence $$\left(\frac{dx}{dt}\right)^2=\mu\left(\frac{1}{x^2}-\frac{1}{a^2}\right) \qquad \text{... (ii)}$$

$\therefore$ At B, i.e. at $x=b$, (velocity)$^2=\mu\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=\frac{\mu(a^2-b^2)}{a^2b^2}$.

or velocity $=-\frac{\sqrt{[\mu(a^2-b^2)]}}{ab}$, $-$ve sign shows that its direction is towards O.

Also from (i) we get $\frac{dx}{dt}=-\sqrt{\mu}\sqrt{\left(\frac{a^2-x^2}{a^2x^2}\right)}$

($-$ve sign shows that x decreases as time t increases)

or $$dt=-\frac{1}{\sqrt{\mu}}\frac{ax\,dx}{\sqrt{(a^2-x^2)}}$$

$\therefore$ Required time from A to B i.e. from $x=a$ to $x=b$

$$=-\frac{a}{\sqrt{\mu}}\int_{x=a}^{b}\frac{x\,dx}{\sqrt{(a^2-x^2)}}=\frac{a}{\sqrt{\mu}}\left[\sqrt{(a^2-x^2)}\right]_a^b=a\sqrt{\left(\frac{a^2-b^2}{\mu}\right)}$$

Hence proved.

Ex. 10. A particle moves in a straight line, its acceleration is directed towards a fixed point O in the line and is always equal to $\mu(a^5/x^2)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, show that it will arrive at O with a velocity $a\sqrt{(6\mu)}$ after time $(8/15)\sqrt{(6/\mu)}$

(Meerut 87, 86; Rohilkhand 87)

Sol. The equation of motion is

$$\frac{d^2x}{dt^2}=-\mu\left(\frac{a^5}{x^2}\right)^{1/3} \text{ or } v\frac{dv}{dx}=-\frac{\mu a^{5/3}}{x^{2/3}}, \quad \because \frac{d^2x}{dt^2}=v\frac{dv}{dx}$$

or $$v\,dv=-\frac{\mu a^{5/3}}{x^{2/3}}dx.$$

Integrating, $\frac{1}{2}v^2=-3\mu a^{5/3}.x^{1/3}+C$, where C is constant.

Initially $x=a$, velocity $v=0$, $\therefore C=3\mu a^{5/3}a^{1/3}$

$\therefore$ $v^2=6\mu a^{5/3}(a^{1/3}-x^{1/3})$

or $(dx/dt)^2=6\mu a^{5/3}(a^{1/3}-x^{1/3})$ ...(i)

or $dx/dt=-\sqrt{(6\mu a^{5/3})}.\sqrt{(a^{1/3}-x^{1/3})}$

or $$dt=\frac{-1}{\sqrt{(6\mu a^{5/3})}}\cdot\frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}}.$$

$\therefore$ Required time from $x=a$ to O, where $x=0$

$$=-\frac{1}{\sqrt{(6\mu a^{5/3})}}\int_{x=a}^{0}\frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}}=\frac{1}{\sqrt{(6\mu a^{5/3})}}\int_{x=0}^{a}\frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}}$$

$$=\frac{1}{\sqrt{(6\mu a^{1/3})}}\int_{\theta=0}^{\pi/2}\frac{6a\sin^5\theta\cos\theta\,d\theta}{\sqrt{(a^{1/3})}\cos\theta}, \text{ putting } x=a\sin^6\theta$$

$$=\frac{6a}{\sqrt{(6\mu a^2)}}\int_0^{\pi/2}\sin^5\theta\,d\theta=\frac{6}{\sqrt{(6\mu)}}\cdot\{\tfrac{4}{5}\cdot\tfrac{2}{3}\}=\{\tfrac{8}{15}\}\sqrt{\left(\frac{6}{\mu}\right)}$$

Also v, the velocity at O *i e.* at $x=0$ is given by

$$v^2=6\mu a^{5/3}(a^{1/3}), \quad \text{putting } x=0 \text{ in (i)}$$

or $$v=a\sqrt{(6\mu)}.$$ Hence proved.

Ex. 11. A particle moves in a straight line with an acceleration towards a fixed point in the straight line which is equal to $\left(\frac{\mu}{x^2}-\frac{\lambda}{x^3}\right)$ at a distance 'x' from the given point, the particle starts from rest at a distance 'a'. Show that it oscillates between this distance and the distance $\lambda a/(2\mu a-\lambda)$ and the periodic time is

$$\frac{2\mu\pi a^3}{(2a\mu-\lambda)^{3/2}}$$

Sol. The equation of motion is

$$v\frac{dv}{dx}=-\left(\frac{\mu}{x^2}-\frac{\lambda}{x^3}\right) \quad ...(i)$$

Integrating both sides of (i) with respect to x, we get

$$\tfrac{1}{2}v^2=\frac{\mu}{x}-\frac{\lambda}{2x^2}+c_1, \quad ..(ii)$$

where c_1 is constant of integration.

Initially $x=a$, $v=0$, so form (ii) we get

$$x=\frac{\mu}{a}-\frac{\lambda}{2a^2}+c_1 \quad \text{or} \quad c_1=\frac{\lambda}{2a^2}-\frac{\mu}{a}$$

$\therefore$ From (ii), we get $\tfrac{1}{2}v^2=\frac{\mu}{x}-\frac{\lambda}{2x^2}+\frac{\lambda}{2a^2}-\frac{\mu}{a}$

or $$v^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right)-\lambda\left(\frac{1}{x^2}-\frac{1}{a^2}\right)$$

or $$v^2=\left(\frac{1}{x}-\frac{1}{a}\right)\left[2\mu-\lambda\left(\frac{1}{x}+\frac{1}{a}\right)\right] \quad ...(iii)$$

The particle comes to rest when $v=0$

i e. when $$\left(\frac{1}{x}-\frac{1}{a}\right)\left[2\mu-\lambda\left(\frac{1}{x}+\frac{1}{a}\right)\right]=0$$

i.e. when $x=a$ or $2\mu-\lambda\left(\frac{1}{x}+\frac{1}{a}\right)=0$, which gives $x=\frac{\lambda a}{2\mu a-\lambda}$.

The two values of x give us the positions of rest, *i.e.* the particle oscillates between $x=a$ and $x=\frac{\lambda a}{2\mu a-\lambda}$.

Again from (iii), we have

$$v^2=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left[\left(\frac{2\mu}{\lambda}-\frac{1}{a}\right)-\frac{1}{x}\right]$$

or
$$v^2=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left[\left(\frac{2\mu a-\lambda}{a\lambda}\right)-\frac{1}{x}\right]$$

$$=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{x}\right), \text{ where } b=\frac{a\lambda}{2\mu a-\lambda}$$

or
$$\left(\frac{dx}{dt}\right)^2=\frac{\lambda}{ab}\cdot\frac{(a-x)(x-b)}{x^2}$$

or
$$\frac{dx}{dt}=-\sqrt{\left(\frac{\lambda}{ab}\right)}\cdot\frac{\sqrt{\{(a-x)(x-b)\}}}{x},$$

the negative sign has been taken as the particle moves towards the origin

or
$$dt=-\sqrt{\left\{\frac{ab}{\lambda}\right\}}\cdot\frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}}.$$

$\therefore$ Time from one position of rest to another, *i.e.* from $x=a$ to $x=b$

$$=-\sqrt{\left\{\frac{ab}{\lambda}\right\}}\int_{x=a}^{b}\frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}}$$

$$=\sqrt{\left\{\frac{ab}{\lambda}\right\}}\int_{y=-(a-b)/2}^{(a-b)/2}\frac{\{\frac{1}{2}(a+b)-y\}\,dy}{\sqrt{[\{\frac{1}{2}(a-b)\}^2-y^2]}},\quad \text{(Note)}$$

putting $x=\frac{1}{2}(a+b)-y$ or $dx=-dy$

$$=\sqrt{\left\{\frac{ab}{\lambda}\right\}}\left[\tfrac{1}{2}(a+b)\sin^{-1}\left\{\frac{y}{\frac{1}{2}(a-b)}\right\}+\sqrt{\left\{\tfrac{1}{4}(a-b)^2-y^2\right\}}\right]_{-(a-b)/2}^{(a-b)/2}$$

$$=\sqrt{(ab/\lambda)}\cdot[\tfrac{1}{2}(a+b)\pi].$$

$\therefore$ The required time $=2$ times time from $x=a$ to $x=b$

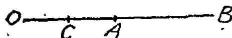
$$=\sqrt{\left\{\frac{ab}{\lambda}\right\}}\cdot(a+b)\pi, \text{ where } b=\frac{a\lambda}{2\mu a-\lambda}$$

$$=\sqrt{\left[\frac{a\,\lambda a}{\lambda(2a\mu-\lambda)}\right]}\cdot\left[a+\frac{\lambda a}{(2a\mu-\lambda)}\right]\pi$$

$$=\pi\left[\frac{2a^2\mu}{2a\mu-\lambda}\right]\cdot\frac{a}{\sqrt{(2a\mu-\lambda)}}=\frac{2a^3\mu\pi}{(2a\mu-\lambda)^{3/2}}.$$ Hence proved.

Ex. 12. A particle starts from rest at a distance a from a fixed point, under the action of a force through the fixed point, the law of which at a distance x is $\mu(1-a/x)$ towards the point when $x>a$, but $\mu[(a^2/x^2)-(a/x)]$ from the same point when $x<a$; prove that the particle will oscillate through a space $(b^2-a^2)/b$.

Sol. Let the fixed point be O and the particle start from B, such that $OB=b$ (given) and let it move along the line BO. Let $OA=a$.



(Fig. 12)

Consider the motion from B to A i.e. for $x > a$.

The equation of motion is $m\dfrac{d^2x}{dt^2}=-\mu\left(1-\dfrac{a}{x}\right)$

or $$v\frac{dv}{dx}=-\frac{\mu}{m}\left(1-\frac{a}{x}\right), \quad \because v\frac{dv}{dx}=\frac{d^2x}{dt^2}$$

or $$v\,dv=-\frac{\mu}{m}\left(1-\frac{a}{x}\right)dx.$$

Integrating, $\frac{1}{2}v^2=-(\mu/m)[x-a\log x]+C$,

where C is constant of integration.

Initially at B, $v=0, x=b$.

$\therefore$ $0=-(\mu/m)[b-a\log b]+C$ or $C=(\mu/m)[b-a\log b]$

$\therefore$ $v^2=(2\mu/m)[b-a\log c-x+a\log x]$. ...(i)

Let V be the velocity of the particle when it reaches A, i.e. at $x=a$. Then from (i), $V^2=(2\mu/m)[b-a\log b-a+a\log a]$. ...(ii)

When the particle crosses A and moves towards O, $x < a$ and the equation of motion becomes $m\dfrac{d^2x}{dt^2}=\mu\left(\dfrac{a^2}{x^2}-\dfrac{a}{x}\right)$,

+ve sign on the right is due to the fact that the law of force is away from O

or $$v\frac{dv}{dx}=\frac{\mu}{m}\left(\frac{a^2}{x^2}-\frac{a}{x}\right), \quad \because v\frac{dv}{dx}=\frac{d^2x}{dt^2}$$

or $$v\,dv=\frac{\mu}{m}\left(\frac{a^2}{x^2}-\frac{a}{x}\right)dx.$$

Integrating, $\dfrac{v^2}{2}=\dfrac{\mu}{m}\left(-\dfrac{a^2}{x}-a\log x\right)+k$,

where k is constant of integration.

At A, $x=a$ and $v=V$ {given by (ii)}.

$$\therefore \frac{V^2}{2}=\frac{\mu}{m}\left[-\frac{a^2}{a}-a\log a\right]+k.$$

Subtracting, $v^2-V^2=\dfrac{2\mu}{m}\left[\dfrac{a^2}{a}+a\log a-\dfrac{a^2}{x}-a\log x\right]$. ...(iii)

Let the particle come to rest at C, such that $OC=c$,

i.e. $v=0$ when $x=c$.

$\therefore$ From (iii), $0-V^2=\dfrac{2\mu}{m}\left[a+a\log a-\dfrac{a^2}{c}-a\log c\right]$

or $$-\frac{2\mu}{m}[b-a\log b-a+a\log a]$$

$$= \frac{2\mu}{m}\left[a + a\log a - \frac{a^2}{c} - a\log c\right], \text{ from (ii),}$$

or $\quad a\log b - a\log a + a - b = a + a\log a - (a^2/c) - a\log c$

or $\quad (a^2/c) + a\log c = b + 2a\log a - a\log b$

$$= b + a\,[\log(a^2/b)]$$

or $\quad \dfrac{a^2}{c} + a\log c = \dfrac{a^2}{(a^2/b)} + a\log\left(\dfrac{a^2}{b}\right)$ (Note)

or $\quad c = (a^2/b)$ i.e. $OC = a^2/b$.

$\therefore$ Required distance through which the particle oscillates

$= BC = OB - OC = b - \dfrac{a^2}{b} = \dfrac{b^2 - a^2}{b}$. Hence proved.

Ex 13. A particle of mass m moving in a straight line is acted upon by an attractive force which is expressed by $(m\mu x/a)$ for $x < a$ and by $(m\mu a^2/x^2)$ for $x > a$ from a fixed origin on the line. I[illegible] particle starts from rest at a distance $2a$ from the origin, [illegible] velocity $\sqrt{(2\mu a)}$ and the [illegible] (Avadh 89)

Sol. Refer Fig. 12 Page 29 of this chapter.

Let O be a fixed origin and the particle start from B, where $OB = 2a$. The particle moves along BO. Let $OA = a$.

Consider the motion from B to A, i.e., for $x > a$

The equation of motion is $m\dfrac{d^2x}{dt^2} = -m\mu\dfrac{a^2}{x^2}$

or $\quad \dfrac{d^2x}{dt^2} = -\dfrac{\mu a^2}{x^2}$ or $2\dfrac{dx}{dt}\cdot\dfrac{d^2x}{dt^2} = \dfrac{-2\mu a^2}{x^2}\dfrac{dx}{dt}$

Integrating, $\left(\dfrac{dx}{dt}\right)^2 = \dfrac{2\mu a^2}{x} + C$, where C is constant of integration.

At B, velocity $= 0$, $x = 2a$.

$$\therefore\ 0 = \frac{2\mu a^2}{2a} + C \quad \text{or} \quad C = -\mu a.$$

$$\therefore\ \left(\frac{dx}{dt}\right)^2 = \frac{2\mu a^2}{x} - \mu a. \qquad \text{...(i)}$$

If V be the velocity of the particle when it reaches A i.e. at $x = a$, then from (i) we get $V^2 = 2\mu a - \mu a = \mu a$. ...(ii)

When the particle crosses A and moves towards O, $x < a$ and the equation of motion becomes

$$m\frac{d^2x}{dt^2} = -\frac{m\mu x}{a} \text{ or } 2\frac{d^2x}{dt^2}\cdot\frac{dx}{dt} = -\frac{2\mu}{a}x\frac{dx}{dt}.$$

Integrating, $\left(\dfrac{dx}{dt}\right)^2 = -\dfrac{\mu}{a}x^2 + D$, where D is constant of integration

At A, $x=a$ and velocity $=V$.

$\therefore\quad V^2=-\mu a+D \quad$ or $\quad D=V^2+\mu a$

or $\quad D=\mu a+\mu a$, from (ii)

or $\quad D=2\mu a$.

$$\therefore\quad (dx/dt)^2=(\mu/a)\,x^2+2\mu a. \qquad \text{...(iii)}$$

At the origin O, $x=0$, and let velocity $=u$.

Then from (iii), we get $u^2=2\mu a$ or $u=\sqrt{(2a\mu)}$.

Also from (i), we get $\dfrac{dx}{dt}=-\sqrt{\left(\dfrac{2\mu a^2}{x}-\mu a\right)}$,

the negative sign is due to the fact that x decreases as t increases

or $\quad \dfrac{dx}{dt}=-\sqrt{\left(\dfrac{2\mu a^2-\mu a x}{x}\right)}=-\sqrt{(a\mu)}\sqrt{\left(\dfrac{2a-x}{x}\right)}$

or $\quad dt=-\dfrac{1}{\sqrt{(a\mu)}}\sqrt{\left(\dfrac{x}{2a-x}\right)}\,dx.$

$\therefore$ Time from B to A, i.e. from $x=2a$ to $x=a$

$$=-\frac{1}{\sqrt{(a\mu)}}\int_{x=2a}^{a}\sqrt{\left(\frac{x}{2a-x}\right)}dx$$

$$=-\frac{1}{\sqrt{(a\mu)}}\int_{\theta=\pi/2}^{\pi/4}\sqrt{\left(\frac{2a\sin^2\theta}{2a\cos^2\theta}\right)}.4a\sin\theta\cos\theta\,d\theta,$$

putting $x=2a\sin^2\theta$

$$=-\frac{2a}{\sqrt{(\mu a)}}\int_{\pi/2}^{\pi/4}2\sin^2\theta\,d\theta=-2\sqrt{\left(\frac{a}{\mu}\right)}\int_{\pi/2}^{\pi/4}(1-\cos 2\theta)\,d\theta$$

$$=-2\sqrt{\left(\frac{a}{\mu}\right)}\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi/4}=-2\sqrt{\left(\frac{a}{\mu}\right)}\left[\left(\frac{\pi}{4}-\frac{1}{2}\right)-\left(\frac{\pi}{2}\right)\right]$$

$$=2\sqrt{(a/\mu)}\left[\tfrac{1}{4}\pi+\tfrac{1}{2}\right] \qquad \text{...(iv)}$$

And from (ii), we get $\left(\dfrac{dx}{dt}\right)^2=2\mu a-\dfrac{\mu}{a}x^2=\dfrac{\mu}{a}(2a^2-x^2)$

or $\quad \dfrac{dx}{dt}=-\sqrt{\left(\dfrac{\mu}{a}\right)}\sqrt{(2a^2-x^2)}$ or $\quad dt=-\sqrt{\left(\dfrac{a}{\mu}\right)}\dfrac{dx}{\sqrt{(2a^2-x^2)}}$

$\therefore$ Time from A to O i.e. from $x=a$ to $x=0$

$$=-\sqrt{\left(\frac{a}{\mu}\right)}\int_{x=a}^{0}\frac{dx}{\sqrt{(2a^2-x^2)}}=-\sqrt{\left(\frac{a}{\mu}\right)}\int_{\theta=\pi/4}^{0}\frac{a\sqrt{2}\cos\theta\,d\theta}{\sqrt{(2a^2\cos^2\theta)}},$$

putting $x=a\sqrt{2}\sin\theta$

$$=\sqrt{(a/\mu)}\int_0^{\pi/4}d\theta=\sqrt{(a/\mu)}\left[\theta\right]_0^{\pi/4}=\tfrac{1}{4}\pi\sqrt{(a/\mu)}. \qquad \text{...(v)}$$

$\therefore$ From (iv) and (v), we get the required time from B to O

$=2\sqrt{(a/\mu)}\left[\tfrac{1}{4}\pi+\tfrac{1}{2}\right]+\tfrac{1}{4}\pi\sqrt{(a/\mu)}=\sqrt{(a/\mu)}\left[\tfrac{3}{4}\pi+1\right].$

Hence proved.

Exercises on Miscellaneous Laws

Ex. 1. A man starts to walk at the rate of 4 ft./sec. and his velocity at any instant is inversely proportional to $(x+10)$, where x is the number of feet he has walked. Find (i) the time he takes to walk 20 yards and (ii) the distance he goes in $\frac{1}{2}$ minute.

[Hint : One yard=3 feet].

Ex. 2. A particle is attracted by a force to a fixed point varying inversely as (distance)^n. Show that if the velocity acquired in falling from infinity to a distance 'b' from the fixed point be equal to the velocity attained in falling from rest at a distance b to a distance $b/16$, then $n=5/4$.

[Hint : Do as Ex. 5 (b) Page 20 of this chapter]

Ex. 3. A particle moves in a straight line OA with an acceleration which is always directed towards O and which varies as the distance of the particle from O. If the particle were at rest at A, find the motion.

Ex. 4. If a particle begins to move directly towards a fixed centre which repels with an intensity$=\mu$ (distance) and with an initial velocity$=\frac{1}{2}\mu$ (distance), prove that it will continually approach the fixed centre but never reach it.

Ex. 5. A particle of mass m moves in a straight line under an attractive force mn^2x towards a fixed point on the line when at a distance x from it. If it is projected with a velocity V towards the centre of force at a distance a. from it, prove that it reaches the centre of force in time $(1/x)\tan^{-1}(xa/V)$.

Ex. 6. A particle of mass m moves on a straight line under an attraction mn^2x towards a point O on the line, where x is the distance from O. If $x=a$ and $dx/dt=0$ when $t=0$, find the period of oscillation.

Harmonic Motion

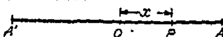
§ 1. Simple Harmonic Motion.

(*Agra 86; Avadh 90, 88; Bundelkhand 86; Gorakhpur 89, 88; Kanpur 87, 86; Meerut 86P; Ranchi 86*)

Definition. *A particle is said to execute Simple Harmonic Motion (or* S.H.M.) *if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.*

The expressions for acceleration, velocity and position of the particle at any instant are obtained as follows :—

Let O be the fixed point and the particle start from rest from the point A, such that $OA=a$. Let after time t the particle be at P, such that $OP=x$.



(Fig. 13)

Then dx/dt and d^2x/dt^2 are the velocity and the acceleration of the particle at time t acting in the direction in which x increases.

Since in this case the acceleration is directed towards the fixed point O and is proportional to the distance of the particle from O, therefore the equation of motion is

$$m.(d^2x/dt^2)=-\lambda x \quad \text{or} \quad d^2x/dt^2=-\mu x, \text{ where } \mu=(\lambda/m)>0$$

or $$v\frac{dv}{dx}=-\mu x, \quad \because \frac{d^2x}{dt^2}=v\frac{dv}{dx} \quad \ldots\text{(i)}$$

The differential equation (i) represents S. H. M. and is used generally as definition of S. H. M. *i.e.* if the equation of motion of a particle is of the form of (i) we atonce say that the particle is executing S.H.M. A system whose position is given by a single co-ordinate which satisfies (i) is known as a **Linear Harmonic Oscillator.**

Integrating (i) with respect to x, we get

$\frac{1}{2}v^2=-\frac{1}{2}\mu x^2+c_1$, where c_1 is constant of integration.

Initially *i.e.* at A, $v=0$ and $x=a$.

$$\therefore \quad 0=-\tfrac{1}{2}\mu a^2+c_1 \quad \text{or} \quad c_1=\tfrac{1}{2}\mu a^2$$

$$\therefore \quad \tfrac{1}{2}v^2=\tfrac{1}{2}\mu(a^2-x^2) \quad \text{or} \quad v^2=\mu(a^2-x^2) \quad \ldots\text{(ii)}$$

or $$dx/dt=-\sqrt{\mu}\sqrt{(a^2-x^2)}, \quad \ldots\text{(iii)}$$

the negative sign is to be taken here as the particle is moving towards O *i.e.* x decreases as t increases.

From (iii), we have $\dfrac{-dx}{\sqrt{(a^2-x^2)}}=\sqrt{\mu}\,dt$

Integrating, $\cos^{-1}(x/a)=\sqrt{(\mu)}\,t+c_2$, ..(iv)

where c_2 is constant of integration.

At A, $x=a$ and $t=0$, so from (iv) we get $c_2=0$ and hence

$$\cos^{-1}(x/a)=\sqrt{(\mu)}\,t \quad \text{or} \quad x=a\cos\{\sqrt{(\mu)}\,t\} \qquad ..(v)$$

The equations (i), (iii) and (v) give the acceleration, velocity and the position of particle at time t.

Nature of Simple Harmonic Motion.

At the point A, $x=a$ and therefore from (i) acceleration $=-\mu a$, which is numerically maximum, as at all other points $x<a$. The negative sign only shows that the acceleration is directed towards O i.e. in the direction in which x decreases as t increases.

At the point O, $x=o$ and therefore from (i) acceleration $=0$ and from (iii) velocity $=-\sqrt{(\mu)}a$, which is numerically maximum.

Hence we find that at O though the acceleration is zero but the velocity being not zero the particle will not stop. The velocity being negative the particle will move beyond O in the direction AO.

Beyond O. As the particle moves beyond O, the velocity goes on decreasing whereas acceleration increases, since x increases numerically and is in the direction of O. Therefore the particle will move under *retardation* and at a stage it will come to rest. Let this be A'. Then from (iii) at A', we get

$$0=-\sqrt{(\mu)}\,\sqrt{(a^2-x^2)} \quad \text{or} \quad x=\pm a.$$

But $x=+a$ corresponds to the point A, hence at A' we have

$$x=-a \text{ i.e. } OA'=a \qquad \therefore \quad OA=a=OA'.$$

At the point A'. $x=-a$ and therefore from (i) acceleration [illegible] in the direc-
[illegible] erefore

Again at O, the acceleration would become zero and velocity will be positive, so the particle will cross O and move towards A.

In this way the particle would go on oscillating between A and A', which are two positions of momentary rest.

Amplitude: The distance AO or OA' i.e. the distance of the centre from one of the positions of momentary rest is called the Amplitude.

Also from (v) the time taken in moving from A to O is given

by $\quad 0=a\cos\{\sqrt{(\mu)}\,t\}$ or $\cos\{\sqrt{(\mu)}\,t\}=0$

or $\quad \sqrt{(\mu)}\,t=\frac{1}{2}\pi$ or $t=\pi/2\sqrt{\mu}$.

$\therefore$ Total time taken in moving from A to A' and back to A $=4\times$ time taken in moving from A to O

$=4(\pi/2\sqrt{\mu})=2\pi/\sqrt{\mu}$ which is called the **time period or period.**

$\therefore$ **Period of motion** $=\dfrac{2\pi}{\sqrt{\mu}}$, which is independent of the amplitude a.

(Agra 86; Avddh 90, 88; Gorakhpur 89, 88; Kanpur 86; Meerut 86P)

If T denotes the time period, then

$$T=2\pi/\sqrt{\mu} \text{ or } T^2=4\pi^2/\mu \text{ or } T^2\mu=4\pi^2=\text{constant}.$$

Periodic Motion. *(Gorakhpur 86)*

If a point moves such that after a certain fixed interval of time it occupies the same position and moves with the same velocity in the same direction, then it is said to have periodic motion.

From the above definition, it is evident that S H M is periodic and its period has been proved above to be $2\pi/\sqrt{\mu}$

Frequency : It is the number of complete oscillations in one second, so that if T denotes the time period and n the frequency, then

$$n.T=1 \quad \text{or} \quad n=\frac{1}{T}=\frac{\sqrt{\mu}}{2\pi}$$ *(Bundlkhand 86)*

Phase and Epoch. From (i), we have $(d^2x/dt^2)+\mu x=0$.

The most general solution of this equation is

$x=a\cos\{\sqrt{(\mu)}\,t+\alpha\}$ **(See Author's Differential Equations)**

The quantity α is called the epoch and the angle $\{\sqrt{(\mu)}\,t+\alpha\}$ is called the argument.

Phase of the motion is the time that has elapsed since the particle was as its maximum distance in the positive direction.

(Bundelkhand 86)

Now let the particle be at its maximum distance after time t_1 (say) then $\cos[\sqrt{(\mu)}\,t_1+\alpha]$ is maximum

or $\sqrt{(\mu)}\,t_1+\alpha=0$, since cos 0 is maximum, or $t_1=-\alpha/\sqrt{(\mu)}$.

Hence the phase at time $t=t-t_1=t+\dfrac{\alpha}{\sqrt{(\mu)}}$.

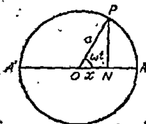
*§ 2. Geometrical Representation of S H.M.

If a particle describes a circle with constant angular velocity, the foot of the perpendicular from the particle on a diameter moves with S.H M.

(Avadh 86)

Let a point P move along the circumference of a circle with centre O, with uniform speed. Let P describe equal arcs in equal times. Let the rate of description of $\angle POA$ be ω.

Let N be the foot of the perpendicular drawn from P on the diameter AOA'.



(Fig. 4)

Let the particle start from A and reach P after time t.

Then $\angle AOP = \omega t$.

If $ON = x$ and $OP = a$, then from $\angle OPN$, we get ..(i)

$$x = a \cos(\omega t) \quad ..(ii)$$

whence $dx/dt = -a\omega \sin(\omega t)$ ..(iii)

and $d^2x/dt^2 = -a\omega^2 \cos(\omega t) = -\omega^2 x$.

The velocity and acceleration of N is given by (ii) and (iii). Also as P moves along the circumference of the circle, N oscillates from A to A' and back to A. Thus the motion of N is periodic. And the periodic time of N is the time taken by P in moving once along the whole circumference of the circle *i e.* the time taken by P to turn through an angle 2π with the uniform rate of ω

$\therefore$ Periodic time of $N = 2\pi/\omega$.

***§ 3. Compounding of two simple harmonic motions of the same period along the same straight line.**

Let the most general displacement of this kind be given by

$$a \cos\{(\sqrt{\mu}t)+\alpha\} \text{ and } b \cos\{(\sqrt{\mu}t)+\beta\}$$

.. See § 1 (Phase and Epoch Page 35 of this chapter)

Then compounding these two S H M.'s we have

$$x = a \cos\{(\sqrt{\mu}t)+\alpha\} + b \cos\{(\sqrt{\mu}t)+\beta\}$$

$$= a\{\cos(\sqrt{\mu}t)\cos\alpha - \sin(\sqrt{\mu}t)\sin\alpha\}$$
$$+ b\{\cos(\sqrt{\mu}t)\cos\beta - \sin(\sqrt{\mu}t)\sin\beta\}$$

$$= \cos(\sqrt{\mu}t)\{a\cos\alpha + b\cos\beta\} - \sin(\sqrt{\mu}t)\{a\sin\alpha + b\sin\beta\}$$

$$= \cos(\sqrt{\mu}t)\{A\cos E\} - \sin(\sqrt{\mu}t)\{A\sin E\}, \quad ..(i)$$

where $\left.\begin{array}{l} A\cos E = a\cos\alpha + b\cos\beta \\ A\sin E = a\sin\alpha + b\sin\beta \end{array}\right\}$ ..(ii)

Squaring and adding (ii) we get

$$A^2 = (a\cos\alpha + b\cos\beta)^2 + (a\sin\alpha + b\sin\beta)^2 \quad ...(iii)$$

or $A = \sqrt{\{a^2 + b^2 + 2ab\cos(\alpha - \beta)\}}$

From (ii) dividing we get $\tan E = \dfrac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}$ ...(iv)

Also from (i) we get $x = A\cos.\{(\sqrt{\mu}t)+E\}$, which shows that the resulting motion is also S H.M. of the same period and A and E (which give amplitude and epoch of the resulting motion) are given by (iii) and (iv).

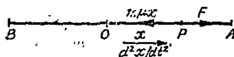
§ 4. Disturbed Harmonic Motion.

Definition : *If a particle moving in a straight line is under the action of another force (not involving the velocity) besides a force to a fixed point and proportional to its distance from this fixed point then the motion is said to be* **Disturbed Harmonic Motion.**

There are following two cases of such a motion :—

Case I. When the disturbing force is constant.

Let f be the contsant acceleration produced by this constant disturbing force F, then $F=mf$ where m is the mass of the particle.



(Fig. 15)

Then the equation of motion of the particle, at time t when it is at a distance x from the fixed point O, is

$m.(d^2x/dt^2)=F-m\mu x,$ where μ is constant. (Note)

or $m.(d^2x/dt^2)=mf-m\mu x;$ or $d^2x/dt^2=-\mu\,[x-(f/\mu)]$ ...(i)

If we now substitute X for $x-(f/\mu)$ *i.e.* if we shift the centre of S. H. M. through a distance f/μ in the direction of the constant force F, the equation (i) reduces to $d^2X/dt^2=-\mu X,$ ...(ii)
which the standard equation of S.H M.

From (ii) we get $X=b\sin(\mu t+\beta),$ ...(iii)
where b and β are arbitrary constants.

$\therefore$ The solution of (i) is $x-(f/\mu)=b\sin(\mu t+\beta),$
putting $X=x-(f/\mu)$

or $x=(f/\mu)+b\sin(\mu t+\beta)$ ...(iv)

This shows that the effect of this constant force is simply to shift the centre of S. H. M. (or the origin) through a definite distance f/μ in the direction of the constant force F.

i.e. the centre of S. H. M. is the point of equilibrium on the point where the constant force F is balanced by the variable force μx.

Note. We shall apply this (in articles to follow) while discussing problems of masses suspended by elastic strings (or springs) where the constant force mg is balanced by the tension in the elastic string (or spring) at the equilibrium position.

Case II When the disturbing force is periodic.

If the disturbing force is periodic force proportional to $\cos pt$, say, then the equation of motion is

$m.(d^2x/dt^2)=-m\mu x+mf\cos(pt)$ (Note)

or $(d^2x/dt^2)+\mu x=f\cos pt.$ ...(v)

Its auxiliary equation is $M^2+\mu=0$ which gives complementary function $=a\sin(\sqrt{\mu}t)+b\cos(\sqrt{\mu}t),$ ..(vi)
where a and b are arbitrary constants.

And particular integral $=\dfrac{1}{D^2+\mu}f\cos pt$

$=[f/(\mu-p^2)]\cos pt,$ if $\mu\neq p^2$

or $-(f/2p)\,t\cos pt,$ if $\mu=p^2$

[See Author's Differential Equations]

$\therefore$ The complete solution of (v) is

$$x=a\sin(\sqrt{\mu}t)+b\cos(\sqrt{\mu}t)+[f/(\mu-p^2)]\cos pt,\ \mu\neq p^2 \quad \ldots(vii)$$

or $$x=a\sin(\sqrt{\mu}t)+b\cos(\sqrt{\mu}t)-(f/2p)\,t\cos pt,\ \text{if } \mu=p^2 \quad \ldots(viii)$$

$\therefore$ When $\mu\neq p^2$, we find from the solution (vii) above that complementary function as well as particular integral are periodic. The complementary function can also be written as $a\sin(\sqrt{\mu}t+\beta)$, where β is constant and this function is periodic, its period being $2\pi/\sqrt{\mu}$. The particular integral is also a periodic function of period $2\pi/p$ *i.e.* of the same period as that of the disturbing force Thus the resulting motion is nothing but the combination of two S. H. M.'s (See § 3 Page 36 of this chapter).

When $\mu=p^2$, from the solution (viii) given above we find that the amplitude of the forced oscillation increases with time as is evident from the particular integral.

When μ is nearly equal to p^2 (*i e.* when the periods of forced and free oscillations are nearly equal) we observe from the solution (vii) above that the amplitude $f/(\mu-p^2)$ of the forced oscillation is very large and such a motion is familair in electromagnetism or sound and is known as resonance

§ 5. Damped Harmonic Oscillation (or Resisted S H. M)

Definition . *If a particle moves in a straight line under the action of a force towards a fixed point in the line, proportional to its distance from the fixed point and the motion is resisted by a force proportional to the velocity, then the particle is said to execute Damped Harmonic Oscillation*

The equation of motion in this case is

$$m\ddot{x}=-m\mu x-2mk\dot{x}, \text{ where } k \text{ and } \mu \text{ are constants.} \quad \ldots(i)$$

or $$\ddot{x}+2k\dot{x}+\mu x=0.$$

Its auxiliary equation is $m^2+2km+\mu=0$ ...(ii)

which gives $m=\frac{1}{2}[-2k\pm\sqrt{(4k^2-4\mu)}]=-k\pm\sqrt{(k^2-\mu)}$.

If $k\neq\sqrt{\mu}$, then solution of (i) can be written as

$$x=Ae^{\{-k+\sqrt{(k^2-\mu)}\}t}+Be^{\{-k-\sqrt{(k^2-\mu)}\}t}$$

or $$x=\{Ae^{\sqrt{(k^2-\mu)}\,t}+Be^{-\sqrt{(k^2-\mu)}\,t}\}e^{-kt} \quad \ldots(iii)$$

If $k=\sqrt{\mu}$, then the solution of (i) can be written as

$$x=(At+B)e^{-kt} \quad \ldots(iv)$$

In (iii) and (iv) A and B are arbitrary constants which can be determined from the initial conditions. Let the initial conditions be

$$x=a,\ t=0 \text{ and } \dot{x}=0. \quad \ldots(v)$$

Now three cases arises which are as follows :

Case I. If $k=\sqrt{\mu}$

From (iv) we get $x=(At+B)e^{-kt}$ and $\dot{x}=Ae^{-kt}-k(At+B)e^{-kt}$

Applyining initial conditions from (v) we get

$a=B,\ 0=A-k(B)$ or $A=k\,B=ka$

$\therefore$ From (iv) we get $x = a\,(1 + k/t)\,e^{-kt}$ ...(vi)

As $t \to \infty$, from (vi) we observe $x \to 0$ without being negative at any stage.

$\therefore$ The particle moves towards the equilibrim position and reaches the same after infinite time. This motion is not periodic and is known as critical damping.

Case. II. If $k > \sqrt{\mu}$.

Let $k^2 - \mu = \lambda^2$, where λ is real.

Then form (iii) we get

$$x = [Ae^{\lambda t} + Be^{-\lambda t}]e^{-kt} = Ae^{(\lambda - k)t} + Be^{-(\lambda + k)t}$$

$$\therefore \quad \dot{x} = A(\lambda - k)e^{(\lambda - k)t} - B\,(\lambda + k)\,e^{-(\lambda + k)t}.$$

Applying initial conditions from (v) we get

$a = A + B$ and $0 = A(\lambda - k) - B(\lambda + k)$ or $\lambda(A - B) = k(A + B) = ka$

or $A + B = a$, $A - B = (ka/\lambda)$ whence $A = \frac{1}{2}(1 + k/\lambda)$, $B = \frac{1}{2}a\,(1 - k/\lambda)$

$$\therefore \quad x = \tfrac{1}{2}a(1 + k/\lambda)e^{(\lambda - k)t} + \tfrac{1}{2}a(1 - k/\lambda)e^{-(\lambda + k)t}$$

$$= \tfrac{1}{2}(a/\lambda)e^{-kt}[(\lambda + k)e^{\lambda t} + (\lambda - k)e^{-\lambda t}]$$

$$= (a/\lambda)e^{-kt}[\lambda \cosh \lambda t + k \sinh \lambda t].$$

As $t \to \infty$, $x \to \infty$ without being negative at any stage.

Hence the particle moves towards the equilibum position but at a rate slower than in case I above. This is called **over damping.**

Case III. If $k < \sqrt{\mu}$.

Let $k^2 - \mu = -\lambda^2$, where λ is real.

Then from (iii) we get

$$x = (Ae^{i\lambda t} - Be^{-i\lambda t})e^{-kt} = [A_1 \cos \lambda t + B_1 \sin \lambda t]\,e^{-kt}$$

$$\therefore \quad \dot{x} = [\lambda A_1 \cos \lambda t + \lambda B_1 \cos \lambda t]\,e^{-kt}$$

$$- k\,[A_1 \cos \lambda t + B_1 \sin \lambda t]\,e^{-kt}$$

Applying initial conditions from (v) we get

$a = A_1$ and $0 = \lambda B_1 - kA_1$ or $B_1 = (k/\lambda)\,A_1 = ka/\lambda$

$$\therefore \quad x = a\,[-\cos \lambda t + (k/\lambda) \sin \lambda t]\,e^{-kt}$$

$$= (a/\lambda)\,[\lambda \cos \lambda t + k \sin \lambda t]\,e^{-kt}$$

This motion is periodic and represents harmonic oscillations [illegible] under damping.

Solved Examples on Harmonic Motion.

Ex. 1. The maximum velocity of a body moving with S.H.M. is 2 unit/sec. and its period is $\frac{1}{5}$ sec. What is its amplitude ?

Sol. Let a be the required amplitude.

The maximum velocity $= a\sqrt{\mu} = 2$ units/sec.

or $\mu a^2 = 4$ ...(i)

Also time period $= \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{1}{5}$ sec. (given)

or $\sqrt{\mu}=10\pi$ or $\mu=100\pi^2$

$\therefore$ From (i) we get $100\pi^2a^2=4$ or $a^2=1/(25\pi^2)$

or $a=1/(5\pi)$ **Ans.**

Ex. 2 (a). A particle starting from rest and moving with a simple harmonic motion of period 18 seconds, travels 10 inches in 3 seconds. Find the amplitude, the max. velocity and the velocity at the end of 3 seconds. (one foot=12 inches)

Sol. If μ be the intensity and a the amplitude of S. H. M. then the time period and the distance of the particle after time t are given by $T=2\pi/\sqrt{(\mu)}$ ...(i)

and $x=a\cos\{\sqrt{(\mu)}\,t\}$ ...(ii)

From (i) we get $\dfrac{2\pi}{\sqrt{(\mu)}}=18$ or $\sqrt{\mu}=\dfrac{2\pi}{18}=\dfrac{1}{9}\pi$ ...(iii)

It is given that the particle travels 10" or 10/12 feet in 3 sec.

$\therefore$ From (iii) we get $\dfrac{10}{12}=a\cos\left\{\dfrac{1}{9}\pi.3\right\}$ (Note)

or $\frac{5}{6}=a\cos(\frac{1}{3}\pi)$ or $\frac{5}{6}=a(\frac{1}{2})$ or $a=5/3$ ...(iv) **Ans.**

$\therefore$ Amplitude='a'$=\frac{5}{3}$ ft.=1 ft. 8 in.

Also max. velocity="$\sqrt{(\mu)}\,a$"$=\dfrac{1}{9}\pi.\dfrac{5}{3}=\dfrac{5\pi}{27}$ ft./sec. **Ans.**

Also from (ii) on differentiating we get

$$dx/dt=-a\sqrt{\mu}\sin\{\sqrt{(\mu)}\,t\}$$

$\therefore$ At the end of 3 seconds, the velocity

$=-a\sqrt{\mu}\sin\{\sqrt{(\mu)}.3\}$

$=-(\frac{5}{3}).(\frac{1}{9}\pi)\sin\{\frac{1}{9}\pi.3\}$, from (iii) and (iv)

$=-\dfrac{5}{27}\pi\sin\left(\dfrac{\pi}{3}\right)=-\dfrac{5}{27}\cdot\dfrac{1}{2}\sqrt{3}\,\pi$ ft./sec. **Ans.**

Ex. 2 (b). At what distance from the centre will the velocity be half of the maximum?

Sol. If μ be the intensity and a the amplitude of S. H. M., then velocity of the particle at a distance x from the centre is given by

$$\boxed{v^2=\mu(a^2-x^2)} \quad ..(i)$$

Also when $x=0$ max. velocity$=\sqrt{(\mu)}\,a$ ..see § 1 Page 33.

$\therefore$ If x_1 be the required distance at which the velocity is $\frac{1}{2}$ the max. velocity *i.e.* $\frac{1}{2}a\sqrt{\mu}$, then from (i)

$$(\tfrac{1}{2}a\sqrt{\mu})^2=\mu(a^2-x_1^2)$$

or $a^2=4(a_1^2-x_1^2)$ or $x_1^2=\frac{3}{4}a^2$ or $x_1=\pm\frac{1}{2}a\sqrt{3}$. **Ans.**

Ex. 3. Show that the particle executing S. H. M. requires $\frac{1}{6}$th of its period to move from the position of maximum displacement to one in which the displacement is half the amplitude.

Sol. The equation of motion of the particle is $\ddot{x}=-\mu x$

Then its period$=2\pi/\sqrt{\mu}=T$ (say) ...(i)

and $(\dot{x})^2=\mu(a^2-x^2)$, where a is the amplitude

or $\dot{x}=-\sqrt{\mu}\sqrt{(a^2-x^2)}$, when particle is moving towards centre

or $$dt=-\frac{dx}{\sqrt{\mu}\sqrt{(a^2-x^2)}}$$

$\therefore$ The time T_1 taken by the particle in moving from the position of the max. displacement *i.e.* $x=a$ to the position when the displacement is half the amplitude *i.e.* $x=\frac{1}{2}a$ is given by

$$T_1=-\frac{1}{\sqrt{\mu}}\int_{x=a}^{a/2}\frac{dx}{\sqrt{(a^2-x^2)}}=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\left(\frac{x}{a}\right)\right]_a^{a/2}$$

$$=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}(\tfrac{1}{2})-\cos^{-1}(1)\right]=\frac{1}{\sqrt{\mu}}\left[\frac{\pi}{3}-0\right]=\frac{\pi}{3\sqrt{\mu}}$$

$=\frac{1}{6}T$, from (i) Hence proved.

Ex. 4. A horizontal shelf is moved up and down with S.H.M. of period 1 sec. What is the amplitude admissible in order that a weight placed on the shelf may not be jerked off ?

Sol. Let μ be the intensity and a the amplitude of S. H. M. then the equation of motion is $d^2x/dt^2=-\mu x$,

whence the max acceleration$=\mu a$

And period$=2\pi/\sqrt{\mu}=1$ sec (given)

or $\mu=4\pi^2$ .. (i)

The weight placed on the shelf will be jerked off when the max acceleration of S. H. M is greater than g (Note)

$\therefore$ If it is not jerked off, the max. acceleration of S H. M. should be at the most equal to g.

i.e. $\mu a=g$ or $4\pi^2 a=g$, from (i)

or $a=g/4\pi^2$. Ans.

Ex. 5. Show that if the displacement of a particle moving in a straight line is expressed by the equation $x=a\cos nt+b\sin nt$, it describes a S. H. M. whose amplitude is $\sqrt{(a^2-b^2)}$ and period is $2\pi/n$.

Sol. Given $x=a\cos nt+b\sin nt$ ...(i)

Therefore $dx/dt=-an\sin nt+bn\cos nt$

and $d^2x/dt^2=-an^2\cos nt-bn^2\sin nt$

$=-n^2(a\cos nt+b\sin nt)$

or $d^2x/dt^2=-n^2x$, from (i) ...(ii)

The equation is of the standard form $d^2x/dt^2=-\mu x$ hence represents S.H.M for which 'μ'$=n^2$.

$\therefore$ Time period $=\dfrac{\text{"}2\pi\text{"}}{\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{(n^2)}}=\dfrac{2\pi}{n}$

Also amplitude of the motion is the value of x when dx/dt is zero. (Remember)

$\therefore$ Equating dx/dt to zero we get $-an\sin nt+bn\cos nt=0$

or $\tan nt=b/a$, which gives

$$\sin nt=\frac{b}{\sqrt{(a^2+b^2)}} \text{ and } \cos nt=\frac{a}{\sqrt{(a^2+b^2)}}$$

$\therefore$ From (i) we have the amplitude

$$=a\left[\frac{a}{\sqrt{(a^2+b^2)}}\right]+b\left[\frac{b}{\sqrt{(a^2+b^2)}}\right]=\frac{a^2+b^2}{\sqrt{(a^2+b^2)}}$$

$=\sqrt{(a^2+b^2)}$. Hence proved.

***Ex. 6 (a).** **The speed v is a point P which moves in a straight line is given by the relation $v^2=a-bx^2$, where x is the distance of the point P from a fixed point on the path, a and b being constants. Show that the motion of P is simple harmonic and determine its amplitude and period.**

Sol. Given $v^2=a-bx^2$.

Differentiating both sides with respect to x, we get,

$$2v\frac{dv}{dx}=-2bx \quad \text{or} \quad \frac{d^2x}{dt^2}=-bx, \quad \because v\frac{dv}{dx}=\frac{d^2x}{dt^2}$$

This is of the standard form $d^2x/dt^2=-\mu x$,

hence represents S. H. M. and its period $=\dfrac{\text{"}2\pi\text{"}}{\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{b}}$.

Also from $v^2=a-bx^2$ we have $x=\sqrt{(a/b)}$ when $v=0$ and we know that the amplitude of the motion is the value of x when v is zero, provided acceleration is zero at $x=0$.

Hence amplitude of the motion $=\sqrt{(a/b)}$

***Ex. 6 (b).** **The speed v of the point P which moves in a line is given by the relation $v^2=a+2bx-cx^2$, where x is the distance of the point P from a fixed point on the path, and a, b, c are constants. Show that the motion is simple harmonic if c is positive and determine the period.**

Sol. Given $v^2=a+2bx-cx^2$

Differentiating both sides with respect to x, we get

$$2v\frac{dv}{dx}=2b-2cx \quad \text{or} \quad \frac{d^2x}{dt^2}=b-cx, \quad \because v\frac{dv}{dx}=\frac{d^2x}{dt^2}$$

or
$$\frac{d^2x}{dt^2}=-c\left(x-\frac{b}{c}\right)$$

Putting $x-b/c=y$, we get $d^2y/dt^2=-cy$.
which is of the form $d^2x/dt^2=-\mu x$, hence represents S H. M. and and its period $=\frac{"2\pi"}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{c}}$. Ans

Ex. 6 (c) The speed v of a particle moving along the axis OX is given by the relation $v^2=n^2(8ax-x^2-12a^2)$. Prove that the motion is simple harmonic, with amplitude $2a$ and that the time taken from $x=4a$ to $x=6a$ is $\pi/(2n)$. What is the periodic time ?

Sol Given $v^2=n^2(8ax-x^2-12a^2)$ ...(i)

Differentiating both sides with respect to x, we get

$2v\frac{dv}{dx}=n^2(8a-2x)$ or $\frac{d^2x}{dt^2}=n^2(4a-x)$, $\because$ $v\frac{dv}{dx}=\frac{d^2x}{dt^2}$

Putting $x-4a=y$ we get $\frac{d^2y}{dt^2}=-n^2y$,

which is of the form $d^2x/dt^2=-\mu x$, hence represents S. H. M. of period $=2\pi/\sqrt{\mu}=2\pi/n$ about $y=0$ *i.e.* $x=4a$. .. (ii)

Also when velocity $v=0$, from (i) we get

$0=n^2(8ax-x^2-12a^2)$ or $x^2-8ax+12a^2=0$

or $(x^2-8ax+16a^2)-4a^2=0$ or $(x-4a)^2-(2a)^2=0$

or $x-4a=\pm 2a$ or $x=4a\pm 2a$ or $x=6a, 2a$ ..(iii)

Now amplitude = The distance between the points where the acceleration is zero (*i.e.* the centre of oscillation) and the velocity is zero $=6a-4a$, from (ii) and (iii) (Note)

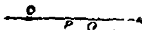
$=2a$. Hence proved.

Again the time taken in moving from $x=4a$ to $x=6a$ is the time taken in moving from the centre of oscillation to the extreme end the particle can reach (*i.e.* the point where the velocity is zero) *i.e.* $\frac{1}{4}$ of period *i.e.* $\frac{1}{4}\left(\frac{2\pi}{n}\right)$ *i.e.* $\frac{\pi}{2n}$.

Ex. 7 A particle is performing a S. H. M. of period T about a centre O and it passes through a point P where OP $=b$ with velocity v in the direction OP, prove that time which elapses before it returns to P is (T/π) $\tan^{-1}(vT/2\pi b)$.

(Avadh 88 ; Gorakhpur 90, 87)

Sol. $OP=b$ (given). Take a point Q on the line OP, such that $OQ=x$.



(Fig. 16)

Then velocity Q is given by

$dx/dt=\sqrt{\mu}\sqrt{(a^2-x^2)}$,

where $OA=a$ is the amplitude of the S H M. .. (i)

Now we are given that velocity $=v$ at P *i.e.* at $x=b$

$\therefore$ From (i) we get $v=\sqrt{\mu}\sqrt{(a^2-b^2)}$. ...(ii)

Also from (i) we get $\dfrac{dx}{\sqrt{(a^2-x^2)}}=\sqrt{(\mu)}\,dt$.

Integrating $\sin^{-1}(x/a)=\sqrt{(\mu)}\,t+C$, where C is an arbitrary constant.

Let $t=0$, when $x=0$, then $C=0$.

$\therefore$ $\sin^{-1}(x/a)=\sqrt{(\mu)}\,t$ or $x=a\sin[\sqrt{(\mu)}\,t]$ ...(iii)

Let t' be the time taken in moving from O to P, then from (iii), we have

$b=a\sin\{(\sqrt{\mu})\,t'\}$ or $t'=(1/\sqrt{\mu})\sin^{-1}(b/a)$ ..(iv)

Also time taken from O to $A=\frac{1}{4}$ (Period) $=\frac{1}{4}\left(\dfrac{2\pi}{\sqrt{\mu}}\right)=\dfrac{\pi}{2\sqrt{\mu}}$

$\therefore$ Time taken in moving from P to $A=\dfrac{\pi}{2\sqrt{\mu}}-t'$.

$\therefore$ Required time $=2\times$ time taken in moving from P to A

$$=2\left(\frac{\pi}{2\sqrt{\mu}}-t'\right)=2\left(\frac{\pi}{2\sqrt{\mu}}-\frac{1}{\sqrt{\mu}}\sin^{-1}\frac{b}{a}\right),\text{ from (iv)}$$

$=(2/\sqrt{\mu})\,[\frac{1}{2}\pi-\sin^{-1}(b/a)]$ ...(v)

Also time period $=2\pi/\sqrt{\mu}=T$ (given) ...(vi)

$\therefore$ $1/\sqrt{\mu}=T/(2\pi)$.

Also let $\frac{1}{2}\pi-\sin^{-1}(b/a)=\theta$

then $\sin^{-1}(b/a)=\frac{1}{2}\pi-\theta$ or $b/a=\sin(\frac{1}{2}\pi-\theta)=\cos\theta$

$\therefore$ $\tan\theta=\dfrac{\sqrt{(a^2-b^2)}}{b}=\dfrac{v}{b\sqrt{\mu}}$, from (ii)

or $\tan\theta=vT/(2\pi b)$, from (vi)

or $\theta=\tan^{-1}(vT/2\pi b)$.

Substituting the values of $1/\sqrt{\mu}$ and θ i.e. $\frac{1}{2}\pi-\sin^{-1}(b/a)$ in (v)

the required time $=\dfrac{2T}{2\pi}\tan^{-1}\left(\dfrac{vT}{2\pi b}\right)=\dfrac{T}{\pi}\tan^{-1}\left(\dfrac{vT}{2\pi b}\right)$

Ex. 8. A particle moves in a straight line and its velocity at a ... the origin is $k\sqrt{(a^2-x^2)}$, where k and a are consts. ... and find the amplitude

...(i)

or $(dx/dt)^2=k^2(a^2-x^2)$

Differentiating both sides with respect to t, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=k^2(-2x)\frac{dx}{dt}$$

or $d^2x/dt^2=-k^2x$, which being of the form of the standard equation of S. H. M. represents S. H. M.

Putting dx/dt equal to zero, we get from (i) the amplitude $x=a$.

Also time period $= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{k^2}} = \frac{2\pi}{k}$. **Ans.**

Ex. 9 (i). Show that in a S. H. M. of amplitude a and period T, the velocity v at a distance x form the centre is given by the relation $v^2T^2=4\pi^2(a^2-x^2)$

(ii) Find the new amplitude if velocity were doubled when the particle is at a distance $a/2$ from the centre, the period remaining unaltered.

Sol. (i). O is the centre and P is any point such that

$OP=x$, $OA=a$=amplitude

Then the equation of S.H.M. is $d^2x/dt^2=-\mu x$ ...(i)

(Fig. 17)

Multiplying both sides by $2dx/dt$ and integrating we get $(dx/dt)^2=-\mu a^2+C$, where C is constant of integration.

At A, $x=a$ and $dx/dt=0$

$\therefore$ $0=-\mu a^2+C$ or $C=\mu a^2$.

$\therefore$ $(dx/dt)^2=\mu(a^2-x^2)$

or $v^2=\mu(a^2-x^2)$, $\because$ velocity $dx/dt=v$ (given)

Also period $=2\pi/\sqrt{\mu}=T$ (given) $\therefore$ $\mu=4\pi^2/T^2$.

$\therefore$ $v^2=(4\pi^2/T^2)(a^2-x^2)$ or $v^2T^2=4\pi^2(a^2-x^2)$.

(ii) Also from $v^2=\mu(a^2-x^2)$ ...(ii)

we find that velocity v_1 at $x=\frac{1}{2}a$ is given by

$v_1^2=\mu(a^2-\frac{1}{4}a^2)$ or $v_1=\sqrt{(\frac{3}{4}\mu a^2)}$ ...(iii)

Now multiplying (i) by $2\,dx/dt$ and integrating we have as in part (i) $(dx/dt)^2=-\mu x^2+C$.

Here at $x=\frac{1}{2}a$ we are given that velocity $dx/dt=2v_1$

$\therefore$ $4v_1^2=-\frac{1}{4}\mu a^2+C$ or $4(\frac{3}{4}\mu a^2)=-\frac{1}{4}\mu a^2+C$, from (iii)

or $C=\frac{1}{4}.13\mu a^2$.

$\therefore$ $(dx/dt)^2=-\mu x^2+\frac{1}{4}(13\mu a^2)=\mu(\frac{1}{4}.13a^2-x^2)$.

Equating dx/dt to zero, we get $x^2=\frac{1}{4}.13a^2$ or $x=\frac{1}{2}a\sqrt{(13)}$.

$\therefore$ Required amplitude $=\frac{1}{2}a\sqrt{(13)}$. **Ans.**

****Ex. 10. A body is attached to one end of an inextensible string and the other end moves in a vertical line with S. H. M. of amplitude a, taking n complete oscillations per second show that the string will not remain tight during the motion unless $n^2<(g/4\pi^2a)$.**

(Meerut 86 P)

Sol. The equation of motion is $d^2x/dt^2=-\mu x$.

$\therefore$ If amplitude be a, then the maximum acceleration $=\mu a$. ...(i)

Also the body makes n complete oscillations per second, so periodic time $=1/n$ sec.

i e. $2\pi\sqrt{\mu}=1/n$ or $\mu=4\pi^2n^2$.

$\therefore$ from (i) maximum acceleration $=4\pi^2n^2a$.

Now the string will remain tight if the maximum acceleration in the downward direction is less than g. (Note)

i.e. $4\pi^2n^2a<g$ or $n^2<g/4\pi^2a$ Hence proved.

***Ex. 11 (a).** A particle is moving with S. H. M. and while making an excursion from one position of rest to the other, its distance from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3; prove that the time of a *complete* oscillation is $2\pi/\theta$, where $\theta=\cos^{-1}\{(x_1+x_3)/2x_2\}$.

Sol Let the particle be at distances x_1, x_2 and x_3 from the middle point of its path after $t, t+1$ and $t+2$ seconds. Then from $x=a\cos\{\sqrt{(\mu t)}\}$, where μ is the intensity and a the amplitude of S. H. M, we have

$$x_1=a\cos\{\sqrt{(\mu)}\,t\} \quad ...(i)$$
$$x_2=a\cos\{\sqrt{(\mu)}\,(t+1)\} \quad ...(ii)$$

and

$$x_3=a\cos\{\sqrt{(\mu)}\,(t+2)\} \quad ...(iii)$$

Adding (i) and (iii) we have

$$x_1+x_3=a[\cos\{\sqrt{(\mu)}\,t\}+\cos\{\sqrt{(\mu)}\,(t+2)\}]$$
$$=2a\cos\{\sqrt{(\mu)}\,(t+1)\}\cos\{\sqrt{(\mu)}\}$$
$$=2x_2\cos\{\sqrt{(\mu)}\}, \text{ from (ii)}$$

or $$[(x_1+x_3)/2x_2]=\cos\sqrt{(\mu)}$$

or $$\sqrt{(\mu)}=\cos^{-1}[(x_1+x_3)/2x_2]=\theta \text{ (given).}$$

$\therefore$ Required time $=$ "$2\pi/\sqrt{\mu}$" $=2\pi/\theta$,

where $\theta=\cos^{-1}\{(x_1+x_3)/2x_2\}$. Hence proved.

* **Ex 11 (b)** At the ends of three successive seconds the distances of a point moving with S. H. M. from its mean position measured in the same direction are 1, 5 and 5. Show that the period of a complete oscillation is $2\pi/\cos^{-1}(3/5)$

Sol. Let μ be the intensity and a be the amplitude of the S. H. M., and x be the distance of the particle from its mean position at time t, then we have $x=a\cos\{\sqrt{\mu}\,t\}$. ...(i)

Then according to the problem we have

$$1=a\cos\{\sqrt{\mu}\,(T-1)\}, \text{ ...(ii) }; \; 5=a\cos\{\sqrt{\mu}.T\} \quad ...(iii)$$

and

$$5=a\cos\{\sqrt{\mu}(T+1)\}, \quad ...(iv)$$

considering three successive seconds $T-1$, T and $T+1$.

Adding (ii) and (iv) we get

$$1+5=a[\cos\{\sqrt{\mu}\,(T-1)\}+\cos\{\sqrt{\mu}\,(T+1)\}]$$

or $6=a\,[2\cos(\sqrt{\mu}T)\cos(\sqrt{\mu})]$

or $3=a\cos(\sqrt{\mu}T)\cos(\sqrt{\mu})=5\cos(\sqrt{\mu})$, from (iii)

or $\cos\sqrt{\mu}=\frac{3}{5}$ or $\sqrt{\mu}=\cos^{-1}\frac{3}{5}$.

∴ The period of a complete oscillation

$=$"$2\pi/\sqrt{\mu}$"$=2\pi/\cos^{-1}(3/5)$. Hence proved.

Ex. 11 (c). At the end of three successive seconds, the distance of a point moving with S. H. M. from its mean position, measured in the same direction are 1, 3 and 4. Show that the period of complete oscillation is $2\pi/\cos^{-1}(5/6)$. *(Meerut 87)*

Sol. Do as Ex. 11 (b) above.

****Ex 11. (d). A particle moves with S. H. M. in a straight line. In the first second after starting from rest, it travels a distance a and in the next second it travels a distance b in the same direction. Prove that the amplitude of the motion is $2a^2/(3a-b)$.**

Sol Let μ be the intensity and c be the amplitude of S H M. then we know that $x=c\cos\{\sqrt{\mu}\,t\}$, ...(i)
where x is the distance of the particle from the middle point of its path (or the centre of force).

After one second from start the distance of the particle from the point of start $=a$ (given) and so the distance of the particle from the centre of force $=c-a$, where c is the amplitude. (Note)

∴ From (i) we get $c-a=c\cos\{\sqrt{(\mu)}.1\}$ ...(ii)

Similarly as in the next second it is given that the distance moved by the particle is b, so its distance from the point of start after 1+1 i.e. 2 seconds is $(a+b)$ and so its distance from the centre of force after 2 seconds is $c-(a+b)$.

∴ From (i) we get $c-(a+b)=c\cos\{\sqrt{(\mu)}.2\}$ ...(iii)

From (iii) we get

$$c-(a+b)=c\cos(2\sqrt{\mu})=c\,\{2\cos^2(\sqrt{\mu})-1\}$$
$$=c\,[2\,\{(c-a)/c\}^2-1],\text{ from (ii)}$$

or $c^2-(a+b)\,c=2\,(c-a)^2-c^2$ or $-(a+b)\,c=2a^2-4ac$

or $c\,(3a-b)=2a^2$ or $c=2a^2/(3a-b)$ Hence proved.

****Ex. 12 (a). A point is moving in a straight line with S.H.M. about a fixed point O of the line The point has a velocity v_1 when its displacement from O is x_1 and a velocity v_2 when its distance from O is x_2. Show that the period of the motion is**

$$2\pi\sqrt{[(x_1^2-x_2^2)/(v_2^2-v_1^2)]}.$$

(Avadh 86; Kanpur 89)

Sol If μ be the intensity and a the amplitude of S H. M. then we know $v^2=\mu\,(a^2-x^2)$ ...(i)

∴ According to the problem we get from (i)

$v_1^2=\mu\,(a^2-x_1^2)$ ...(ii) and $v_2^2=\mu\,(a^2-x_2^2)$ ...(iii)

Subtracting (ii) from (iii), we get

$$v_2^2-v_1^2=\mu(x_1^2-x_2^2) \text{ or } \mu=(v_2^2-v_1^2)/(x_1^2-x_2^2).$$

$\therefore$ Required period $=\dfrac{2\pi}{\sqrt{(\mu)}}=2\pi\sqrt{\left(\dfrac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}$. Hence proved.

Ex 12 (b). In S. H. M., if the velocities at distances b and c from the centre of force be respectively u and v, then prove that the frequency n of oscillation is given by $4\pi^2n^2(b^2-c^2)=v^2-u^2$.

Hint : As in Ex. 12 (a) above we can prove that

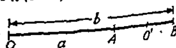
$$v^2-u^2=\mu(b^2-c^2) \quad ...(i)$$

Also frequency $n=\sqrt{\mu}/(2\pi)$ or $4\pi^2n^2=\mu$

or $4\pi^2n^2=(v^2-u^2)/(b^2-c^2)$ or $v^2-u^2=4\pi^2n^2(b^2-c^2)$

Ex 13. A body moving in a straight line OAB with S. H M has zero velocity when at the points A and B whose distances from O are a and b respectively and has velocity when half way between them. Show that the complete period is $\pi(b-a)/v$.

Sol Let O' be the middle point of AB, then O' is the centre of oscillation.



(Fig. 18).

$\therefore$ Amplitude of motion
$=\frac{1}{2}(AB)=\frac{1}{2}(b-a)$.

Also we know that the velocity at the middle point of the two extremities in the case of S. H. M. $=$"$\sqrt{(\mu)}\,a$", where μ is the intensity and a the amplitude of S H.M.

$\therefore$ In this case $v=\sqrt{(\mu)}\left(\dfrac{b-a}{2}\right)$ or $\sqrt{(\mu)}=\dfrac{2v}{(b-a)}$.

$\therefore$ Required point $=\dfrac{2\pi}{\sqrt{(\mu)}}=\dfrac{2\pi}{2v/(b-a)}=\dfrac{\pi(b-a)}{v}$.

*Ex. 14. A point executes S. H. M. such that in two of its positions the velocities are u, v and the corresponding accelerations α, β; show that the distance between the positions is $(v^2-u^2)/(\alpha+\beta)$.

And the amplitude of the motion is

$$\{(v^2-u^2)(\alpha^2v^2-\beta^2u^2)\}^{1/2}/(\beta^2-\alpha^2)$$

And find the time period. *(Avadh 89, 87)*

Sol. Let at distances x_1 and x_2 from the centre the velocities be u and v and accelerations be α and β respectively.

If μ be the intensity and a the amplitude of S. H. M. then we have $d^2x/dt^2=-\mu x$ ...(i) and $(dx/dt)^2=\mu(a^2-x^2)$ .. (ii)

$\therefore$ From (i), we have $\alpha=-\mu x_1$ ...(iii)

and $\beta=-\mu x_2$. ...(iv)

And from (ii), we get $u^2=\mu(a^2-x_1^2)$ ...(v)

and $v^2=\mu(a^2-x_2^2)$. ...(vi)

Adding (iii) and (iv), we get $\alpha+\beta=-\mu\,(x_1+x_2)$ ...(vii)

Also from (v) and (vi), we get $v^2-u^2=\mu\,(x_1^2-x_2^2)$. ...(viii)

Dividing (viii) by (vii), we get $\dfrac{v^2-u^2}{\alpha+\beta}=\dfrac{\mu\,(x_1^2-x_2^2)}{-\mu\,(x_1+x_2)}=(x_2-x_1)$

i.e. the required distance between the positions $=x_2-x_1$

$=(v^2-u^2)/(\alpha+\beta)$.

Again from (iii) and (iv), we get $\beta^2-\alpha^2=\mu^2\,(x_2^2-x_1^2)$. ...(ix)

From (viii) and (ix), we get $\dfrac{(v^2-u^2)}{(\beta^2-\alpha^2)}=\dfrac{\mu^2\,(x_1^2-x_2^2)}{\mu\,(x_1^2-x_2^2)}=-\dfrac{1}{\mu}$.. (x)

From (v), $a^2=(u^2/\mu)+x_1^2=(u^2/\mu)+(\alpha^2/\mu^2)$, from (iii)

$=\dfrac{u^2\,(v^2-u^2)}{(\beta^2-\alpha^2)}+\dfrac{\alpha^2\,(v^2-u^2)^2}{(\beta^2-\alpha^2)}$, from (x)

$=[(v^2-u^2)/(\beta^2-\alpha^2)^2]\,[-u^2\,(\beta^2-\alpha^2)+\alpha^2\,(v^2-u^2)]$

$=(v^2-u^2)\,(\alpha^2v^2-u^2\beta^2)/(\beta^2-\alpha^2)^2$

or amplitude $=a=\{(v^2-u^2)\,(v^2\alpha^2-u^2\beta^2)\}^{1/2}/(\beta^2-\alpha^2)$

And the time period $=\dfrac{2\pi}{\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{\{(\alpha^2-\beta^2)/(v^2-u^2)\}}}$

$=2\pi\sqrt{[(v^2-u^2)/(\alpha^2-\beta^2)]}$ **Ans.**

Ex 15. A particle of mass m is attracted towards a fixed point O with a force $m\mu$ times the distance from O. If initially it is projected towards O with a velocity v from a point distance c from O, *find the amplitude of its oscillation.*

Sol. The equation of motion is

$m.d^2x/dt^2=-m\mu x$ or $d^2x/dt^2=-\mu x$.

Multiplying both sides by 2 (dx/dt) and integrating, we have

$(dx/dt)^2=-\mu x^2+C$, where C is constant of integration

$\because$ At $x=c$, $dx/dt=v$ (given).

$\therefore$ $v^2=-\mu c^2+C$ or $C=v^2+\mu c^2$.

$\therefore$ $(dx/dt)^2=v^2+\mu\,(c^2-x^2)$.

Equating dx/dt to zero, we get $v^2+\mu\,(c^2-x^2)=0$

or $v^2+\mu c^2=\mu x^2$ or $x=[(v^2+\mu c^2)/\mu]^{1/2}$.

$\therefore$ Required amplitude $=\{(v^2/\mu)+c^2\}^{1/2}$. **Ans.**

Ex. 16 In a S. H. M. of period $2\pi/\omega$ if the initial displacement be x_0 and the initial velocity u_0 prove that

(a) Amplitude $=\sqrt{(x_0^2+(u_0^2/\omega^2))}$

(b) position at time t

$=\sqrt{(x_0^2+u_0^2/\omega^2)}\cos\omega\,[t-(1/\omega)\tan^{-1}(u_0/\omega x_0)]$

and (c) time to the position of the rest $=(1/\omega)\tan^{-1}(u_0/\omega x_0)$.

Sol. *The equation of motion is*

$d^2x/dt^2=-\mu x$, ...(i)

where μ is the intensity of S. H. M.

$\therefore$ Time period $=2\pi/\sqrt{\mu}=2\pi/\omega$ (given) or $\mu=\omega^2$.

Hence from (i), we get $d^2x/dt^2=-\omega^2 x$. ...(ii)

The general solution of (ii) is $x=A\cos(\omega t+B)$. ...(iii)

Initially $x=x_0$ an $t=0$, so $x_0=A\cos B$. ...(iv)

Differentiating (iii), we get $dx/dt=-A\omega\sin(\omega t+B)$

Initially $dx/dt=u_0$ and $t=0$, so $u_0=-A\omega\sin B$.

$\therefore$ From (iv) and (v), we get $-\omega\tan B=u_0/x_0$

or $$B=-\tan^{-1}(u_0/\omega x_0).$$ ...(vi)

(a) We know in (iii) A is amplitude and B is the phase.

From (iv) and (v), $x_0^2+(u_0^2/\omega^2)=A^2$. (Squaring and adding) ...(vii)

or $$A=\sqrt{\{x_0^2+(u_0^2/\omega^2)\}}.$$

(b) From (iii), position at time t

$=A\cos(\omega t+B)$

$=\sqrt{\left\{x_0^2+\frac{u_0^2}{\omega^2}\right\}}\cos\left[\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right]$, from (vi) and (vii)

$=\sqrt{\left\{x_0^2+\frac{u_0^2}{\omega^2}\right\}}\cos\omega\left[t-\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right]$

(c) And at the position of rest $dx/dt=0$

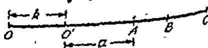
i.e., $-A\omega\sin(\omega t+B)=0$ or $\omega t+B=0$

or $t=-(B/\omega)=(1/\omega)\tan^{-1}(u_0/\omega x_0)$, from (vi). Hence proved.

****Ex. 7. If in a S. H. M. u, v, w be the velocities at distances a, b, c, from a fixed point on the straight line which is not the centre of the force; show that the period T is given by the equation**

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b)=\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$ *(Kanpur 88)*

Sol. Let O be the centre of force and O' the fixed point from which the distance of the points A, B and C are a, b and c respectively. The velocities at these points are given to be u, v and w. Let $OO'=k$. Then the distances of A, B and C from O are $a+k$, $b+k$ and $c+k$ respectively.



(Fig. 19)

The velocity V at a distance x from the centre of force is given by $$V^2=\mu(A^2-x^2),$$ ...(i)

where μ is the intensity and A the amplitude of S. H. M.

$\therefore$ At $x=a+k$, we get $u^2=\mu[A^2-(a+k)^2]$ from (i)

or $$u^2/\mu=A^2-a^2-k^2-2ak$$

or $$[(u^2/\mu)+a^2]+2ak+(k^2-A^2)=0. \quad ...(ii)$$

Similarly $$[(v^2/\mu)+b^2]+2bk+(k^2-A^2)=0. \quad ...(iii)$$

and $$[(w^2/\mu)+c^2]+2ck+(k^2-A^2)=0. \quad ...(iv)$$

Eliminating $2k$ and (k^2-A^2) from (ii), (iii) and (iv), we get

$$\begin{vmatrix} (u^2/\mu)+a^2 & a & 1 \\ (v^2/\mu)+b^2 & b & 1 \\ (w^2/\mu)+c^2 & c & 1 \end{vmatrix}=0$$

or $$\frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}+\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}=0$$

or $$\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}=-\mu\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

or $$\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}=\mu\,(b-c)\,(c-a)\,(a-b),$$
on evaluating the determinant on the right.

(See Author's Algebra or Matrices)

Also periodic time $T=\dfrac{2\pi}{\sqrt{\mu}}$ or $\mu=\dfrac{4\pi^2}{T^2}$.

$$\therefore \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}=\frac{4\pi^2}{T^2}\,(b-c)\,(c-a)\,(a-b)$$

Hence proved.

or $\Sigma u^2\,(b-c)=(4\pi^2/T^2)\,(b-c)\,(c-a)\,(a-b)$,

expanding the determinant.

or $4\pi^2\,(b-c)\,(c-a)\,(a-b)=T^2\,\Sigma\,[u^2\,(b-c)]$

Ex. 18. *A particle oscillates with S. H. M. of amplitude a and periodic time T. Find the expression of the velocity v in terms of (ii) a, T and x ; (ii) a, T and t and also prove that*

$$\int_0^T v^2\,dt=\frac{2\pi^2 a^2}{T}$$

Sol. If μ be the intensity of S.H.M., we have

$$v^2=\mu\,(a^2-x^2), \quad ...(i)$$

$$x=a\cos\sqrt{(\mu)}\,t \quad ...(ii).$$

and $$T=2\pi/\sqrt{\mu}. \quad ...(iii)$$

(i). From (iii), $\mu=4\pi^2/T^2$. ...(iv)

$\therefore$ From (i), $v^2=(4\pi^2/T^2)(a^2-x^2)$. Ans

(ii). From (i) and (ii),

$$v^2=\mu\,[a^2-a^2\cos^2\sqrt{(\mu)}\,t]=\mu a^2\sin^2\sqrt{(\mu)}\,t$$

$$=\frac{4\pi^2}{T^2}a^2\sin^2\left(\frac{2\pi t}{T}\right), \text{ from (iv)} \quad \text{Ans.}$$

(iii). $\int_0^T v^2\,dt$

$$=\int_0^T\frac{4\pi^2}{T^2}a^2\sin^2\left(\frac{2\pi t}{T}\right)dt=\frac{2a^2\pi^2}{T}\int_0^T 2\sin^2\left(\frac{2\pi t}{T}\right)dt$$

$$=\frac{2a^2\pi^2}{T^2}\int_0^T\left(1-\cos\frac{4\pi t}{T}\right)dt=\frac{2a^2\pi^2}{T}\left[t-\frac{T}{4\pi}\sin\frac{4\pi t}{T}\right]_0^T$$

$$=\frac{2a^2\pi^2}{T^2}\left[T-\frac{T}{4\pi}\sin 4\pi\right\}-\left\{0\right\}\Big]=\frac{2a^2\pi^2}{T}.$$

Ex. 19. A particle starts from rest under an acceleration k^2x directed towards a fixed point and *after time t* another particle starts from the same position under the same acceleration. *Show* that the particles will collide at time $[(\pi/k)+\frac{1}{2}t]$ after the start of the first particle provided $t<2\pi/k$

Sol. The equation of motion is $d^2x/dt^2=-k^2x$. .. (i)

$\therefore$ The time period $=$ "$(2\pi/\sqrt{\mu})$" $=2\pi/k$.

$\therefore$ The condition $t<2\pi/k$ i.e. $t<$ time period indicates that the second particle starts before the first has made one complete oscillation.

Let two particles collide after time t' of the start of 2nd particle.

$\therefore$ Before collision the first particle was in motion for a time $(t'+t)$.

From (i), $\quad x=a\cos kt$. .. (ii)

$\therefore$ If the particles collide then the distance moved by the first particle in time $(t'+t)$ = distance moved by 2nd particle in time t'

or $\quad a\cos k\,(t'+t)=a\cos kt'$.

$\therefore\ k\,(t'+t)=kt'$ or $k\,(t'+t)=2\pi-kt'$ (Note)

But $k\,(t'+t)=kt'$ gives $t=0$, which is against hypothesis

$\therefore\ k\,(t'+t)=2\pi-kt'$ or $2kt'=2\pi-kt$ or $t'=(\pi/k)-\frac{1}{2}t$.

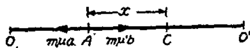
Required time $=t'+t=\frac{\pi}{k}-\frac{t}{2}+t=\frac{\pi}{k}+\frac{t}{2}$.

Hence proved.

****Ex. 20.** A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their

intensity being μ, μ' ; the particle is displaced slightly towards one of them. Show that the time of a small oscillation is $2\pi/\sqrt{(\mu+\mu')}$.
(*Agra 86; Rohilkhand 88*)

Sol. O and O' are the centres of force. Let A be the position of equilibrium of the particle of mass m (say). Let $OA=a$ and $O'A=b$. Then the forces acting on the particle at A are $m\mu a$ towards O and $m\mu' b$ towards O'.



(Fig 20)

∵ The particle is in equilibrium at A

∴ $m\mu a=m\mu' b$. ...(i)

Let the particle be slightly displaced towards O and let it go. Let P be any displaced position of the particle, such that $AP=x$. The forces acting on the particle at P are $m.\mu\,(a-x)$ towards O and $m\mu'\,(b+x)$ towards O'.

∴ The equation of motion at P is

$$m.\frac{d^2x}{dt^2}=m\mu\,(a-x)-m\mu'\,(b+x) \text{ or } \frac{d^2x}{dt^2}=(\mu a-\mu' b)-(\mu+\mu')\,x$$

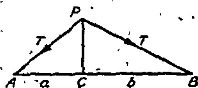
or $\quad d^2x/dt^2=-(\mu+\mu')\,x$, ∵ from (i) $\mu a=\mu' b$.

∴ Required time $=\dfrac{\text{"}2\pi\text{"}}{\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{(\mu+\mu')}}$.

***Ex. 21. A particle of mass m is attached to a light wire which is stretched tightly between two fixed points with a tension T. If a, b be the distances of the particle from the two ends, prove that the period of small transverse oscillation of mass m is**

$2\pi/\sqrt{[T\,(a+b)/mba]}$. (*Avadh 88; Gorakhpur 87*)

Sol. A and B are the given fixed points. The light wire is stretched between A and B. Let C be the point where the particle of mass m is attached. Then $AC=a$ and $BC=b$ (given).



(Fig. 21)

Let the particle be given a small transverse displacement *i.e.* a displacement at right angles to AB. Since the displacement is small therefore tension in any displaced position remains practically the same. Let P be the position of the particle at time t, such that $CP=x$.

Then $BP=\sqrt{(b^2+x^2)}$ and $AP=\sqrt{(a^2+x^2)}$.

Therefore the resultant force on the particle in the direction of $PC=T\cos\angle BPC+T\cos\angle APC$

$$=T.\frac{x}{\sqrt{(b^2+x^2)}}+T.\frac{x}{\sqrt{(a^2+x^2)}}, \text{ see figure}$$

$$=T.x\,[(b^2+x^2)^{-1/2}+(a^2+x^2)^{-1/2}]$$

$$=T.x\left[\frac{1}{b}\left(1+\frac{x^2}{b^2}\right)^{-1/2}+\frac{1}{a}\left(1+\frac{x^2}{a^2}\right)^{-1/2}\right]$$

$$=T.x\left[\frac{1}{b}\left(1-\frac{x^2}{2b^2}+\dots\right)+\frac{1}{a}\left(1-\frac{x^2}{2a^2}+\dots\right)\right]$$

$$=T.x\left(\frac{1}{b}+\frac{1}{a}\right), \text{ neglecting higher powers of } x.$$

$$=\frac{T(a+b)}{ab}\,x.$$

∴ The equation of motion is

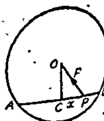
$$m.\left(\frac{d^2x}{dt^2}\right)=-\frac{T(a+b)}{ab}\,x \text{ or } \frac{d^2x}{dt^2}=-\left[\frac{T(a+b)}{mab}\right]x,$$

which is in the standard form $d^2x/dt^2=-\mu x$ of S H.M.

Hence the required period $=\frac{2\pi}{\sqrt{\mu}}=2\mu\Big/\sqrt{\left[\frac{T(a+b)}{mab}\right]}$

Ex. 22. Assuming that the gravity inside the earth varies as the distance from its centre, show that a train, starting from rest and moving under gravity only, would take the same time to traverse smooth straight airless tunnel between any two points of the earth's surface. Find the time.

Sol. Let O be the centre of the earth, AB the tunnel and C the mid-point of AB.



(Fig. 22)

Let at time t the position of the train be at P, such that $PC=x$. Let F be the force acting on the train at P due to gravity.

Then $F=\lambda\ OP$, acting towards O.

If m be the mass of the train, then the equation of motion is

$$m\,(d^2x/dt^2)=-F\cos\angle OPC \qquad \text{...(i)}$$

[illegible] of CP and the resol-

[illegible] OPC, ∵ $F=\lambda.OP$

$$=-\frac{\lambda}{m}\,OP.\frac{x}{OP}, \text{ see figure}$$

or $\qquad d^2x/dt^2=-(\lambda/m)\,x, \qquad$ ...(ii)

which is of the form of standard equation of S.H.M.

Hence the train moves in S H. M. between A and B with C, the middle point of AB, as origin.

∴ Time from B to $A=\frac{1}{2}$ (Time period)

$$=\tfrac{1}{2}\left(\frac{2\pi}{\sqrt{\mu}}\right)=\frac{\pi}{\sqrt{(\lambda/m)}}\text{, from (ii)}$$

$$=\pi\sqrt{(m/\lambda)}. \quad \text{...(iii)}$$

To find (m/λ) we use the condition that on the surface of the earth, the magnitude of acceleration $=g$

i.e. from (ii) $g=(\lambda/m)\,r$, where r is the radius of the earth

or $9{\cdot}8=(\lambda/m).(6400\times1000)$, since $g=9{\cdot}8$ m/sec^2 and $r=6400$ km.

or $\lambda/m=(9{\cdot}8)/(6400\times1000)$.

$\therefore$ From (iii), time from A to B

$=\pi\sqrt{[(6400\times1000)/(9{\cdot}8)]}$ sec. $=(3{\cdot}1416)\sqrt{(640000)}$ sec. nearly

$=(3{\cdot}1416)\times800$ sec. nearly $=42$ minutes nearly.

Exercises on S. H. M.

Ex. 1. A particle moves in a straight line with S.H.M. of periodic time 2 seconds. If it starts from rest at a distance of 13 cms. from the centre of the path, show that the greatest velocity and the velocity acquired by it when it has just described 8 cms. are respectively $?\pi$ and 1.12π cm./sec.

Ex. 2 A particle moves with S. H. M. if when at a distance of 3 and 4 cms. from the centre of its path its velocities are 8 and 6 cm /sec ; find its period, maximum velocity and acceleration.

Ans. π secs, 10 cm./sec, 20 cm/sec^2.

Ex. 3. A particle possesses two S. H. M.'s in perpendicular directions, having the same period but differing in phase and ampl[illegible]

I[illegible]
unifo[illegible]
midd[illegible]
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cot^2 $\frac{1}{2}$.
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Ex. 5. A particle of mass M is moving along the axis of x, [illegible] , it [illegible] 14 [illegible] 16 [illegible] the amplitude of the path.

Ex. 6. A particle is projected with velocity u directed away from a fixed point at a distance b from the point of projection. If the acceleration be μ times the distance from the fixed point and always directed towards it, find the amplitude of the S H.M.

Ex. 7. A point moving with S. H. M. has a period of oscillation π secs and its greatest acceleration is 5 m/sec^2. Find the amplitude and the velocity when the particle is at a distance 1 metre from the centre of oscillation Ans. $1\frac{1}{4}$ metre ; $1\frac{1}{2}$ m /sec.

Ex 8. A point moving with S. H. M. has a velocity of 4 m. per sec. when passing through the centre of its path and its period is π seconds. What is its velocity when it has described one metre from the position in which its velocity is zero ?

Ex. 9. Write the correct answer :

If $\frac{d^2x}{dt^2} = -\mu x$, the periodic time is

(i) $4\pi/\sqrt{\mu}$; (ii) $2\pi/\sqrt{\mu}$; (iii) $\pi/2\sqrt{\mu}$; (iv) none of these.

Ans. (ii).

Ex. 10. Show that in a S. H. M., the average speed and the average acceleration (in magnitude) are obtained by multiplying their maximum values by 0·637. (*Agra 87*)

Ex. 11. A point moving with S.H.M. has a period of oscillation of π seconds, and the greatest acceleration of 5 ft/sec². Find the amplitude and the velocity when the particle is at a distance of 1 foot from the centre of oscillation. (*Ranchi 86*)

§ 6. Hooke's Law.

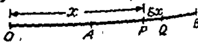
Statement. *The tension of an elastic string is proportional to its extension beyond its natural length.*

$\therefore$ If a be the natural length and l the extended length of an elastic string, then extension produced is $l-a$ and if T be tension in the string, then $T = \frac{\lambda}{a}(l-a)$, .. (A)

where the constant λ is called the modulus of elasticity.

Theorem *Prove that the work done against the tension in stretching a light elastic string is equal to the product of its extension and the mean of the initial and final tensions.*

Proof. Let l be the natural length of the string. Let it be extended from A to B such that $OA = a$ and $OB = b$. We are to find out the work done against the tension in stretching the string from A to B. Let λ be the modulus of elasticity.



(Fig. 23)

Let $OP = x$ and $PQ = \delta x$. This distance PQ being small, the tensions at P and Q are practically equal. Tension in the string, when it is stretched upto P

$= (\lambda/l)(x-l)$, acting in the direction PO.

$\therefore$ Work done in stretching from P to Q against the tension

$= (\lambda/l)(x-l)\,\delta x$. (*Note*)

$\therefore$ Work done against the tension in stretching from A to B

$$= \int_{x=a}^{b} (\lambda/l)(x-l)\,dx = (\lambda/l)\left[\frac{(x-l)^2}{2}\right]_a^b$$

$=(\lambda/2l)\,[(b-l)^2-(a-l)^2]$

$=(\lambda/2l)\,[\{(b-l)+(a-l)\}\,\{(b-l)-(a-l)\}]$

$=(\lambda/2l)\,(b-a)\,[(b-l)+(a-l)]$

$=(b-a).\tfrac{1}{2}\,[(\lambda/l)\,(b-l)+(\lambda/l)\,(a-l)]$. (i)

Also initial tension = tension at $A=(\lambda/l)\,(a-l)$

And final tension = tension at $B=(\lambda/l)\,(b-l)$

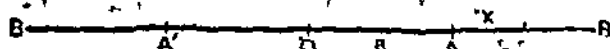
Extension produced $=AB=(b-a)$.

∴ From (i), work done against the tension in stretching from A to B = (extension produced) × (mean of initial and final tensions).

§ 7. Particle attached to an end of a horizontal elastic string.

One end of an elastic string whose modulus of elasticity is 'λ' and whose natural length is 'a' is tied to a fixed point on a smooth horizontal table, and the other end is tied to a mass 'm' lying on the table. The particle is pulled to a distance where the extension of the string becomes 'b' and then let go; describe the character of motion and find the period of one complete oscillation.

Let $OA=a$ be the natural length of the string. Let O be the fixed end and a particle of mass m be attached at A. Let this



(Fig. 24)

particle be pulled to B from A such that $AB=b$ and then let go. Let P be the displaced position of the particle at time t, such that $AP=x$.

Then tension in the string at $P=(\lambda/a)\,x$, acting towards O.

∴ The equation of motion is

$$m.\frac{d^2x}{dt^2}=-\frac{\lambda}{a}\,x \quad\text{or}\quad \frac{d^2x}{dt^2}=-\frac{\lambda}{am}\,x, \qquad ..(i)$$

which is of the form of the standard equation of S. H. M.

At B, the velocity being zero, AB is the amplitude of S.H.M.

∴ Time of describing BA

$=\tfrac{1}{4}$ (time period) $=\tfrac{1}{4}\,(2\pi/\sqrt{\mu})$, where $\mu=\lambda/am$, from (i)

$=\tfrac{1}{2}\pi\sqrt{(am/\lambda)}$.

From (i) we have $v\,\dfrac{dv}{dx}=-\dfrac{\lambda}{am}\,x$, since $\dfrac{d^2x}{dt^2}=v\,\dfrac{dv}{dx}$

Integrating with respect to x, we get

$$\tfrac{1}{2}v^2=-(\lambda/am).\tfrac{1}{2}x^2+c, \qquad ..(iii)$$

where c is constant of integration.

At B, velocity $v=0$ and $x=b$, so from (iii)

$0=-(\lambda/am).\tfrac{1}{2}b^2+c$ or $c=(\lambda/am).\tfrac{1}{2}b^2$

4. From (iii) we have $v^2=(\lambda/am)(b^2-x^2)$. ...(iv)

Since the particle is moving from B towards O *i.e.* x decreases as t increases, so from (iv) we get

$$\frac{dx}{dt}=v=-\sqrt{\left(\frac{\lambda}{am}\right)}\sqrt{b^2-x^2)} \quad ..(v)$$

Now when the particle reaches A, $x=0$ and from (i) and (v) we have acceleration $=\frac{d^2x}{dt^2}=0$ and velocity $=\frac{dx}{dt}=-\sqrt{\left(\frac{\lambda}{am}\right)}b$.

Hence at A, the tension being zero, there is no acceleration whereas the velocity is $-\sqrt{(\lambda/am)}\,b$, which being negative shows that the particle would move in the direction of OA with an uniform velocity $\sqrt{(\lambda/am)}\,b$ and not stop at O. With this uniform velocity the particle would cross O, the string remaining loose and will go beyond O. The string will be again taut when it reaches A', where $OA'=a$ *i.e.* natural length. Beyond A' it would move under retardation due to tension in the string which wou'd try to pull it towards O. Thus the particle, after some time would again come to momentary rest at B' (say) such that $A'B=AB=b$. At B' the velocity will be zero but acceleration will be $(\lambda b/am)$ towards O, so the particle would begin to move towards O. Thus the particle would oscillate between B and B'. (Correct the figure)

Between A and A' the string will remain loose and therefore there will be no tension in it so the particle moves with uniform velocity $\sqrt{(\lambda/am)}\,b$.

∴ The time of describing the distance AO

$$=\frac{OA}{\text{Velocity}}=\frac{a}{\sqrt{(\lambda/am)}\,b}=\sqrt{\left(\frac{am}{\lambda}\right)}\cdot\frac{a}{b} \quad ...(vi)$$

∴ Time of moving from B to O

$=$time of moving from B to A + time of moving from A to O

$=\frac{1}{2}\pi\sqrt{(am/\lambda)}+\sqrt{(am/\lambda)}(a/b)$, from (ii) and (vi).

∴ Required time of one complete oscillation

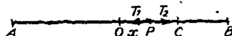
$=4$ (time of moving from B to O)

$$=4\left[\frac{\pi}{2}\sqrt{\left(\frac{am}{\lambda}\right)}+\sqrt{\left(\frac{am}{\lambda}\right)}\cdot\frac{a}{b}\right]=2\sqrt{\left(\frac{am}{\lambda}\right)}\left(\pi+\frac{2a}{b}\right).$$

Solved Examples on Horizontal Elastic Strings.

****Ex. 1. A light elastic string whose modulus of elasticity is λ is stretched to double its length ; and is tied to two fixed points distance $2a$ apart. A particle of mass m, tied to its middle point is displaced in the line of the string through a distance equal to half its distance from the fixed points and released. Find the time of a complete oscillation and the maximum velocity acquired in the subsequent motion.** *(Purvanchal 89)*

Sol. Let A and B be the fixed points, such that $AB=2a$ and let O be its middle point. Then $OA=a$. Let the particle of mass m be displaced from O to C, such that $OC=\frac{1}{2}a$. Let P be the displaced position of the particle at time t, such that $OP=x$. At P, the forces acting on the particle are tensions T_1 and T_2 in the parts PA and PB acting in the directions as shown in the figure.



(Fig. 25)

Natural length of OA or $OB=\frac{1}{2}a$

Then T_1 = tension in the string AP

$$=\frac{\lambda}{\frac{1}{2}a}\left[AP-\tfrac{1}{2}a\right]=\frac{\lambda}{\frac{1}{2}a}\left[(a+x)-\tfrac{1}{2}a\right]$$

And T_2 = tension in the string PB

$$=\frac{\lambda}{\frac{1}{2}a}\left[PB-\tfrac{1}{2}a\right]=\frac{\lambda}{\frac{1}{2}a}\left[(a-x)-\tfrac{1}{2}a\right]$$

Also acceleration at $P=d^2x/dt^2$, acting in the direction OP.

∴ The equation of motion is

$$m\cdot\frac{d^2x}{dt^2}=T_2-T_1=\frac{\lambda}{\frac{1}{2}a}\left[(a-x)-\tfrac{1}{2}a\right]-\frac{\lambda}{\frac{1}{2}a}\left[(a+x)-\tfrac{1}{2}a\right]$$

$$=(2\lambda/a)\left[a-x-\tfrac{1}{2}a-a-x+\tfrac{1}{2}a\right]$$

or
$$\frac{d^2x}{dt^2}=-\frac{4\lambda}{am}x, \qquad \ldots(i)$$

which shows that the motion is simple harmonic.

∴ Time of one complete oscillation $=$ "$\frac{2\pi}{\sqrt{\mu}}$" $=\frac{2\pi}{\sqrt{(4\lambda/am)}}$

$=\pi\sqrt{(am/\lambda)}$ **Ans.**

Also the amplitude of S.H.M. $=OC=\frac{1}{2}a$

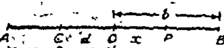
(∵ amplitude is the value of x, when $dx/dt=0$)

∴ Maximum velocity acquired $=$ "$\sqrt{\mu}\times$ amplitude"

$=\sqrt{(4\lambda/am)}\,\frac{1}{2}a=\sqrt{(\lambda a/m)}$. **Ans.**

Ex. 2. An elastic string of natural length $(a+b)$ where $a>b$ and modulus of elasticity λ has a particle of mass m attached to it to a distance a from one end which is fixed to a point A of a smooth horizontal plane. The other end of the string is fixed to a point B so that the string ... at B and ... a distance ...

Sol. Let O be the point where the particle of mass m is attached such that $OA=a$ and $OB=b$. When the particle is held at B, the portion



(Fig. 26)

AO of the string is stretched whereas the portion OB is slack. When the particle is released from B the particle moves towards O due to the tension in the portion AO which is stretched upto B. There will be no tension in the portion BO of the string till the particle reaches O.

Let particle be at P, such that $OP=x$ and lies between O and B after time t of its release from B. At P the only force acting on the particle is the tension in the string $=(\lambda/a)(x)$, acting towards O. Also the acceleration at P is d^2x/dt^2 acting in the direction of x increasing i.e. OP.

∴ The equation of motion is

$$m.\frac{d^2x}{dt^2}=-\frac{\lambda}{a}(x) \text{ or } \frac{d^2x}{dt^2}=-\frac{\lambda}{am}(x), \quad ...(i)$$

which is of the form of standard equation of S.H.M.

∴ Time from B to $O=\frac{1}{4}$ (Periodic time)

$$=\frac{1}{4}\left[\frac{2\pi}{\sqrt{(\lambda/am)}}\right]=\frac{\pi}{2}\sqrt{\left(\frac{am}{\lambda}\right)} \quad ...(ii)$$

From (i) multiplying both sides by $2(dx/dt)$ and integrating, we get

$(dx/dt)^2=-(\lambda/am)(x^2)+C$, where C is constant.

At B, $\frac{dx}{dt}=0$ and $x=b$, so $0=-\frac{\lambda}{am}(b^2)+C$ or $C=\frac{\lambda b^2}{am}$

∴ $(dx/dt)^2=(\lambda/am)(b^2-x^2)$

∴ velocity of the particle when it reaches O i.e. at $x=0$

$=\sqrt{(\lambda/am)}\, b$

When the particle crosses O and moves towards the left of O, the string OA becomes slack and OB becomes taut. So as before, the equation of motion becomes $d^2x/dt^2=-(\lambda/bm)(x)$, ...(iii) changing a by b in (i).

This is also of the form of standard equation of S. H. M. Hence the particle will come to momentary rest at C (say) such that $OC=d$ (say).

∴ Time from O to $C=\frac{1}{4}$ (Periodic time) $=\frac{1}{4}[2\pi/\sqrt{(\lambda/bm)}]$

$=\frac{1}{2}\pi\sqrt{(bm/\lambda)}$ ...(iv)

Also from (iii) we can find as before

$(dx/dt)^2=-(\lambda x^2/bm)+k$, where k is constant of integration.

At O, $x=0$ and $\frac{dx}{dt}=\sqrt{(\lambda/am)}\, b$, so we get $\frac{\lambda}{am}b^2=0+k$

Hence $\left(\frac{dx}{dt}\right)^2=\frac{\lambda}{am}b^2-\frac{\lambda}{bm}x^2$

At C, $dx/dt=0$ and $x=d$

Then $0=\frac{\lambda}{am}b^2-\frac{\lambda}{bm}d^2$ or $d^2=\frac{b^3}{a}$ or $d=\frac{b\sqrt{b}}{\sqrt{a}}$

$\therefore$ Required distance $=BC=BO+OC$

$=b+d=b+(b\sqrt{b}/\sqrt{a})=b(\sqrt{a}+\sqrt{b})/\sqrt{a}$

And required time period $=2$ (time taken from B to C) (Note)

$=2$ [(time taken in moving from B to O)

+(time taken in moving from O to C)]

$=2\,[\frac{1}{2}\pi\sqrt{(am/\lambda)}+\frac{1}{2}\pi\sqrt{(bm/\lambda)}]$, from (ii) and (iv)

$=\pi\sqrt{(m/\lambda)}\,[\sqrt{a}+\sqrt{b}]$. Hence proved.

***Ex 3** A particle of mass 'm' executes S. H. M. in the line joining the points A and B on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T, show that the time of an oscillation is $2\pi\sqrt{\left[\frac{mll'}{T(l+l')}\right]}$, *where l, l' are the extensions of the strings beyond their natural lengths.*

Sol. Let OA and OB be the elastic string whose natural lengths are l_1 and l_2 respectively Let O be the position of equilibrium of mass m Let λ_1 and λ_2 be the modulus of elasticity of the strings OA and OB respectively.

x
A O P B

(Fig. 27)

Then in the position of equilibrium,
the tension in the string OA=tension in the string OB.

i.e. $\frac{\lambda_1}{l_1}(l)=\frac{\lambda_2}{l_2}(l')=T$ (given), ...(i)

where l and l' are the extensions in OA and OB.

Let at time t, P be the displaced position of the particle of mass m, such that $OP=x$. Then tension in the string PB $=(\lambda_2/l_2)(l'-x)$, acting in the direction PB and tension in the string $PA=(\lambda_1/l_1)(l+x)$, acting in the direction PA.

Also the acceleration $=\frac{d^2x}{dt^2}$ acting in the direction OP.

$\therefore$ The equation of motion of the particle of mass m is

$$m\frac{d^2x}{dt^2}=\frac{\lambda_2}{l_2}(l'-x)-\frac{\lambda_1}{l_1}(l+x)=\left(\frac{\lambda_2}{l_2}l'-\frac{\lambda_1}{l_1}l\right)-\left(\frac{\lambda_2}{l_2}+\frac{\lambda_1}{l_1}\right)x \quad ...(ii)$$

$$=(T-T)-\left(\frac{T}{l'}+\frac{T}{l}\right)x, \text{ from (i)}$$

or $\frac{d^2x}{dt^2}=-\frac{T(l+l')}{mll'}x$, which is of the form of the standard equation of S. H. M.

Hence required time period $=\frac{2\pi}{\sqrt{\mu}}=2\pi\sqrt{\left[\frac{mll'}{T(l+l')}\right]}$.

Ex. 4. Two light elastic strings are fastened to a mass m and their other ends to fixed points so that the ... taut. The modulus of each is λ, the tension T and length ... Show that the period of an oscillation along the line of the ... is $2\pi\,[mab/\{(T+\lambda)(a+b)\}]^{1/2}$

Sol. Let O be the position of equilibrium of mass m. Let A and B be two points where the other ends of the strings are attached. $AO=a$ and $OB=b$ (given). Let the natural lengths of these be l_1 and l_2 respectively.

A O x P

(Fig. 28)

Then in the position of equilibrium,

tension in the string OA = tension in the string OB

or $(\lambda/l_1)(a-l_1)=(\lambda/l_2)(b-l_2)=T$ (given),

whence we get $(\lambda a/l_1)-\lambda=$ or $\lambda a/l_1=T+\lambda$

or $l_1=\lambda a/(T+\lambda)$.

Similarly $l_2=\lambda b/(T+\lambda)$

Let at time t, the displaced position of the particle be ... such that $OP=x$.

The force acting on the particle at P are tension in the ... $PB=(\lambda/l_2)(b+x-l_2)$, acting in the direction PB and tension ... PA ...

$$m\,(d^2x/dt^2)=(\lambda/l_2)(b+x-l_2)-(\lambda/l_1)(a+x-l_1)$$

$$=[(\lambda/l_2)(b-l_2)-(\lambda/l_1)(a-l_1)]-\lambda x\,[(1/l_2)+(1/l_1)]$$

$$=-\lambda x\left[\left(\frac{T+\lambda}{\lambda a}\right)+\left(\frac{T+\lambda}{\lambda b}\right)\right],\text{ from (i), (ii) and (iii)}$$

or $d^2x/dt^2=-[(T+\lambda)(a+b)/mab]\,x$, which is of the fo... standard equation of S...

$$\text{Hence required period}=\frac{``2\pi"}{\sqrt{\mu}}=2\pi\sqrt{\left\{\frac{mab}{(T+\lambda)(a+b)}\right\}}$$

Ex. 5. A particle is performing S.H.M. in the line joinin... points A and B on a smooth lane and is connected with ... points by elastic strings o... lengths a and a', the mod... elasticity being λ and λ' r... Show that the periodic ... is

and $$m\frac{d^2x}{dt^2}=\left(\frac{\lambda'}{a'}l'-\frac{\lambda}{a}l\right)-\left(\frac{\lambda}{a}\right)x$$

or $$\frac{d^2x}{dt^2}=-\left[\left(\frac{\lambda}{a}+\frac{\lambda'}{a'}\right)\bigg/m\right]x, \text{ from (A)}$$

∴ Required time period

$$=\frac{\text{"}2\pi\text{"}}{\sqrt{\mu}}=2\pi\sqrt{\left[m\bigg/\left\{\frac{\lambda}{a}+\frac{\lambda'}{a'}\right\}\right]}$$ Hence proved.

Exercises on Horizontal Elastic Strings

Ex. 1. Choose the correct answer :

An elastic string is stretched from its natural length l to $2l$ The tension T is given by

(i) λ/l ; (ii) $2\lambda/l$; (iii) λ ; (iv) $\lambda/2$ *(Ranchi 86)*

Ex. 2. In the above example, choose the correct answer from the following :—

(i) λ/l, (ii) λ ; (iii) λ ; (iv) l/λ. **Ans.** (ii)

***Ex. 3.** Prove that the energy of a stretched elastic string is equal to half the product of tension and extension.

Ex. 4. Prove that the modulus of elasticity of an elastic string is equal to the force which would stretch a light string to twice its natural length.

[Hint : Put $l=2a$ in result (A) of § 6 Page 56]

Ex. 5. One and of an elastic string, whose modulus of elasticity is λ, is fixed to a point on a smooth horizontal table and the other end is tied to a particle of mass m lying on the table. The particle is pulled to twice the natural length l from the point of attachment of the string, and is then let go. Show that the time of a complete oscillation is $2(\pi+2)\sqrt{(lm/\lambda)}$.

(Hint : See § 7 on Page 57)

**§ 8. Particle suspended by an elastic string (Vertical elastic string).

A light elastic string of natural length 'a' and modulus of elasticity 'λ' is suspended by one end, to the other end is tied particle of weight 'mg', the particle is slightly pulled down and released, find the motion. *(Kanpur 90; Ranchi 86)*

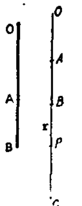
Let O be the fixed point from which the elastic string is suspended. Le $OA=a$ be the natural length of string. At this stage if at A a particle of weight mg is attached to the string, the string will be stretched downwards. Let the particle be at B, in the position of equilibrium, such that $AB=d$ (say).

At B, the force acting on the particle are the tension $(\lambda/a)(d)$ in the string acting vertically upwards and the weight mg of the particle acting vertically downwards.

$\therefore$ At B, we get $mg=(\lambda/a)(d)$ ...(i)

Now the particle is dragged vertically downwards to a point C and then released. Let $BC=b$

A C, the tension in the string is greater than that at B *i e*. the weight mg of the particle [from (i)]. Hence the particle will begin to move vertically upwards.



(Fig. 29)

Let P be the position of the particle after time t of its release from C such that $BP=x$

At P the tension in the string $=(\lambda/a)(d+x)$, acting vertically upwards.

Also the acceleration of the particle at P is d^2x/dt^2, acting in the sense of x increasing *i e*. vertically downwards.

$\therefore$ The equation of motion is

$m\, d^2x/dt^2=mg$ - Tension in the string

or $$m\frac{d^2x}{dt^2}=mg-\frac{\lambda}{a}(d+x)=-\frac{\lambda}{a}(x), \text{ from (i)}$$

or $$d^2x/dt^2=-(\lambda/am)(x), \quad \text{...(ii)}$$

which is of the form of the standard equation of S. H. M.

Here centre of oscillation is B, from which x is measured and amplitude $=BC=b$.

Note 1. The equation (ii) holds till the tension in the string exists *i e* the string remains stretched.

If the particle rises above A (this case will happen when $BC>AB$) the string will become slack when it reaches A on its upwards journey from C and part of motion above A will be free motion under gravity.

Note 2. If $BC<AB$, the particle will not rise upto A and in this case the whole motion is simple harmonic and its period

$$=\frac{2\pi}{\sqrt{(\lambda/am)}}=2\pi\sqrt{\left(\frac{d}{g}\right)}, \text{ from (i)}$$

Note 3. In the case of a spring it is to be remembered that the law of compression is the same as that of extension thus the tension operates even when the particle rises above A. Hence in this case the equation (ii) holds throughout the motion and the motion is simple harmonic of period $2\pi\sqrt{(d/g)}$ as in Note 2 above.

Solved Examples on Vertical Elastic Strings

**Ex. 1. An elastic string without weight of which the unstretched length is l and modulus of elasticity is ng is suspended by one

end and a mass m is attached to the other end. Show that the time period of a small vertical oscillation is
$2\pi\sqrt{(ml/ng)}$.

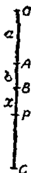
Sol. As in § . . . .
m is suspended . . . . . .
elasticity λ, then the time period

$$=\frac{2\pi}{\sqrt{(\lambda/lm)}}=2\pi\sqrt{\left(\frac{ml}{\lambda}\right)}.$$

Here λ = modulus of elasticity $=ng$

∴ Required time period $=2\pi\sqrt{[(ml)/(ng)]}$. Hence proved.

Ex. 2. A mass 'm' hangs from a fixed point by a light spring and is given a small vertical displacement. Show that the motion is simple harmonic. If 'l' is the length of the spring when the system is in equilibrium and 'n' the number of oscillations per second, show that the natural length of the spring is $l-(g/4\pi^2n^2)$.



(Fig. 30)

Sol. Let O be the fixed point from which the spring is hung. Let $OA=a$ be the natural length of the spring. Let B be the position of equilibrium, then $OB=l$ (given).

∴ AB = extension produced $=l-a$.

Let λ be the modulus of elasticity. At the position of equilibrium B,

the weight of the particle = tension in the spring

or $mg=(\lambda/a)(l-a)$. ...(i)

Let the particle be given a small vertical displacement and let it be at P after time t, such that $BP=x$. At P, the force acting on the particle of mass m are its weight mg acting vertically downwards and the tension $(\lambda/a)(l-a+x)$ in the string, acting vertically upwards. Also the acceleration of the particle at P is d^2x/dt^2 acting in the direction in which x increases i.e. BP.

∴ The equation of motion is

$$m\frac{d^2x}{dt^2}=mg-\frac{\lambda}{a}(l-a+x)=mg-\frac{\lambda}{a}(l-a)-\frac{\lambda}{a}x$$

$$=-(\lambda/a)(x), \text{ from (i)}$$

or $$\frac{d^2x}{dt^2}=-\left(\frac{\lambda}{am}\right)x, \quad ...(ii)$$

which is of the form of standard equation of S. H. M.

Hence the motion is simple harmonic and its time period

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{(\lambda/am)}}=2\pi\sqrt{\left(\frac{am}{\lambda}\right)}. \quad ...(iii)$$

Also time for one complete oscillation $=1/n$ second (given)

∴ From (iii), we get $(1/n)=2\pi\sqrt{(am/\lambda)}$.

or $1/n^2=4\pi^2am/\lambda$ or $\lambda=4\pi^2amn^2$

or $amg/(l-a)=4\pi^2amn^2$, from (i) putting value of λ

$$n^2(l-a) \quad \text{or} \quad l-a=g/4\pi^2n^2 \quad \text{or} \quad a=l-(g/4\pi^2n^2).$$

∴ At B3 A light elastic string of natural length l is hung by one

Now then the other end are tied successively particles of masses m_1

downwards ! If t_1 and t_2 be the periods and c_1, c_2 the statical extensions

Let BC=ponding to these two weights, prove that

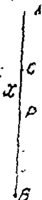
AC $$g(t_1^2-t_2^2)=4\pi^2(c_1-c_2).$$

Sol. As in § 8 Note 2 Page 64, we can prove that if d be the statical extension corresponding to a mass m, then the time period $=2\pi\sqrt{(d/g)}$.

∴ Here $t=2\pi\sqrt{(c_1/g)}$ and $t_2=2\pi\sqrt{(c_2/g)}$.

∴ $t_1^2-t_2^2=4\pi^2\cdot[c_1/g-c_2/g]$ or $g(t_1^2-t_2^2)=4\pi^2(c_1-c_2)$.

Ex. 4. A heavy particle is attached to one point of a uniform light elastic string. The ends of the string are attached to two points in a vertical line. Show that the period of a vertical oscillation in which the string remains taut is $2\pi\sqrt{(mh/2\lambda)}$, where λ is the coefficient of the elasticity of the string and h the harmonic mean of the unstretched lengths of the two parts of the string. *(Avadh 89)*



Sol. A and B are two points in a vertical line to which the ends of the string are attached. Let C be the point where particle of mass m, is attached. Let $CA=a$ and $CB=b$. Let l_1 and l_2 be the natural length of the portions CA and CB respectively.

At C forces acting on the particle of mass m are (i) its weight mg, acting vertically downwards (ii) the tension in the string $CB=(\lambda/l_2)(b-l_2)$ acting vertically downwards and (iii) the tension in the string $CA=(\lambda/l_1)\cdot(a-l_1)$, acting vertically upwards. Also C being the position of equilibrium the upward force on the particle = downward force on it.

i.e. $$(\lambda/l_1)(a-l_1)=mg+(\lambda/l_2)(b-l_2). \quad \ldots(i)$$ (Fig. 31)

Let the particle be now given a small vertical downward displacement and then let go. The particle will begin to move in the upward direction. Let P be the position of the particle at time t, such that $CP=x$. Then the resultant force on the particle at P in the vertically downward direction.

$$=mg+\text{tension in } PB-\text{tension in } PA$$

$$=mg+(\lambda/l_2)(b-x-l_2) \quad (\lambda/l_1)(a+x-l_1)$$

$$=\left[mg+\frac{\lambda}{l_2}(b-l_2)-\frac{\lambda}{l_1}(a-l_1)\right]-\lambda x\left(\frac{1}{l_2}+\frac{1}{l_1}\right)$$

$$=-\lambda x\,[(l_1+l_2)/l_1l_2], \text{ from (i).}$$

Also the acceleration of the particle at P is d^2x/dt^2 acting in the vertically downwards direction.

∴ From Newton's second law of motion, we have

$$m\frac{d^2x}{dt^2}=-\lambda x\left(\frac{l_1+l_2}{l_1l_2}\right) \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{2\lambda}{mh}x,$$

where h = harmonic mean of l_1 and $l_2 = 2l_1l_2/(l_1+l_2)$.

∴ The motion is simple harmonic and its period

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{(2\lambda/mh)}}=2\pi\sqrt{\left(\frac{mh}{2\lambda}\right)}.$$

****Ex. 5 (a). A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time. $\sqrt{(a/g)}\,[\frac{4}{3}\pi+2\sqrt{3}]$ where a is the natural length of the string.**

(Gorakhpur 86)

Sol. O is the fixed end of the string. $OA=a$ is its natural length. Let B be the position of the particle of mass m (say) when it is in equilibrium. Let $AB=d$. Then at B,

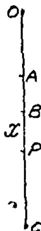
the weight of the particle = tension in the string

or $mg=(\lambda/a)\,d$ or $mg=(mg/a)\,d$, ∵ $\lambda=mg$ (given)

or $d=a$ ..(i)

Now the string is drawn vertically downwards to a point C, such that $OC=4a$ (given) and then let go. Let P be the position of the particle after time t such that $BP=x$.

At P the force acting on the particle are (i) its weight mg acting vertically downwards and (ii) the tension $(\lambda/a)\,(d+x)$ in the string acting vertically upwards. Also the acceleration of the particle is d^2x/dt^2 acting in the sense of x increasing *i.e.* vertically downwards.



(Fig. 32)

∴ From Newton's second law of motion, we have the equation of motion $m\frac{d^2x}{dt^2}=mg-\frac{\lambda}{a}(d+x)$

$=mg-(mg/a)\,(a+x)$, ∵ $\lambda=mg$ and from (i)

or $m\frac{d^2x}{dt^2}=-\frac{mg}{a}x$ or $\frac{d^2x}{dt^2}=-\frac{g}{a}x$ ...(ii)

Multiplying both sides by $2\,dx/dt$ and integrating, we have $(dx/dt)^2=-(g/a)\,x^2+C$, where C is constant of integration.

At C, $x=2a$, $dx/dt=0$.

∴ $0=-(g/a)\,(4a^2)+C$ or $C=(g/a)\,(4a^2)$.

∴ $(dx/dt)^2=(g/a)\,[4a^2-x^2]$

or $dx/dt=-\sqrt{(g/a)}\sqrt{(4a^2-x^2)}$, .(iii)

gn is due to the fact that as time increases x decre-

∴ At B,
Now th $dt = -\sqrt{\left(\frac{a}{g}\right)} \cdot \frac{dx}{\sqrt{(4a^2-x^2)}}$

downwargrating, $t=\sqrt{(a/g)} \cos^{-1}(x/2a)+k$, where k is constant.

Let BG C, $\quad x=2a \quad$ and $\quad t=0$.

$\therefore \quad 0=\sqrt{(a/g)} \cos^{-1}(1)+k \quad$ or $\quad k=0$.

$\therefore \quad t=\sqrt{(a/g)} \cos^{-1}(x/2a)$. .. (iv)

If t_1 be the time taken in moving from C to A, then putting $x=-a$ in (iv), we have $t_1=\sqrt{(a/g)} \cos^{-1}(-\frac{1}{2})$.

or $\quad t=\sqrt{(a/g)}\, \frac{2}{3}\pi$. .. (v)

Also when particle reaches A, *i.e.* $x=-a$, we have from (ii) the velocity at $A=-\sqrt{(g/a)}\sqrt{(4a^2-a^2)}=-\sqrt{(3ag)}$ *i.e.* the velocity at A is $\sqrt{(3ag)}$, acting vertically upwards (*i.e.* in the sense of x decreasing).

At A string becomes slack and tension vanishes so the particle moves as a free particle under gravity when it rises above A till the string is again taut.

Let the particle rise above A for time t_2 (say) till its velocity becomes zero, then from "$v=u+ft$", we get

$0=\sqrt{(3ga)}-gt_2 \quad$ or $\quad t_2=\sqrt{(3a/g)}$ .. (vi)

∴ Time taken by the particle in moving from C to the point where it comes to momentary rest $=(t_1+t_2)$.

Conditions being same it will take the same time to come back to C from its position of momentary rest

∴ Required time $=2\,(t_1+t_2)=2\sqrt{(a/g)}\,\frac{2}{3}\pi+\sqrt{(3a/g)}$, from (v) and (vi)

$=\sqrt{(a/g)}\,[\frac{4}{3}\pi+2\sqrt{3}]$. Hence proved.

Ex. 5 (b). One end of an elastic string is fixed and to the other end is fastened a particle heavy enough to stretch the string to double its natural length a. The string is drawn vertically down till it is four times its natural length and then let go. Describe the motion. *(Gorakhpur 85)*

Hint. Do as Ex. 5 (a) Page 62 of this chapter.

Here in the position of equilibrium at B the weight of the body = tension in the string. *i.e.* $mg=(\lambda/a)(2a-a)$, as the string is stretched to double its natural length a.

or $\quad \lambda=mg$, where λ is the modulus of elasticity.

***Ex. 6. A heavy particle attached to a fixed point by an elastic string hangs freely, stretching the string by a quantity e. It is drawn by an additional distance f and then let go; determine the height to which it will rise if $f^2-e^2=4ae$; e being unstretched length of the string.**

Sol. O is the fixed point to which the particle of ... is attached by the elastic string hanging freely. Let ... natural length. Let B the position of equilibrium of th... such that $AB=e$ (given).

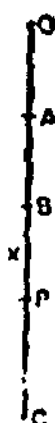
$\therefore$ At B, wt. of the particle = tension in the string.

i.e. $mg=(\lambda/a)(e)$. ...(i)

Now the particle is drawn to C, such that $BC=f$ (given) and then let go. Let P be the position of the particle at time t. Then the equation of motion is

$$m\frac{d^2x}{dt^2}=mg-\frac{\lambda}{a}(e+x)=-\frac{\lambda}{a}x, \text{ from (i)}$$

or $d^2x/dt^2=-(\lambda/am).x$ ...(ii)



Multiplying both sides by $2dx/dt$ and integrating, we get $(dx/dt)^2=-(\lambda/am)x^2+k$, where k is constant of integration.

At C, $x=f$ and $dx/dt=0$.

$\therefore$ $0=-(\lambda/am)f^2+k$ or $k=(\lambda/am)f^2$.

$\therefore$ $(dx/dt)^2=(\lambda/am)(f^2-x^2)$

or $dx/dt=-\sqrt{(\lambda/am)}\sqrt{(f^2-x^2)}$, ...(iii)

(negative sign shows that x decreases as t increases).

At A, $x=-e$.

$\therefore$ From (iii) velocity at A

$=-\sqrt{(\lambda/am)}\sqrt{(f^2-e^2)}$

$=-\sqrt{(\lambda/am)}\sqrt{(4ae)}$, $\because f^2-e^2=4ae$ (Fig. 33).

$=-\sqrt{(g/e)}\sqrt{(4ae)}$, from (i) $\lambda/am=g/e$

$=-\sqrt{(4ag)}$.

The string becomes slack at A, *i.e.* tension vanishes and above A the particle moves freely under gravity. Let the particle rise to a height d above A till its velocity vanishes, the from $v^2=u^2+2fs$ we have $0=4ag-2gd$ or $d=2a$.

$\therefore$ Required height $=CA+d=CB+BA+d=f+e+2a$. Ans.

Ex. 7. A light elastic string whose natural length is a has one end fixed to a point O and to the other end is attached a weight which in equilibrium would produce an extension e. Show that if the weight be let fall from rest at O, it will come to stay instantaneously at a point distant $\sqrt{(2ae+e^2)}$ below the position of equilibrium.

Sol. $OA=a$ is the natural length of the string. Let m be the mass of the particle and B its position of equilibrium, such that $AB=e$. Then at B,

weight of the particle = tension of the string,

or $mg=(\lambda/a)(e)$. ...(i)

∴ At A weight is dropped from rest at O,
Now then from O to A will be due to gradownwardly as there will be no tension, the string being slack. Let v be the velocity with which the particle reaches A.

O O A A B B P C

Then for the motion from O to A, from "$v^2=u^2+2fs$" we have

$v^2=0+2ga$ or $v=\sqrt{(2ga)}$

At A, the string becomes taut and for the motion below A, the tension of the string comes into play. Let P be any displaced position of the weight such that $BP=x$. Then at P, the force acting on the particle being its weight mg, acting vertically downwards and tension $(\lambda/a)(e+x)$, acting vertically upwards, the equation of motion is

$$m.\frac{d^2x}{dt^2}=mg-\frac{\lambda}{a}(e+x)=-\frac{\lambda}{a}x, \text{ from (i)}$$

$$=-\frac{mg}{e}x, \text{ since } \lambda=-\frac{mga}{e} \text{ from (i)}$$

or $$d^2x/dt^2=-(g/e)\,x. \quad \text{..(ii)}$$

Multiplying both sides by $2dx/dt$ and integrating we have

$$(dx/dt)^2=-(g/e)\,x^2+k,$$

where k is constant of integration.

At A, $dx/dt=v=\sqrt{(2ag)}$ and $x=-e$

∴ $2ag=-(g/a)(e)^2+k$ or $k=(2a+e)\,g$

∴ $(dx/dt)^2=(2a+e)\,g-(g/e)\,x^2$. ...(iii)

Let the particle come to instantaneous rest at C, such that $BC=d$ (say)

Then from (iii) we get $0=(2a+e)\,g-gd^2/e$

or $d^2=(2ae+e^2)$ or $d=\sqrt{(2ae+e^2)}$. Hence proved.

Ex. 8. A light elastic string of natural length 4 m has one end fixed at a point A and the other attached to a stone, which in equilibrium extends the string by 1 metre. If the stone be dropped from A, find the maximum extension produced.

Sol. Proceed exactly as in last example.

Here $a=4$m. ; $d=1$m.

Required maximum extension $=a+e+d'=4+1+3=8$ m. **Ans.**

Ex 9. A light elastic string of natural length l has one end fixed at a point A, and the other attached to a stone, the weight of which in equilibrium would extend the string to a length l_1; show that if the stone be dropped from rest at A, it will come to an instantaneous rest at a depth $\sqrt{(l_1^2-l_2^2)}$ below the equilibrium position.

Sol. $AB=l$ is the natural length of the string. Let, mass of the stone and its position of the equilibrium, s.

$$AC=l_1 \text{ or } BC=(l_1-l).$$

Then at C, $mg=(\lambda/l)(l_1-l)$, .. (i)

where λ is the modulus of elasticity.

If the stone be dropped from rest at A, the motion from A to B will be due to gravity only as there will be no tension, the string being slack. If v be the velocity of the particle when it reaches B then from "$v^2=u^2+2fs$" we have

$$v^2=0+2gl \text{ or } v=\sqrt{(2gl)} \quad ...(ii)$$

At B, the string becomes taut and for the motion below B, the tension of the string comes into play. Let P be any displaced position of the stone, such that $CP=x$. Then at P, the force acting on the stone being its weight mg acting vertically downwards and tension $(\lambda/l)(l_1-l+x)$ in the string acting vertically upwards, the equation of motion is $m\dfrac{d^2x}{dt^2}=mg-\dfrac{\lambda}{l}(l_1-l+x)$

$$=-(\lambda/l)\,x, \text{ from (i)}$$

or $$\frac{d^2x}{dt^2}=-\frac{\lambda}{ml}.x=-\frac{gx}{(l_1-l)}, \text{ from (i)}$$

Multiplying both sides by $2\,(dx/dt)$ and integrating we get $\left(\dfrac{dx}{dt}\right)^2=-\dfrac{g}{(l_1-l)}x^2+k$, where k is constant. (Fig. 6)

At B, $dx/dt=v=\sqrt{(2gl)}$ from (ii) and $x=-(l_1-l)$

$$\therefore\; 2gl=-[g/(l_1-l)](l_1-l)^2+k$$

or $$k=[2gl+g(l_1-l)].$$

$$\therefore\; (dx/dt)^2=2gl+g(l_1-l)-[g/(l_1-l)]\,x^2 \quad ...(iii)$$

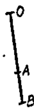
If the particle comes to instantaneous rest at D such that $CD=d$ then from (iii) we get

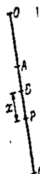
$$0=2gl+g(l_1-l)-(gd^2/(l_1-l))$$

or $$d^2=(l_1^2-l^2) \text{ or } d=\sqrt{(l_1^2-l^2)}.$$ Hence proved.

*Ex. 10. A heavy particle of mass m is attached to one end of an elastic string of natural length l; whose other end is fixed at O. The particle is then let fall from rest at O. Show that, part of the motion is simple harmonic and that, if the greatest depth of the particle below O is $l\cot^2\frac{1}{2}\theta$, the modulus of elasticity of the string is $\frac{1}{2}mg\tan^2\theta$ and that the particle attains this depth in time $\sqrt{(2l/g)}[1+(\pi-\theta)\cot\theta]$, where θ is a positive acute angle.

$\therefore$ At.]. $OA\ (=l)$ is the natural length of Now tring. Let B be the position of equi- downwatum such that $AB=e$ (say).





(Fig. 36)

Then as in Ex. 7 Page 70, we can prove that motion from O to A, will be under gravity only and for the motion below A, we can have $\frac{d^2x}{dt^2}=-\frac{g}{e}\ x$, (see result (ii) of Ex 7 Page 69) which shows that this motion is simple harmonic.

Also we can show that $mg=(\lambda/l)\ (e)$

...See result (i) of Ex. 7 Page 69 .. (A)

or $\lambda=mlg/e$

Also if the particle comes to instantaneous rest at C, then if $BC=d$, we can have as in Ex. 7 Page 70, $d=\sqrt{(2le+e^2)}$ ...(B)

Here we are given $OC=l\cot^2\frac{1}{2}\theta$ or $d=l\cot^2\frac{1}{2}\theta-l-e$

or $l+e+d=l\cot^2\frac{1}{2}\theta$

or $\sqrt{(2le+e^2)}=l\cot^2\frac{1}{2}\theta-e$, from (B)

Squaring both sides and simplifying we get

$2e\cot^2\frac{1}{2}\theta=l\ (1+\cot^4\frac{1}{2}\theta-2\cot^2\frac{1}{2}\theta)$

or $2e=l\left[\frac{\cot^2\frac{1}{2}\theta-1}{\cot\frac{1}{2}\theta}\right]^2=l\left[\frac{\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}\right]^2$

or $2e=l\ [(2\cos\theta)/\sin\theta]^2=4l\cot^2\theta$ or, $e=2l\cot^2\theta$

$\therefore$ From (A) modulus of elasticity $=\lambda=mlg/e=\frac{1}{2}mg\tan^2\theta$.

Let time from O to A be t_1 then considering the motion from O to A under gravity only we have $l=\frac{1}{2}gt_1^2$, from $s=ut+\frac{1}{2}ft^2$,

i e. $t_1=\sqrt{(2l/g)}$.

Also from $d^2x/dt^2=-(g/e)\ x$ on integrating after multiplying both sides by $2\ (dx/dt)$ we get $(dx/dt)^2=-(g/e)\ x^2+c$.

At A, $dx/dt=$ velocity at $A=\sqrt{(2gl)}$ and $x=-e$.

$\therefore\ 2gl=-ge+c$ or $c=g\ (2l+e)$

$\therefore\ (dx/dt)^2=-(g/e)\ x^2+g\ (2l+e)$

or $\frac{dx}{dt}=\sqrt{\left(\frac{g}{e}\right)}\left[(2el+e^2)-x^2\right]^{1/2}$,

taking the +ve sign as x increases when t increases

or $\frac{dx}{dt}=\sqrt{\left[\frac{g}{(2l\cot^2\theta)}\right]}\left[4l^2\ (\cot^2\theta+\cot^4\theta)-x^2\right]^{1/2}$, $\because\ e=2l\cot^2\theta$

$=\sqrt{(g/2l)}\tan\theta\sqrt{[(2l\cot\theta\operatorname{cosec}\theta)^2-x^2]}$

or $dt=\sqrt{\left(\frac{2l}{g}\right)}\cot\theta\frac{dx}{\sqrt{[(2l\cot\theta\operatorname{cosec}\theta)^2-x^2]}}$

∴ Time from A to C

$$=\sqrt{\left(\frac{2l}{g}\right)}\cos\theta\int_{x=-e}^{a}\frac{dx}{\sqrt{[(2l\cot\theta\operatorname{cosec}\theta)^2-x^2]}},$$

where $d=(2el+e^2)=(2l\cot\theta\operatorname{cosec}\theta)$

$$=\sqrt{(2lg)}\cot\theta\left[\sin^{-1}\{x/(2l\cot\theta\operatorname{cosec}\theta)\}\right]_{2l\cot^2\theta}^{2l\cot\theta\operatorname{cosec}\theta}$$

$=\sqrt{(2lg)}\cot\theta\,[\sin^{-1}(1)+\sin^{-1}\{(\cot\theta/\operatorname{cosec}\theta)\}$

$=\sqrt{(2l/g)}\cot\theta\,[\frac{1}{2}\pi+\sin^{-1}(\cos\theta)]$

$=\sqrt{(2l/g)}\cot\theta\,[\frac{1}{2}\pi+\sin^{-1}\{\sin(\frac{1}{2}\pi-\theta)\}]$

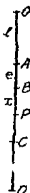
$=\sqrt{(2l/g)}\cot\theta\,[\frac{1}{2}\pi+(\frac{1}{2}\pi-\theta)]=\sqrt{(2l/g)}\,(\pi-\theta)\cot\theta$

∴ Required time = time from O to A + time from A to C

$=\sqrt{(2l/g)}+\sqrt{(2l/g)}\,\{(\pi-\theta)\cot\theta\}$

$=\sqrt{(2l/g)}\,[1+(\pi-\theta)\cot\theta]$. Hence proved.

Ex. 11. **A light elastic string of natural length l and modulus of elasticity λ, is hung by one end. To the other end is tied a particle of mass m. *Find the time of travel from a distance x_1 to x_2 in the vertical from the point of suspension.***



Sol. $OA=l$ is the natural length of the string and O is the point of suspension. Let B be the position of equilibrium when mass m is tied to the end A, such that $AB=e$. Then at B,

tension in the string = weight of the particle.

i.e. $\frac{\lambda}{l}(e)=mg$ or $e=\frac{mgl}{\lambda}$ ...(i)

Let P be any position of the particle in its downward motion, such that $BP=x$.

Then at P, the equation of motion is (Fig. 37)

$$m\frac{d^2x}{dt^2}=mg-\frac{\lambda}{l}(AP)=mg-\frac{mg}{e}(e+x),\text{ from (i)}$$

or $$m\frac{d^2x}{dt^2}=-\frac{mg}{e}x\quad\text{or}\quad\frac{d^2x}{dt^2}=-\frac{g}{e}x \quad ...(ii)$$

Multiplying both sides by $2\,(dx/dt)$ and integrating, we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{e}x^2+C \quad ...(iii)$$

If the particle starts with velocity V from the point B *i.e.* $x=0$. (Note)

Then from (iii), we get $V^2=0+C$ or $C=V^2$

∴ From (iii) we get $(dx/dt)^2=V^2-(g/e)\,x^2$

or $\quad (dx/dt)^2=(g/e)\,[a^2-x^2]$, where $a^2=V^2e/g$ ...(iv)

or $\quad \dfrac{dx}{dt}=\sqrt{\left(\dfrac{g}{e}\right)}\sqrt{(a^2-x^2)}$ or $dt=\sqrt{\left(\dfrac{e}{g}\right)}\dfrac{dx}{\sqrt{(a^2-x^2)}}$ ...(v)

Now let C and D be two points, such that

$OC=x_1$ and $OD=x_2$

Then $BC=OC-(l+e)=x_1-l-e$

and $\quad OB=OD-(l+e)=x_2-l-e$

$\therefore$ From (v) required time in moving from C to D

$$=\sqrt{\left(\frac{e}{g}\right)}\int_C^D \frac{dx}{\sqrt{(a^2-x^2)}}=\sqrt{\left(\frac{e}{g}\right)}\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_{x_1-l-e}^{x_2-l-e}$$

$$=\sqrt{\left(\frac{ml}{\lambda}\right)}\left[\sin^{-1}\left(\frac{x_2-l-e}{a}\right)-\sin^{-1}\left(\frac{x_1-l-e}{a}\right)\right]$$

Ans.

where $e=mgl/\lambda$, $a=V\sqrt{(e/g)}$.

****Ex. 12.** ***A light elastic string AB of length l is fixed at A and is such that if a weight ω be attached to B, the string will be stretched to length 2l. If a weight $\omega/4$ be attached to B and let fall from the level of A, prove that [i] the amplitude of S. H. M. that ensues is $3l/4$, [ii] the distance through which it falls is $2l$ and [iii] the period of oscillation is $\sqrt{(l/4g)}\,(4\sqrt{2}+\pi+2\sin^{-1}\frac{1}{3})$.***

Sol. Let λ be the modulus of elasticity.

If a weight ω is attached at B, we are given that the extension in the string$=2l-l=l$

$\therefore \quad \omega=(\lambda/l)\,(l) \quad$ or $\quad \omega=\lambda$ ...(i)

($\because$ In the position of equilibrium weight of the particle=tension in the string).

Now if a weight $\omega/4$ is attached at B, let the particle be in the position of equilibrium at C, such that $BC=d$ (say). Then at C as

weight of the particle=tension in the string.

$\therefore \quad \frac{1}{4}\omega=(\lambda/l)\,(d) \quad$ or $\quad \frac{1}{4}\omega=(\omega/4l)\,(d)$, from (i)

or $\quad d=\frac{1}{4}l$. ...(ii)

Now the particle is dropped from A and the motion from A to B is under gravity only, string being slack.

If v be the velocity of the particle at B then from "$v^2=u^2+2fs$" we have $v^2=0+2gl$ or $v=\sqrt{(2gl)}$ and from "$v=u+ft$" we find t_1, the time taken in moving from A to B, is given by $v=0+gt_1$

or $\quad t_1=v/g=\sqrt{(2lg)}/g=\sqrt{(2l/g)}$ ...(iii)

A B C P D

(Fig. 38)

As the particle falls below B, the tension of the elastic string comes into play. Let P be the position of the particle at time t, such that $CP=x$.

(Here time is measured from the instant the particle crosses B and x is measured from C downwards). (Note)

At P, the forces acting on the particle are its weight $\frac{1}{4}\omega$ acting vertically downwards and the tension $(\lambda/l)(d+x)$, acting vertically upwards. Also the acceleration of the particle at P is d^2x/dt^2 acting vertically downwards i.e. in the sense of x increasing.

$\therefore$ From Newton's second law of motion, we have

$$m\frac{d^2x}{dt^2}=\tfrac{1}{4}\omega-\frac{\lambda}{l}(d+x),\text{ where } m=\text{mass of the particle}$$

or $$\left(\frac{\omega}{4g}\right)\frac{d^2x}{dt^2}=\frac{\omega}{4}-\frac{\lambda d}{l}-\frac{\lambda}{l}x, \quad \because m=\tfrac{1}{4}\omega$$

$$=-(\omega/l)\,x,\text{ from (i) and (ii)}$$

or $$\frac{d^2x}{dt^2}=-\left(\frac{4g}{l}\right)x, \qquad \text{.. (iv)}$$

which shows the motion below B is simple harmonic.

Multiplying both sides by $(2\,dx/dt)$ and integrating, we get

$$(dx/dt)^2=-(4g/l)\,x^2+k,\text{ where } k \text{ is constant.}$$

At B, $dx/dt=v=\sqrt{(2gl)}$ and $x=-d=-(\frac{1}{4}l)$, from (ii)

$\therefore$ $2gl=-(4g/l)(l^2/16)+c$ or $k=9gl/4$,

$\therefore$ $(dx/dt)^2=(9gl/4)-(4g/l)\,x^2=(4g/l)\,[(9l^2/16)-x^2]$

or $$dx/dt=2\sqrt{(g/l)}\,\sqrt{[(\tfrac{3}{4}l)^2-x^2]}. \qquad \text{.. (v)}$$

(here as t increases, x also increases, hence +ve sign of dx/dt).

Let the particle come to momentary rest at D, such that $CD=b$ (say), then from (v), $0=2\sqrt{(g/l)}\sqrt{\{(\frac{3}{4}l)^2-b^2\}}$

or $$b=\tfrac{3}{4}l=\text{amplitude}.$$

$\therefore$ Total distance fallen through

$=AD=AB+BC+CD=l+d+b=l+\frac{1}{4}l+\frac{3}{4}l=2l$

Again from (iv), time taken by the particle in moving from C to $D=\frac{1}{4}$ (time period) $=\frac{1}{4}[2\pi\sqrt{(l/4g)}]=\frac{1}{4}\pi\sqrt{(l/g)}$. ...(vi)

From (v) we get $dt=\frac{1}{2}\left(\frac{l}{g}\right)\frac{dx}{\sqrt{\{(\frac{3}{4}l)^2-x^2\}}}$.

$\therefore$ time taken in moving from B to C i.e. from $x=-d$ to $x=0$

$$=\tfrac{1}{2}\sqrt{\left(\frac{l}{g}\right)}\int_{x=-d}^{0}\frac{dx}{\sqrt{\{(\frac{3}{4}l)^2-x^2\}}}=\tfrac{1}{2}\sqrt{\left(\frac{l}{g}\right)}\left[\sin^{-1}\left(\frac{4x}{3l}\right)\right]_{-l/4}^{0} \qquad \because d=\tfrac{1}{4}l$$

$$=\tfrac{1}{2}\sqrt{(l/g)}\,\sin^{-1}(\tfrac{1}{3}). \qquad \text{...(vii)}$$

$\therefore$ Required period $=2$ [time taken in moving from A to D]

$=2$ [time from A to B + time from B to C + time from C to D]

$= 2\,[\sqrt{(2l/g)} + \frac{1}{2}\sqrt{(l/g)}\sin^{-1}(\frac{1}{3}) + \frac{1}{2}\pi\sqrt{(l/g)}]$, from (iii), (iv) and (vii)

$= \sqrt{(l/g)}\,[2\sqrt{2} + \sin^{-1}(\frac{1}{3}) + \frac{1}{2}\pi]$

$= \sqrt{(l/4g)}\,[4\sqrt{2} + 2\sin^{-1}(\frac{1}{3}) + \pi]$. Hence proved.

Ex. 13 (a). A heavy particle of mass 'm' is attached to one of an elastic string of natural length l, where modulus of elasticity is equal to the weight of the particle and the other end is fixed at O, the particle let fall from rest at O. Show that a part of the motion is simple harmonic and that the greatest depth of the particle below O is $(2+\sqrt{3})\,l$. Show that this depth is attained in time

$$\sqrt{(l/g)}\,\{\sqrt{2} + \pi - \cos^{-1}(1/\sqrt{3})\} \text{ seconds.}$$

Sol Let λ be the modulus of elasticity, then ...(i)

$$\lambda = mg \text{ (given)}$$



Let $OA = l$, be the natural length of the string. Let $AB = b$ (say) be the statical extension in the string. Then at B,

the weight of the particle = the tension in the string i.e. $mg = (\lambda/l)$

or $mg = (mg/l)\,b$, from (i)

or $\therefore b = l$ ...(ii)

Now the particle is let fall from rest at O and the motion from O to A is under gravity only, string being slack (Fig. 39)

$\therefore$ If v be the velocity of particle when it reaches A, then from "$v^2 = u^2 + 2fs$", we have $v^2 = 0 + 2gl$ or $v = \sqrt{(2gl)}$ and from $v^2 = u + ft$" we find t_1, the time taken in moving from O to A, is given by ...(iii)

$$v = 0 + gt_1 \quad \text{or} \quad t_1 = v/g = \sqrt{(2gl)}/g = \sqrt{(2l/g)}.$$

As the particle falls below A, the tension of the elastic string comes into play. Let P be the position of the particle at time t, such that $BP = x$ (Here time is measured from the instant the particle crosses A).

At P, the forces acting on the particle are its weight mg acting [illegible] $(b+x)$, acting vertically [illegible] e at P is d^2x/dt^2 acting [illegible] asing.

$\therefore$ From Newton's second law of motion, we have

$$m\,\frac{d^2x}{dt^2} = mg - \left(\frac{\lambda}{l}\right)(b+x) = mg - \left(\frac{mg}{l}\right)(l+x), \text{ from (i), (ii)}$$

or

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{l}\right)x,$$

which shows that the motion below A is simple harmonic.

Multiplying both sides by $2\,dx/dt$ and integrating, we have

$$(dx/dt)^2 = -(g/l)\,x^2 + C_1, \text{ where } C_1 \text{ is constant.}$$

At A, $x=-b=-l$ and $dx/dt=v=\sqrt{(2gl)}$.

$\therefore \quad 2gl=-(g/l)\, l^2+C_1 \quad \text{or} \quad C_1=3gl$.

$\therefore \quad (dx/dt)^2=3gl-(g/l)\, x^2=(g/l)\,(3l^2-x^2)$

or $$\frac{dx}{dt}=\sqrt{\left(\frac{g}{l}\right)}\sqrt{[(l\sqrt{3})^2-x^2]}. \qquad \text{...(iv)}$$

(Here as t increases, x increases, hence positive sign of (dx/dt).

Let the particle come to momentary rest at C, such that $BC=d$ (say).

Then from (iv), we get $0=\sqrt{(g/l)}\sqrt{\{(l\sqrt{3})^2-d^2\}}$ or $d=l\sqrt{3}$.

$\therefore$ Total distance fallen below $O=OC=OA+AB+BC$
$=l+b+d=l+l+l\sqrt{3}=l\,(2+\sqrt{3})$.

Again from (iv), the time taken by the particle in moving from B to C.

$=(1/4)$ (time period)$=(1/4)\,[2\pi\sqrt{(g/l)}]=\frac{1}{2}\pi\sqrt{(l/g)}$. ...(v)

Also from (v), $dt=\sqrt{\left(\frac{l}{g}\right)}\dfrac{dx}{\sqrt{\{(l\sqrt{3})^2-x^2\}}}$

$\therefore$ Time taken in moving from A to B i.e. $x=-l$ to $x=0$

$$=\int_{x=-l}^{0}\sqrt{\left(\frac{l}{g}\right)}\frac{dx}{\sqrt{\{(l\sqrt{3})^2-x^2\}}}=\sqrt{\left(\frac{l}{g}\right)}\left[\sin^{-1}\left(\frac{x}{l\sqrt{3}}\right)\right]_{-l}^{0}$$

$=(l/g)\,[\sin^{-1}(1/\sqrt{3})]=(l/g)\,[\frac{1}{2}\pi-\cos^{-1}(1/\sqrt{3})]$

[$\because$ If $\theta=\sin^{-1}(1/\sqrt{3})$ or $\sin\theta=1/\sqrt{3}$ or $\cos(\frac{1}{2}\pi-\theta)=1/\sqrt{3}$

or $\frac{1}{2}\pi-\theta=\cos^{-1}(1/\sqrt{3})$ then $\theta=\frac{1}{2}\pi-\cos^{-1}(1/\sqrt{3})$].

$\therefore$ Required time

$=$time from O to $A+$time from A to $B+$time from B to C

$=\sqrt{(2l/g)}+\sqrt{(l/g)}\,[\frac{1}{2}\pi-\cos^{-1}(1/\sqrt{3})]+\frac{1}{2}\pi\sqrt{(l/g)}$

$=\sqrt{(l/g)}\,[\sqrt{2}+\pi-\cos^{-1}(1/\sqrt{3})]$. Hence proved.

[illegible]

to rest.

Sol. Do as Ex. 13 (a) above. Here we are given
'l'$=a=$'b' of the last Example.

And therefore at B, we shall get $mg=(\lambda/a)\,(a)$ or $\lambda=mg$.

Rest is the same as Ex. 13 (a) above and we are to find only the total distance fallen below O.

Ex. 14. A heavy particle is attached to an inextensible string to a fixed point from which the particle is allowed to fall freely; when the particle is in its lowest position the string is of twice its natural length. Prove that the modulus is four times the weight of the particle and find the time during which the string is extended beyond its natural length.

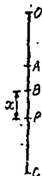
Sol. Let O be the fixed point from which the particle is let fall. Let m be the mass of the particle and λ the required modulus.

Let OA $(=a)$ be the natural length of the string and AB $(=b$, say) be the statical extension.

Then at B,

weight of the particle = tension in the string

i.e. $mg=(\lambda/a)\,(b)$ or $\lambda=mg/ab$ ...(i)



(Fig. 40)

Now particle is let fall from O and the motion from O to A is under gravity only, the string being slack.

If v be the velocity with which the particle reaches A, then from "$v^2=u^2+2fs$", we have $v^2=0+2ag$ or $v=\sqrt{(2ga)}$.

As the particle falls below A, the tension of the string comes into play. Let P be position of the particle at time t (measured from the instant the particle crosses A) such that $BP=x$. At P, the force acting on the particle are its weight mg acting vertically downwards and tension $(\lambda/a)\,(b+x)$ in the string acting vertically upwards. Also acceleration of the particle is (d^2x/dt^2) acting vertically downwards.

$\therefore$ From Newton's second law of motion, we have

$$m\,\frac{d^2x}{dt^2}=mg-\frac{\lambda}{a}\,(b+x) \quad \text{or} \quad \frac{d^2x}{dt^2}=-\left(\frac{g}{b}\right)x, \qquad \text{...(ii)}$$

$\because$ $\lambda=mga/b$,

which shows that the motion below A is simple harmonic.

Integrating (ii), we get $(dx/dt)^2=-(g/b)\cdot x^2+k$, where k is constant,

At A, $x=-b$ and $dx/dt=v=\sqrt{(2ga)}$.

$\therefore$ $2ga=-gb+k$ or, $k=2ga+gb$.

$\therefore$ $(dx/dt)^2=-(g/b)\,x^2+2ga+gb.$ ...(iii)

If C be the lowest position of the particle, then $OC=2a$ (given)

i.e., $BC=2a-OB=2a-(a+b)=a-b.$

Also at C, $dx/dt=0.$

$\therefore$ From (iii), we get $0=-(g/b)\,(a-b)^2+2ga+gb$

or $b^2+2ab-(a-b)^2=0$

or $b^2+2ab-a^2+2ab-b^2=0$ or $b=\frac{1}{4}a$, $\because$ $a\neq 0$

$\therefore$ From (i), $\lambda=mga/b=4m$, ...(iv) Hence proved.

$\therefore$ From (iii), $(dx/dt)^2=-4\,(g/a)\,x^2+2ga+\frac{1}{4}ag$, $\because$ $b=\frac{1}{4}a$

or $$\left(\frac{dx}{dt}\right)^2=\frac{9ag}{4}-\frac{4g}{a}x^2=\frac{4g}{a}\left[\frac{9a^2}{16}-x^2\right]$$

or $$dx/dt=\sqrt{(4g/a)}\,\sqrt{\{(\tfrac{3}{4}a)^2-x^2\}}$$

or $$dt = \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\frac{dx}{\sqrt{\{(\frac{3}{4}a)^2 - x^2\}}} \quad ...(v)$$

∴ Required time = time from A to C

i.e. from $x = -b$ to $x = (a-b)$,

$$= \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\int_{x=-b}^{a-b}\frac{dx}{\sqrt{\{(\frac{3}{4}a)^2 - x^2\}}}, \text{ from (v)}$$

$$= \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\int_{x=-a/4}^{3a/4}\frac{dx}{\sqrt{\{(\frac{3}{4}a)^2 - x^2\}}}, \quad \because b = \tfrac{1}{4}a$$

$$= \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}\left[\sin^{-1}\left(\frac{4x}{3a}\right)\right]_{-a/4}^{3a/4} = \frac{1}{2}\sqrt{\left(\frac{a}{g}\right)}[\sin^{-1}(1) + \sin^{-1}(\tfrac{1}{3})]$$

$$= \tfrac{1}{2}\sqrt{(a/g)}\,[\tfrac{1}{2}\pi + \sin^{-1}(\tfrac{1}{3})] = \tfrac{1}{2}\sqrt{(a/g)}\,[\pi - \cos^{-1}(\tfrac{1}{3})]$$

$$[\because \sin^{-1}(\tfrac{1}{3}) = \tfrac{1}{2}\pi - \cos^{-1}(\tfrac{1}{3})].$$

Ex. 15. A particle of mass m is attached to one end of an elastic string, of natural length g and modulus of elasticity $2mg$, whose other end is fixed at O. The particle is let fall from A when A is vertically above O and OA $= a$. Show that its velocity will be zero at B, where OB $= 3a$; calculate also time from A to B.

Sol. Let $OC = a$ be the natural length of the string and $CD = d$ be the statical extension.

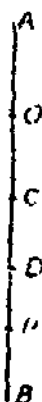
Let λ be the modulus of elasticity. Then $\lambda = 2mg$ (given) .. (i)

At D, for equilibrium, we have

$$mg = \left(\frac{\lambda}{a}\right)d = \left(\frac{2mg}{a}\right)d, \text{ from (i)}$$

or $$d = \tfrac{1}{2}a \quad .. \text{(ii)}$$

The particle is let fall from A, where $OA = a$ (natural length).



(*Fig.* 41)

∴ The motion from A to C will be under gravity only, the string being loose from A to C.

∴ If v be the velocity of the particle when it reaches C, then from "$v^2 = u^2 + 2fs$", we have $v^2 = 0 + 2g\,(AC) = 2g\,(2a)$ or $v = 2\sqrt{(ag)}$.

Also if t_1 be the time from A to C, then from "$v = u + ft$", we have

$$v = 0 + gt_1 \text{ or } t_1 = v/g = 2\sqrt{(a/g)} \quad .. \text{(iii)}$$

As the particle moves below B, the string becomes taut and tension begins to act. Let at time t (time being measured from the instant the particle crosses C) the particle be at P, such that $DP = x$. At P, the forces acting on the particle are its weight mg acting vertically upwards. Also the acceleration of the particle is d^2x/dt^2 acting vertically downwards.

The equation of motion is

$$m\left(\frac{d^2x}{dt^2}\right) = mg - \frac{\lambda}{a}(d+x) = -\frac{\lambda}{a}x \text{ from (ii) and (i)}$$

or $\qquad \dfrac{d^2x}{dt^2}=-\left(\dfrac{2g}{a}\right)x$, from (i) $\qquad$ ..(iv)

which shows motion is simple harmonic below C.

Integrating (iv) we get $\left(\dfrac{dx}{dt}\right)^2=-\dfrac{2g}{a}x^2+k$, where k is const.

At C, $(dx/dt)=v=2\sqrt{(ag)}$ and $x=-d=-\frac{1}{2}a$

$\therefore \quad 4ag=-(2g/a)(-\frac{1}{2}a)^2+k \quad$ or $\quad k=\frac{9}{2}ag$ $\qquad$ ..(v)

$\therefore \quad (dx/dt)^2=\frac{9}{2}ag-2(g/a)x^2$

At B, $(dx/dt)=0$ (given) and let $DB=x_1$

Then $\frac{9}{2}ag-(2g/a)x_1^2=0 \quad$ or $\quad x_1=\frac{3}{2}a$

$\therefore \quad OB=OC+CD+DB=a+d+x_1=a+\frac{1}{2}a+\frac{3}{2}a=2a.$ $\qquad$ Hence proved.

Also from (v), $dx/dt=\sqrt{(2g/a)}\sqrt{\{(\frac{9}{4}a^2)-x^2\}}$

or $\qquad dt=\sqrt{\left(\dfrac{a}{2g}\right)}\cdot\dfrac{dx}{\sqrt{\{(\frac{3}{2}a)^2-x^2\}}}$

$\therefore$ time from C to B i.e. from $x=-d=-\frac{1}{2}a$ to $x=\frac{3}{2}a$

$$=\sqrt{\left(\frac{a}{2g}\right)}\int_{x=-a/2}^{3a/2}\frac{dx}{\sqrt{\{(\frac{3}{2}a)^2-x^2\}}}=\sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}\left(\frac{2x}{3a}\right)\right]_{-a/2}^{3a/2}$$

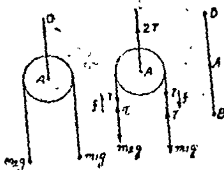
$=\sqrt{(a/2g)}\,[\sin^{-1}(\frac{1}{3})+\sin^{-1}(1)]=\sqrt{(a/2g)}\,[\frac{1}{2}\pi+\sin^{-1}(\frac{1}{3})]$

$\therefore$ Required time $=$ time from A to $C+$ time from C to B

$=2\sqrt{(a/g)}+\sqrt{(a/2g)}\,[\frac{1}{2}\pi+\sin^{-1}(\frac{1}{3})]$ $\qquad$ Ans.

$=\frac{1}{2}\sqrt{(a/2g)}\,[4\sqrt{2}+\pi+2\sin^{-1}(\frac{1}{3})]$

[illegible] light pulley is suspended from a fixed [illegible] nd modulus of elasticity ng. [illegible] a light inextensible string passing round the pulley, [illegible] pulley executes simple harmonic motion about a centre whose depth below the point of suspension is $l(1+2M/n)$, where M is the harmonic mean between m_1 and m_2.



(Fig. 42)

Sol. Let O be the fixed point from which the pulley is suspended. Let λ be the modulus of elasticity then

$$\lambda = ng \text{ (given)} \quad \ldots(i)$$

Consider the motion of masses m_1 and m_2. Let f be the acceleration of the system and T the tension in the inextensible string round the pulley. Then the equation of motion (if $m_1 > m_2$) are $m_1 f = m_1 g - T$ and $m_2 f = T - m_2 g$

whence $T = \dfrac{2m_1 m_2}{(m_1 + m_2)} g = Mg$, where $M = \dfrac{2m_1 m_2}{m_1 + m_2}$.

$\therefore$ Pressure on the pulley due to the masses m_1 and m_2

$$= 2T = 2Mg$$

$\therefore$ The question now reduces to the motion of a mass of $2M$ hanging from one of an elastic spring OA of natural length l whose other end is fixed at O.

Let B be its position of equilibrium such that $AB = d$ (say). Then at B, the weight of the mass $2M$ = Tension in the spring.

i.e. $$2Mg = \left(\frac{\lambda}{l}\right) d = \left(\frac{ng}{l}\right) d, \text{ from (i)}$$

or $$d = \frac{2lM}{n}$$

In case of vertical spring whose one end is fixed we know that the motion is simple harmonic about the position of equilibrium of the mass attached to the other end.

Hence required depth $= OB = OA + AB = l + d$

$$= l + \left(\frac{2lM}{n}\right) = l\left[1 + \frac{2M}{n}\right]$$ Hence proved.

Ex. 17. Two bodies of masses M and M', are attached to the lower end of an elastic string, whose upper end is fixed and hang at rest, M' falls off; show that distance of M from the upper end of string at time t is $a + b + c \cos [\sqrt{(g/b)}\, t]$, where '$a$' is the unstretched length of the string, 'b' and 'c' the distance by which it would be stretched when supporting M and M' respectively.

(Avadh 90, 87; Gorakhpur 86)

Sol. Let P be the fixed point from which the elastic string is suspended. OA ($= a$) is the natural length of the string and λ its modulus of elasticity.

When only mass M is hanging the string will be stretched upto B, such that $AB = b$.

Also at B, we have $Mg = (\lambda/a)\, b$ or $\lambda = Mag/b$. ...(i)

When both the masses are hanging the string will be stretched further a distance c i.e. if C be the position of equilibrium when both the masses are hanging, then $BC = c$ or $AB = b + c$.

Now mass M' falls off and therefore the mass M will try to get back to its equilibrium position viz. B.



(Fig. 43)

Let P be the position of the particle at time t, such that $BP=x$.

Then at P, the forces acting on the mass M are its weight Mg acting vertically downwards and the tension $(\lambda/a)(b+x)$ in the string acting vertically upwards. Also the acceleration of the particle is $\frac{d^2x}{dt^2}$, acting in the vertically downwards sense *i.e.* the sense in which x increases.

∴ From Newton's second law of motion, we have

$$M.\frac{d^2x}{dt^2}=Mg-\left(\frac{\lambda}{a}\right)(b+x)$$

$$=Mg-\left\{\frac{Mg}{a}\right\}(b+x), \text{ from (i)}$$

or

$$\frac{d^2x}{dt^2}=-\left\{\frac{g}{b}\right\}x,$$

which shows that the motion is simple harmonic.

From (ii) we have $v\frac{dv}{dx}=-\frac{g}{b}x.$

Integrating with respect to x, we get

$\frac{1}{2}v^2=-(g/b).\frac{1}{2}x^2+C_1$, where C_1 is constant of integration.

At C, $v=0$ and $x=c$ so we get $0=-(g/b).\frac{1}{2}c^2+C_1$ or $C_1=-gc^2/2b$

$$\therefore\quad v^2=\frac{g}{b}(c^2-x^2)\cdot \text{ or } \frac{dx}{dt}=-\sqrt{\left\{\frac{g}{b}\right\}}\sqrt{(c^2-x^2)},$$

the negative sign shows that as time increases x decreases.

or

$$dt=-\sqrt{\left\{\frac{b}{g}\right\}}.\frac{dx}{\sqrt{(c^2-x^2)}}.$$

Integrating, $t=\sqrt{(b/g)}\cos^{-1}(x/c)+c_2$, where c_2 is constant.

Initially at C, $x=c$, $t=0$ ∴ $c_2=0$

∴ $t=\sqrt{(b/g)}\cos^{-1}(x/c)$ or $x=c\cos\{\sqrt{(g/b)}\,t\}$

∴ Required distance $=OP=a+b+x$

$$=a+b+c\cos\{\sqrt{(g/b)}\,t\}.$$

Ex. 18. *A mass of 10 lbs hanging at the end B of a light spring produces an extension of 4". A mass of 5 lbs. is suddenly added at B. Find how far B will fall before coming to rest and when it will first return to its initial position.*

Sol. Let O be the fixed point from which the spring is suspended. Let $OA=a$ be the natural length of the string. Let the

mass of 10 lbs. be at B in the position of equilibrium, such that $AB=4''$ (given). Then at B we have

$$10g=(\lambda/a).4 \text{ or } 10\times 32=4\lambda/a, \quad \because \quad g=32'/\text{sec}^2$$

or $\lambda=80a$. ...(i).

Now let a mass of 5 lbs. be attached at B. Let C be the new position of equilibrium, such that $BC=d$ inches (say).

At C, $(10+5)\, g=(\lambda/a)\,[4+d]$. (Note)

or $15\times 32=80\,[4+d]$, from (i)

or $6=4+d$ or $d=2''$ ...(ii)

Let the total mass 15 lbs. hanging from B be at P at time t such that $CP=x$.

Then at P, we have

$$15\left(\frac{d^2x}{dt^2}\right)=15g-\left(\frac{\lambda}{a}\right)(4+d+x)$$

$$=15\,(32)-80\,(6+x), \text{ from (ii)}$$

(Fig. 44)

or $$\frac{d^2x}{dt^2}=-\frac{16}{3}x. \qquad ...(iii)$$

[illegible] d integrating we get

[illegible] nstant of integration.

[illegible] $\therefore \quad C=(64/3)$

$\therefore \quad (dx/dt)^2=(16/3)\,(4-x^2).$ ...(iv)

If D be the point where the particle comes to rest, then at D, $dx/dt=0$ and x (*i.e.* CD) $=y$ (say).

$\therefore$ From (iv) we get $4-y^2=0$ or $y=2$ *i.e.* $CD=2''$ **Ans.**

Also from (iii), time from C to $D=\frac{1}{4}\,(2\pi/\sqrt{\mu})$

$$=\frac{1}{4}\cdot\left[2\pi\Big/\left\{\frac{16}{3}\right\}\right]=\frac{3\pi}{32}\text{ sec} \qquad ...(v)$$

Also from (iv) we have time from C to B

$$=\frac{3}{16}\int_{x=0}^{-2}\frac{dx}{\sqrt{(4-x^2)}} \qquad \text{(Note)}$$

$$=\frac{3}{16}\left\{\sin^{-1}(x/2)\right\}_0^{-2}=\frac{3}{16}\,[\sin^{-1}(-1)]=\frac{3}{16}\left\{\frac{3}{2}\pi\right\}$$

$=(9/32)\,\pi$ sec. ...(vi)

$\therefore$ Required time = time from B to D and back to B

$=2$ (time from D to B)

[illegible]

Exercises on Vertical Elastic Strings

Ex. 1. A light elastic string of natural length a and modulus of elasticity λ is suspended by one end, to the other end is tied a

particle of weight mg. Show that the time of a small vertical oscillation is $2\pi\sqrt{(am/\lambda)}$.

Ex. 2. In Ex 2 Page 65, show that when the string is extended to double its natural length, the tension is $m\,(4\pi^2 n^2 l - g)$.

Ex. 3. A light lastic string of natural length 4 feet is hanging vertically from one end. A mass of 6 lbs is now tied to the other end and is left. Find the time of one oscillation of the mass if the modulus of elasticity of the string is equal to 18 lbs. weight.

Ex. 4 A light elastic string of natural length a has one extremity fixed at a point O and other attached to a body of mass m. The equilibrium length of the string with body attached is a sec θ. Show that if the body be dropped from rest from O, it will come to instantaneous rest at a depth a tan θ below the position of equilibrium.

(Hint : See **Ex. 7 Page 69**).

Ex. 5. ne end of an elastic string is fixed at A, the other end is fastened a particle heavy enough to stretch the string to double its natural length. If the particle is dropped from A, find the distance it will descend before coming to rest.

Ex. 6 One end of a light elastic string of natural length a and modulus of elasticity $2mg$ is attached to fixed point O and the other end to a particle of mass m. The particle initially held at rest, is let fall. Show that the greatest extension of the string is $a\,(1+\sqrt{5})/2$ during the motion and, show that the particle will reach back O again after a time $(\pi+2-\tan^{-1} 2)\sqrt{(2a/g)}$. *(Gorakhpur 90)*

(Hint. : See **Ex. 12 Page 74**).

Ex. 7. A body is suspended from a fixed point by a light elastic string of natural length l whose modulus of elasticity is equal to the weight of the body and makes vertical osccillations of amplitude a. Show that if as the body falls through its equilibrium position it picks up another body of equal weight, the amplitude of oscillation is $(l^2+\frac{1}{2}a^2)^{1/2}$.

Ex. 8. A particle is executing S. H. M. between two points A and B. If the period of oscillation be 2π and if v be the velocity of the particle at any point P on its path, then show that

$$v^2 = AP \,.\, BP.$$

Ex. 9. 'The periodic time of a S.H.M. depends on its amplitude'. Is it true or false ?

Ex. 10. A vertical spring extends through a distance l when a given weight is attached to its lowest point. The weight is pulled down a further distance a, where $a < l$ and let go. Find the period of the S.H.M. that ensues and show that the maximum velocity of the weight is $a\sqrt{(g/l)}$.

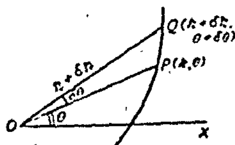
Kinematics
(Two Dimensions)

§ 1. Angular Velocity.

Definition. *If a point P moves in a plane and if O be a fixed point and OX be a fixed line through O in the plane, then the angular velocity of P about O (or of the line OP) is the rate of change of the angle XOP.*

(*Meerut 94*)

Let P be the position of the moving particle at any time t and let $\angle POX = \theta$. Let Q be the position of the particle at time $(t + \delta t)$ and let $\angle QOX = \theta + \delta\theta$.



(Fig. 1 a)

Then in time δt, the angle turned through by the particle about $O = \delta\theta$

$\therefore$ Average rate of the changing of angle about $O = \dfrac{\delta\theta}{\delta t}$.

$\therefore$ **The angular velocity of the point P** $= \lim_{\delta t \to 0} \dfrac{\delta\theta}{\delta t} = \dfrac{d\theta}{dt} = \dot{\theta}$

Similarly angular acceleration of the point P is defind as the rate of change of angular velocity of P about O

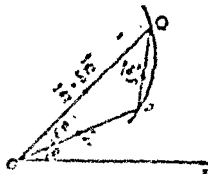
$\therefore$ **Angular acceleration of the point P** $= \dfrac{d}{dt}\left(\dfrac{d\theta}{dt}\right) = \dfrac{d^2\theta}{dt^2} = \ddot{\theta}$.

In vector notations :

The angular velocity of the point P can be proved as above to be $\dot{\theta}$

That is a vector quantity represented by a vector $\vec{\omega}$ at O, whose magnitude is $\dot{\theta}$ and direction is perpendicular to the plane POQ, such that $\vec{OP}$, $\vec{OQ}$ and $\vec{\omega}$ form a right handed system.

Also angular acceleration of the point P is given by $\dot{\vec{\omega}}$ and its magnitude is given by [illegible] $\vec{\omega}$ [illegible]



(Fig. 1 b)

§ 2. Rate of change of unit vector.

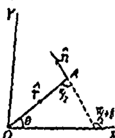
Let $\hat{\mathbf{r}}$ denote the unit vector $\overrightarrow{OA}$ such that $OA=1$ and $\angle AOX=\theta$, where OX and OY are mutually perpendicular and fixed lines in the plane.

Let $\mathbf{i}, \mathbf{j}$ be the unit vectors along OX and OY respectively

Then $\hat{\mathbf{r}}=(\cos\theta)\,\mathbf{i}+(\sin\theta)\,\mathbf{j}$

$\therefore\ \frac{d}{dt}(\hat{\mathbf{r}})=(-\sin\theta)\,\dot{\theta}\,\mathbf{i}+(\cos\theta)\,\dot{\theta}\,\mathbf{j}$

$=[(-\sin\theta)\,\mathbf{i}+(\cos\theta)\,\mathbf{j}]\,\dot{\theta}$ ...(i)



(Fig. 2)

Also if $\hat{\mathbf{n}}$ be the unit vector perpendicular to OA, then it makes an angle $\frac{1}{2}\pi+\theta$ with OX and so we have

$\hat{\mathbf{n}}=[\cos(\frac{1}{2}\pi+\theta)]\,\mathbf{i}+[\sin(\frac{1}{2}\pi+\theta)]\,\mathbf{j}$ ...(ii)

$=(-\sin\theta)\,\mathbf{i}+(\cos\theta)\,\mathbf{j}$

$\therefore$ From (i), we get $\frac{d}{dt}(\hat{\mathbf{r}})=\dot{\theta}\,(\hat{\mathbf{n}})$ (iii)

Here $\hat{\mathbf{n}}$ is in the sense in which θ increases.

Also from (ii), $\frac{d}{dt}(\hat{\mathbf{n}})=(-\cos\theta)\,\dot{\theta}\,\mathbf{i}+(-\sin\theta)\,\dot{\theta}\,\mathbf{j}$

$=-\theta\,[(\cos\theta)\,\mathbf{i}+(\sin\theta)\,\mathbf{j}]$

$\therefore$ From (i), we get $\frac{d}{dt}(\hat{\mathbf{n}})=-\dot{\theta}\,\hat{\mathbf{r}}$ ...(iv)

** § 3. Relation between angular and linear velocities.

Theorem. If v be the velocity of a point P moving in a plane curve and (r,θ) its polar coordinates referred to a fixed point O in the plane, then the angular velocity of P about O is equal to vp/r^2, where p is the perpendicular drawn from O to the tangent at P. *(Meerut 95)*

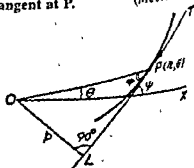
Let the radius vector OP make an angle ϕ $(=\angle OPL)$ with the tangent at P to the given curve. Then if $OL\ (=p)$ be the perpendicular from O to the tangent at P, then from $\Delta\,POL$ we get

$p=r\sin\phi.$

Also we know that

$r\frac{d\theta}{ds}=\sin\phi$ or $\frac{d\theta}{ds}=\frac{\sin\phi}{r}=\frac{p}{r^2}$ ...(i)

$(\because\ p=r\sin\phi)$



(Fig. 3 a)

$\therefore$ Angular velocity of P

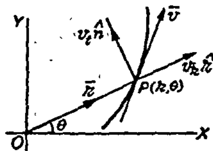
$=\frac{d\theta}{dt}=\frac{d\theta}{ds}\cdot\frac{ds}{dt}=\frac{p}{r^2}v$ ... from (i) and $v=\frac{ds}{dt}.$

Also $p = r \sin \phi$, so $\frac{pv}{r^2} = \frac{(r \sin \phi) \cdot v}{r^2} = \frac{v \sin \phi}{r}$

$\therefore$ Angular velocity of $P = \frac{vp}{r^2} = \frac{v \sin \phi}{r}$

In Vector Notations :

Let the angular velocity of the moving point P about O be $\dot{\theta}$, where θ is the angle which OP makes with the axis OX. Let $\dot{\theta} = \omega$ (say).



(Fig. 3 b)

Let the position vector of the point P referred to O be $\mathbf{r}$, then if $\vec{v}$ be the linear velocity of P we have

$$\vec{v} = \frac{d\mathbf{r}}{dt} \quad \text{...(i)}$$

Also $\mathbf{r} = r\hat{\mathbf{r}}$, where $r = |\mathbf{r}|$ and $\hat{\mathbf{r}}$ is the unit vector in the direction of OP.

From (i) we get $\vec{v} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d}{dt}(\hat{\mathbf{r}})$

or $\vec{v} = \dot{r}\hat{\mathbf{r}} + r(\dot{\theta}\hat{\mathbf{n}})$, see § 2 page 2, ...(ii)

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to OP.

From (ii) we get $\vec{v} \cdot \hat{\mathbf{n}} = (\dot{r})\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} + (r\dot{\theta})\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}$

or $\vec{v} \cdot \hat{\mathbf{n}} = r\dot{\theta}$, since $\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} = 0$ and $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$

$= r\omega, \quad \because \dot{\theta} = \omega$

or $r\omega = \vec{v} \cdot \hat{\mathbf{n}}$ ...(iii)

If the tangent at P makes an angle ϕ with the radius vector OP, then the direction of $\vec{v}$ (*i.e.* the direction of the tangent at P) makes an angle $\frac{1}{2}\pi - \phi$ with $\hat{\mathbf{n}}$.

$\therefore$ From (iii) we get the angular velocity of P

$$= \omega = \frac{\vec{v} \cdot \hat{n}}{r} = \frac{v \cdot 1 \cos(\frac{1}{2}\pi - \phi)}{r} = \frac{v \sin \phi}{r} = \frac{v(r \sin \phi)}{r^2}$$

or $\omega = vp/r^2$.

Note. The rate of change of direction of motion of the particle is $d\psi/dt$ or $\dot{\psi}$.

Solved Examples on § 1–§ 3.

Ex. 1. A body rotates with uniform angular acceleration α. If ω is the angular velocity when the body has turned through an angle θ from rest, show that $\omega^2 = 2\alpha\theta$.

Sol. Given $d^2\theta/dt^2 = \alpha$ ...(i) and $d\theta/dt = \omega$ ...(ii)

Multiplying both sides of (i) by $2\,d\theta/dt$ and integrating we get,

$(d\theta/dt)^2 = 2\alpha\theta + C$, where C is any constant.

Initially $d\theta/dt = 0$ where $\theta = 0$, $\therefore C = 0$

$\therefore (d\theta/dt)^2 = 2\alpha\theta$ or $\omega^2 = 2\alpha\theta$, from (ii)

***Ex. 2. A particle falls down a straight line $x = a$, starting from the axis of x. If the distance from the axis of x be $\frac{1}{2}ft^2$ at time t, find the angular velocity and the acceleration of the line joining the particle to the origin. How far has the particle dropped when the angular acceleration becomes zero?**

Sol. The particle starts falling from A along the line AP, whose equation is $x = a$, i.e. $OA = a$. Let P be the position of the particle at time t. Then

$AP = \frac{1}{2}ft^2$. Let $\angle POA = \theta$,

$\therefore$ In ΔPOA,

$$\tan\theta = \frac{AP}{OA} = \frac{ft^2}{2a} \qquad \text{...(i)}$$

or $\theta = \tan^{-1}(ft^2/2a)$

(Fig. 4)

Differentiating both sides of (i) with respect to t, we get angular velocity $= d\theta/dt$

$$= \frac{1}{1 + (ft^2/2a)^2} \times \left(\frac{2ft}{2a}\right) = \frac{4aft}{4a^2 + f^2t^4}$$

Differentiating again, we get angular acceleration $= d^2\theta/dt^2$

$$= \frac{(4a^2 + f^2t^4)(4af) - (4aft)(4f^2t^3)}{(4a^2 + f^2t^4)^2} = \frac{4af(4a^2 - 3f^2t^4)^2}{(4a^2 + f^2t^4)^2}$$

If angular acceleration be zero, then $4a^2 - 3f^2t^4 = 0$

or $t^4 = 4a^2/3f^2$ or $t^2 = 2a/f\sqrt{3}$.

The required distance $= AP = \frac{1}{2}ft^2$, when $t^2 = 2a/f\sqrt{3}$

$= \frac{1}{2}f[2a/(f\sqrt{3})] = a/\sqrt{3}$ Ans.

****Ex. 3. Prove that the angular acceleration of the direction of motion of a point moving in a plane is $\dfrac{v}{\rho}\dfrac{dv}{ds} = \dfrac{v^2}{\rho^2}\dfrac{d\rho}{ds}$.**

(*Bundelkhand* 91, *Lucknow* 91)

Sol. Let P be the position of the particle at time t and let the direction of motion at P (i.e. the direction of the tangent at P) make an angle ϕ with the axis of x.

$$\therefore \rho = \frac{d\psi}{ds} \text{ and } v = \frac{ds}{dt}$$

Now $\dfrac{d\psi}{dt} = \dfrac{d\psi}{ds}\,\dfrac{ds}{dt} = \dfrac{1}{\rho}v$;

$d\psi/dt = v/\rho$

Differentiating both sides with respect to t, we have

$$\frac{d^2\psi}{dt^2}=\frac{1}{\rho}\frac{dv}{dt}+v\left(-\frac{1}{\rho^2}\frac{d\rho}{dt}\right)$$

$$=\frac{1}{\rho}\frac{v\,dv}{ds}-\frac{v}{\rho^2}\frac{d\rho}{ds}\frac{ds}{dt} \qquad \because \frac{dv}{dt}=\frac{v\,dv}{ds}$$

$$=\frac{1}{\rho}\cdot\frac{v\,dv}{ds}-\frac{v^2}{\rho^2}\cdot\frac{d\rho}{ds}$$

Hence proved.

****Ex. 4. Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.**

(Rolihkhand 92)

Sol. Referred to the focus S as pole, the pedal equation of the path of the projectile (which is a parabola) is $p^2=ar$. ...(i)

Let P be any position of the projectile. Let $SP=r$, then PL, the perpendicular from P to the directrix $=SP=r$.

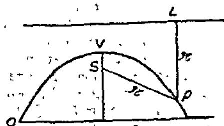
If v be the velocity of the projectile at P, then we know that the velocity of the projectile at any point of its parabolic path is that due to a fall from the level of the directrix.

$\therefore \quad v=\sqrt{(2gr)}$. ..(ii)

Also angular velocity of P

$=\dfrac{vp}{r^2}$ ... § 3 Page 2

$=\dfrac{\sqrt{(2gr)}\sqrt{(ar)}}{r^2}$, from (i) and (ii)



(Fig. 5)

$=\sqrt{(2ga)}/r$, which varies inversely as the distance of P from the focus S. Hence proved.

****Ex. 5. A particle describes a parabola with uniform speed, show that its angular velocity about the foucs, at any point P, varies inversely as $(SP)^{3/2}$.**

Sol. Let the velocity of the particle be v, which is given to be constant. Let the pedal equation of the parabola referred to focus S as pole be $p^2=ar$.

Then the required angular veldcity of any point P

$$=\frac{vp}{r^2}=\frac{v\sqrt{(ar)}}{r^2}=\frac{v\sqrt{a}}{r^{3/2}}=\frac{v\sqrt{a}}{(SP)^{3/2}}, \qquad \because SP=r$$

where v is constant.

$\therefore$ The angular velocity of P about the focus varies inversely as $(SP)^{3/2}$.

Exercises on § 1—§ 3

Ex. 1. A point moves with constant velocity in a circle, find an expression for its angular velocity about any point in the plane of the circle.

Ans. Angular velocity $= vp/r^2 = pk/r^2$.

[**Hint:** If a be the radius of the circle and b the distance of the pole from the centre, then the pedal equation of the circle is $r^2 + a^2 = 2ap + b^2$].

Ex. 2. If a point moves along a circle, prove that its angular velocity about any point on the circle is half of that about the centre.

Ex. 3. The angular velocity ω is given by **Ans.**

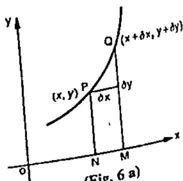
(i) θ, (ii) $\dot{r}$, (iii) $r\dot{\theta}$, (iv) $r\ddot{\theta}$

***Ex. 4.** A point is descibing a circle of radius a with velocity V. ω and ω' are its angular velocities about two points which are inverse w respect to the circle, prove that $\omega + \omega' = V/a$. (*Kumaun*

MOTION IN A PLANE CURVE

§ 4. Velocity and acceleration parallel to co-ordinate axes (cartesian).

The particle is moving on a plane and describing a curved path. To investigate the motion.



(Fig. 6 a)

Ox and Oy are the co-ordinate axes lying in the same plane in which the particle is moving. At time t and $t + \delta t$, let the positions of the particle be at $P(x, y)$ and $Q(x + \delta x, y + \delta y)$.

In moving from P to Q, the displacements parallel to Ox and Oy are δx and δy respectively. Also the time taken in the displacement $= \delta t$.

$\therefore$ The average rate of displacement parallel to x axis $= \delta x/\delta t$.

The rate of displacement parallel to x-axis $= \lim_{\delta t \to 0} \dfrac{\delta x}{\delta t} = \dfrac{dx}{dt}$

i e **The velocity at P parallel to x-axis** $= dx/dt = \dot{x}$

Similarly, **the velocity at P parallel to y-axis** $= dy/dt = \dot{y}$.

The magnitude of the resultant velocity $= \sqrt{(\dot{x}^2 + \dot{y}^2)}$ and if this resultant velocity makes an angle ψ with x-axis, then $\tan \psi = \dot{y}/\dot{x}$.

Acceleration parallel to x-axis. Let u and v be the components of velocity of the particle at P at time t and let $u + \delta u$ and $v + \delta v$ be the components of velocity at Q at time $t + \delta t$, parallel to the co-ordinate axes.

$\therefore$ Average rate of change of velocity parallel to x-axis $= \delta u/\delta t$.

$\therefore$ The rate of change of velocity parallel to x-axis

$$= \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt}$$

$\therefore$ **The acceleration at P parallel to x-axis**

$$=\frac{\mathbf{du}}{\mathbf{dt}}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{\mathbf{d^2x}}{\mathbf{dt^2}}=\ddot{x}. \ldots \qquad \because u=\frac{dx}{dt}$$

Similarly the acceleration at P parallel to y-axis $=\frac{\mathbf{d^2y}}{\mathbf{dt^2}}=\ddot{y}$.

$\therefore$ The magnitude of the resultant acceleration

$$=\sqrt{[(\ddot{x})^2+(\ddot{y})^2]}$$

and if this resultant acceleration makes an angle α with x-axis, then

$$\tan\alpha=\ddot{y}/\ddot{x}$$

Solved Examples on § 4.

Ex. 1. (a). The position of a moving point at time t given by $x=a\cos t$ and $y=a\sin t$. Find its path, velocity and acceleration. (*Meerut* 94)

Sol. Given $x=a\cos t$ .. (i) and $y=a\sin t$. ...(ii)

To obtain the equation of the path, t should be eliminated between (i) and (ii). Thus squaring and adding (i) and (ii) we get the equation of the path as $x^2+y^2=a^2$.

Also from (i) and (ii) differentiating with respect to t, we get

$$\dot{x}=-a\sin t \text{ and } \dot{y}=a\cos t. \qquad ...(i)$$

Hence required velocity $=\sqrt{(\dot{x}^2+\dot{y}^2)}=\sqrt{(a^2\sin^2 t+a^2\cos^2 t)}=a$ making an angle $\tan^{-1}(\dot{y}/\dot{x})$ *i.e.* $\tan^{-1}(-\cot t)$ or $\tan^{-1}[\tan(\frac{1}{2}\pi+t)]$ or $(\frac{1}{2}\pi+t)$ with x-axis.

Also from (ii) differentiating again with respect to t, we have

$$\ddot{x}=-a\cos t \text{ and } \ddot{y}=-a\sin t.$$

Hence required acceleration $=\sqrt{\{(\ddot{x}^2)+(\ddot{y}^2)\}}$

$$=\sqrt{(a^2\cos^2 t+a^2\sin^2 t)}=a,$$

making an angle $\tan^{-1}(\ddot{y}/\ddot{x})$ or $\tan^{-1}(\tan t)$ or t with x-axis.

Ex. 1. (b). The co-ordinates of a moving point at time t are given by $x=at^2$, $y=2at$. Find the magnitude and direction of its resultant velocity and acceleration at time t.

Sol. Given $x=at^2$, $y=2at$. ...(i)

Differentiating with respect to t, we get $\dot{x}=2at$, $\dot{y}=2a$ ...(ii)

$\therefore$ magnitude of the resultant velocity

$$=\sqrt{(\dot{x}^2+\dot{y}^2)}=\sqrt{[(2at)^2+(2a)^2]}=2a\sqrt{(t^2+1)}$$

This velocity makes with x-axis an angle

$$=\tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)=\tan^{-1}\left(\frac{2a}{2at}\right)=\tan^{-1}\left(\frac{1}{t}\right)$$

Again from (ii) differentiating with respect to t, we get.

$$\ddot{x}=2a, \ddot{y}=0. \qquad ...(ii)$$

$\therefore$ magnitude of resultant acceleration

$$=\sqrt{[(\ddot{x})^2+(\ddot{y})^2]}=\sqrt{[(2a)^2+(0)^2]}=2a$$

and this resultant makes with x-axis an angle

$$=\tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)=\tan^{-1}\left(\frac{2a}{0}\right)=\tan^{-1}\infty=\frac{1}{2}\pi.$$ Ans.

***Ex. 1. (c). A particle starts from the origin and the components of its velocity parallel to the axes of coordinates at time 't' are $2t+3$ and $4t$; find the path.**

Sol. Given $\dot{x}=2t+3$ and $\dot{y}=4t$.

To find path, t should be eliminated from these equations.

From $\dot{x}=2t+3$ or $dx=(2t+3)\,dt$

On integrating, we get $x=t^2+3t+C_1$

Initially $x=0, t=0$ (given). $\therefore C_1=0.$

$\therefore$ $x=t^2+3t.$ ...(i)

From $\dot{y}=4t$ or $dy=4t\,dt.$

On integrating, we get $y=2t^2+C_2.$

Initially $y=0, t=0$ (given). $\therefore C_2=0.$

$\therefore$ $y=2t^2.$ ...(ii)

From (ii) we get $t^2=\frac{1}{2}y$ or $t=\sqrt{(\frac{1}{2}y)}.$

Substituting this in (i), we get the required equation of the path as

$$x=[\sqrt{(\tfrac{1}{2}y)}]^2+3[\sqrt{(\tfrac{1}{2}y)}]$$

or $x=\frac{1}{2}y+3\sqrt{(\frac{1}{2}y)}$ or $2x-y=6\sqrt{(\frac{1}{2}y)}$

or $(2x-y)^2=36(\frac{1}{2}y)=18y$ or $4x^2+y^2-4xy-18y=0.$ Ans

Ex. 1. (d). If $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$ and θ increases at a uniform rate ω, prove that the velocity of the point is $a\omega\theta$ and find the inclination of the velocity to the axis of x.

Sol. Given $x=a(\cos\theta+\theta\sin\theta)$ ...(i) $y=a(\sin\theta-\theta\cos\theta)$ ...(ii)

and $d\theta/dt=\omega.$ ...(iii)

Differentiating (i) and (ii) with respect to t, we get

$\dot{x}=a(-\sin\theta+\theta\cos\theta+\sin\theta)\,\dot{\theta}=a\omega\theta\cos\theta$ ...(iv)

and $\dot{y}=a(\cos\theta+\theta\sin\theta-\cos\theta)\,\dot{\theta}=a\omega\theta\sin\theta.$ ...(v)

$\therefore$ Required velocity $=\sqrt{(\dot{x}^2+\dot{y}^2)}$

$=\sqrt{[(a\omega\theta\cos\theta)^2+(a\omega\theta\sin\theta)^2]}$

$=\sqrt{[a^2\omega^2\theta^2(\cos^2\theta+\sin^2\theta)]}=a\omega\theta.$

And inclination of direction of velocity to the axis of x

$=\tan^{-1}(\dot{y}/\dot{x})=\tan^{-1}(a\omega\theta\sin\theta/a\omega\theta\cos\theta)$

$=\tan^{-1}(\tan\theta)=\theta.$ Ans.

***Ex. 1. (e). The co-ordinates of a moving point at time t are given by $x=a(2t+\sin 2t)$, $y=a(1-\cos 2t)$; prove that its acceleration is constant and find the direction of motion at time t.**

Sol. Given $x = a(2t + \sin 2t)$ ...(i) $y = a(1 - \cos 2t)$ ..(ii)

From (i), we get

$\dot{x} = a(2 + 2\cos 2t) = 2a(1 + \cos 2t) = 2a(2\cos^2 t)$

or $\dot{x} = 4a\cos^2 t$ ...(iii)

and $\ddot{x} = 4a(-2\cos t \sin t) = -4a \sin 2t$ ...(iv)

From (ii), we get $\dot{y} = 2a \sin 2t$ ..(v)

and $\ddot{y} = 4a\cos 2t$ (vi)

$\therefore$ The acceleration of the moving point at time t

$= \sqrt{[(\ddot{x})^2 + (\ddot{y})^2]}$

$= \sqrt{[16a^2 \sin^2 2t + 16a^2 \cos^2 2t]}$, from (iv) and (vi)

$= 4a =$ constant Hence proved

Also if the direction of the motion of the particle at time t makes an angle ψ with x-axis, then

$\tan\psi = \dot{y}/\dot{x} = (2a \sin 2t)/(4a\cos^2 t)$, from (iii), (v)

$= \tan t$,

or $\psi = t$, *i e*, the direction of motion makes an angle t with x-axis.

Ex. 2 (a). A point moves in a plane, its velocities parallel to the axes of x and y being $u + ey$ and $v + ex$ respectively; show that it moves in a conic section.

Sol. Given that $dx/dt = u + ey$ and $dy/dt = v + ex$.

Dividing we get $\dfrac{dx/dt}{dy/dt} = \dfrac{u + ey}{v + ex}$ or $\dfrac{dx}{dy} = \dfrac{u + ey}{v + ex}$

or $(v + ex)\,dx = (u + ey)\,dy$.

Integrating we have $vx + \frac{1}{2}ex^2 = uy + \frac{1}{2}ey^2 + \frac{1}{2}C$,

where $\frac{1}{2}C$ is constant of integration

or $ex^2 - ey^2 + 2vx - 2uy - C = 0$, which being an equation of second degree represents a conic. Hence proved

Ex. 2 (b). Find the path of a particle whose velocities parallel to the axes of x and y are respectively $2 + 3y$ and $4 + 5x$.

[**Hint :** Do as Ex. 2 (a) above]. **Ans.** $5x^2 - 3y^2 + 8x - 4y + c = 0$.

Ex. 3 (a). A particle is moving with a constant velocity parallel to the axis of y and a velocity proportional to y parallel to the axis of x; prove that it will describe a parabola.

Sol. Given that $\dot{y} = k$ ... (i) and $\dot{x} = \lambda y$, ...(ii)

where k and λ are constants.

Integrating (i), we get $y = kt + C$, where C is any constant

Let $y = 0$ when $t = 0$. Then $C = 0$ and we get $y = kt$ ...(iii)

From (ii) $\dot{x} = \lambda k t$, $\because$ from (iii) $y = kt$

or $\dot{x} = \alpha t$, where $\alpha = \lambda k$

Integrating $x = \frac{1}{2}\alpha t^2 + B$, where B is any constant.

Let $x = 0$, when $t = 0$, then $B = 0$ and we have $x = \frac{1}{2}\alpha t^2$ ..(i

Eliminating t between (iii) and (iv), we get the equation of the path of the particle as

$x = \frac{1}{2}\alpha (y/k)^2$ or $y^2 = \mu x$, where $\mu = (2k^2/a)$, which is the standard equation of parabola. Hence prove

****Ex. 3 (b). A particle moves in a plane under a constant acceleration μa parallel to OX and an acceleration $-2\mu y$ parallel to O where OX and OY are rectangular axes. If the particle starts from re at a point $(0, a)$ find the path.**

Sol. Given $\ddot{x} = \mu a$ .. (i) and $\ddot{y} = -2\mu y$ ..(

Integrating (i), we get, $\dot{x} = \mu a t + C$, where C is constant.

Initially $\dot{x} = 0, t = 0$ and so we have $C = 0$.

$\therefore$ We have $\dot{x} = \mu a t$. .. (i

Again integrating (iii), we get, $x = \frac{1}{2}\mu a t^2 + C_1$, where C_1 is constant

Initially $t = 0, x = 0$ and so we get $C_1 = 0$.

$\therefore$ We have $x = \frac{1}{2}\mu a t^2$. .. (i

Multiplying both sides of (ii) by $2\dot{y}$, we get $2\dot{y}\ddot{y} = -4\mu y\dot{y}$.

Integrating, we get $(\dot{y})^2 = -2\mu y^2 + C_2$.

Initially $\dot{y} = 0, y = a$, so $0 = -2\mu a^2 + C_2$ or $C_2 = 2\mu a^2$.

$\therefore$ $(\dot{y})^2 = -2\mu y^2 + 2\mu a^2 = 2\mu(a^2 - y^2)$

or $\dot{y} = -\sqrt{(2\mu)}\sqrt{(a^2 - y^2)}$, negative sign is taken as $\dot{y}$ is negative

or $$\sqrt{(2\mu)}\, dt = \frac{-dy}{\sqrt{(a^2 - y^2)}}$$

Integrating we get $\sqrt{(2\mu)}\, t + C_3 = \cos^{-1}(y/a)$.

Initially $y = a, t = 0$ and so $C_3 = 0$

$\therefore$ We have $\sqrt{(2\mu)}\, t = \cos^{-1}(y/a)$

or $y = a\cos\{\sqrt{(2\mu)}\, t\}$ .. (

Eliminating t between (iv) and (v), the required equation of the pa

is $y = a\cos\sqrt{\{\sqrt{(2\mu)}\sqrt{(2x/\mu a)}\}} = a\cos(2\sqrt{(x/a)})$ A

or $y = a\cos\{2\sqrt{(x/a)}\}$.

***Ex. 3 (c). Prove that the parabola $y^2 = 4ax$ can be describ under a constant force parallel to the axis of y and a force proportion to y parallel to the axis to x.**

Sol. Since force is always proportional to the acceleration in th direction by virtue of Newton's II Law of Motion, so we are given

$\ddot{y} = k$ ...(i) and $\ddot{x} = \lambda y$, .. (

where k and λ are constants

Integrating (i) with respect to t, we get $\dot{y}=kt+C$, where C is constant.

Let $\dot{y}=0$ when $t=0$, then $C=0$ and we get $\dot{y}=kt$

Again integrating with respect to t, we get $y=\frac{1}{2}kt^2+d$, where d is constant

Let $y=0$ when $t=0$, then $d=0$ and we get $y=\frac{1}{2}kt^2$ .. (iii)

$\therefore$ From (ii), we get $\ddot{x}=\frac{1}{2}\lambda kt^2$.

Integrating w.r. to t, we get $\dot{x}=\frac{1}{6}\lambda kt^3+\mu$, where μ is constant.

Let $x=0$ when $t=0$, then $\mu=0$ and we get $\dot{x}=(1/6)\lambda kt^3$.

Again integrating w r. to t, we get $x=(1/24)\lambda kt^4+\beta$, where β is constant.

Let $x=0$ when $t=0$, then we get $\beta=0$ and we get

$$x=(1/24)\lambda kt^4. \qquad \text{...(iv)}$$

Eliminating t between (iii) and (iv) we get the equation of the path as $x=(1/24)\lambda k(2y/k)^2$ or $y^2=(6k/\lambda)x$, which is of the form $y^2=4ax$, and hence the path of the particle is the parabola $y^2=4ax$.

****Ex. 4. A particle is acted on by a force parallel to the axis of y whose acceleration (always towards the axis of x) is μy^{-2}, and when $y=a$, it is projected parallel to axis of x with velocity $\sqrt{(2\mu/a)}$; prove that it will describe a cycloid.** *(Meerut 91)*

Sol. Given that $d^2y/dt^2=\mu/y^2$ (Note) ...(i)

and $$dx/dt=\sqrt{(2\mu/a)}. \qquad \text{...(ii)}$$

There being no force parallel to x-axis, the velocity parallel to x-axis will remain constant throughout the motion.

Multiplying both sides of (i) by dy/dt and integrating, we get

$(dy/dt)^2=(2\mu/y)+C$, where C is any constant.

Initially, $dy/dt=0$ and $y=a$.

$\therefore$ $0=(2\mu/a)+C$ or $C=-(2\mu/a)$.

$$\therefore \left(\frac{dy}{dt}\right)^2=2\mu\left(\frac{1}{y}-\frac{1}{a}\right)=\frac{2\mu(a-y)}{ay}$$

or $$dy/dt=-2\mu\sqrt{[(a-y)/ay]} \qquad \text{...(iii)}$$

($-$ ve sign is due to the fact that y decreases as t increases.)

Dividing (iii) by (ii) we get $dy/dx=-\sqrt{[(a-y)/y]}$

or $\int dx=-\int\sqrt{[y/(a-y)]}\,dy=\int 2a\cos^2\theta\,d\theta$,

putting $y=a\cos^2\theta$ or $dy=-2a\cos\theta\sin\theta\,d\theta$

or $x=a\int(1+\cos 2\theta)\,d\theta=a[\theta+\frac{1}{2}\sin 2\theta]+k$,

where k is constant of integration.

or, $x=a[\theta+\frac{1}{2}\sin 2\theta]+k$

Initially $x=0$, $y=a$ or $a\cos^2\theta=a$ or $\cos\theta=1$ or $\theta=0$.

$\therefore \quad k=0$

Hence $\quad x=\frac{1}{2}a\,[2\theta+\sin 2\theta]$ ...(iv)

And $\quad y=a\cos^2\theta=\frac{1}{2}a\,[1+\cos 2\theta]$ ...(v)

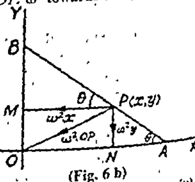
(iv) and (v) being parametric equations of a cycloid, the path is a cycloid

***Ex. 5. A rod moves with its ends sliding on rectangular axes OX and OY. If x, y are the coordinates at any instant, of a point P on the rod and if the angular velocity ω of the rod is constant, show that the component acceleration of P along the axes are $-x\omega^2$ and $-y\omega^2$, and that the resultant acceleration is $OP.\,\omega^2$ towards O.** (Kumaun 90)

Sol. Let $AB=(a+b)$ be the rod whose ends A and B are sliding on x, and y axes respectively Also let $\angle OAB=\theta$, then according to the problem $d\theta/dt=\omega$ or $\dot{\theta}=\omega$ ...(i)

Let $AP=b$, then $BP=a$, where P is a fixed point on the rod AB.

From $P(x,y)$ draw PN and PM perpendiculars to x and y-axes respectively.



(Fig. 6 b)

Then $\quad x=PM=PB\cos\theta=a\cos\theta.$ ...(ii)

And $\quad y=PN=AP\sin\theta=b\sin\theta.$ ...(iii)

From (ii) differentiating, we get $\dot{x}=-a\sin\theta\,\dot{\theta}$

or $\quad \dot{x}=-a\omega\sin\theta$, from (i)

Again differentiating, we get

$\ddot{x}=-a\omega\cos\theta\,\dot{\theta}=-a\omega^2\cos\theta$, from (i) ...(iv)

or $\quad \ddot{x}=-\omega^2x$, from (ii)

$\therefore$ acceleration parallel to x-axis $=-\omega^2x$,

i.e. $\quad \omega^2x$ is in the direction PM

Similarly from (iii), we get $\dot{y}=b\cos\theta\,\dot{\theta}$

or $\quad \dot{y}=b\omega\cos\theta$, from (i).

Again differentiating, we get $\ddot{y}=-b\omega\sin\theta\,\dot{\theta}$ ...(v)

or $\quad \ddot{y}=-b\omega^2\sin\theta$, from (i)

or $\quad \ddot{y}=-\omega^2y$, from (iii)

$\therefore$ acceleration parallel to y-axis $=-\omega^2y$

i.e. $\quad \omega^2y$ is in the direction PN

Magnitude of resultant acceleration of P

$=\sqrt{[(\ddot{x})^2+(\ddot{y})^2]}=\sqrt{[(-\omega^2x)^2+(-\omega^2y)^2]}$, from (iv), (v) (Note)

$=\omega^2\sqrt{(x^2+y^2)}=\omega^2.OP$

This resultant acceleration makes an angle

$\tan^{-1}(\ddot{y}/\ddot{x})$ or $\tan^{-1}[(-\omega^2 y)/(-\omega^2 x)]$ or $\tan^{-1}(y/x)$

i.e. $\tan^{-1}(\tan PON)$ *i.e.* $\angle PON$ with x-axis

i.e. this resultant acceleration acts along OP.

Also from the figure it is evident that the resultant acceleration viz. the resultant of acc. $\omega^2 x$ along PM and acc. $\omega^2 y$ along PN will be in the direction PO *i.e.* towards O. Hence proved

Ex. 6. A particle is acted on by a force parallel to the axis of y whose acceleration is λy, and is initially projected with a velocity $a\sqrt{\lambda}$ parallel to the axis of x at a point where $y = a$. Prove that it will describe a catenary. (*Meerut* 91 *S*)

Sol. Given $d^2y/dt^2 = \lambda y$, ...(i) and $d^2x/dt^2 = 0$, ...(ii)

Integrating (ii), we get $dx/dt = C$, where C is constant

$= a\sqrt{\lambda}$ (initially given)

Integrating again $x = a\sqrt{\lambda}\, t + k$, where k is any constant.

Let $x = 0$ when $t = 0$, $\therefore$ $k = 0$, and we have $x = a\sqrt{\lambda}\, t$. ...(iii)

Multiplying both sides of (i) by $2\, dy/dt$ and integrating, we get

$(dy/dt)^2 = \lambda y^2 + A$, where A is constant.

Initially $y = a$ and $dy/dt = 0$, $\therefore$ $0 = \lambda a^2 + A$ or $A = -\lambda a^2$

$\therefore$ $(dy/dt)^2 = \lambda(y^2 - a^2)$

or $\dfrac{dy}{dt} = \sqrt{\lambda}\sqrt{(y^2 - a^2)}$ or $dt = \dfrac{1}{\sqrt{\lambda}} \cdot \dfrac{dy}{\sqrt{(y^2 - a^2)}}$

Integrating again, $t = [1/\sqrt{\lambda}] \cosh^{-1}(y/a) + B$, where B is any constant.

Initially $t = 0$, $y = a$, $\therefore$ $B = 0$, so we have

$t = [1/\sqrt{\lambda}] \cosh^{-1}(y/a)$ or $y = a\cosh(t\sqrt{\lambda})$

or $y = a\cosh(x/a)$ since from (iii), $t\sqrt{\lambda} = (x/a)$,

which is the required equation of a catenary. Hence proved.

****Ex. 7. A particle moves in the curve $y = a \log\sec(x/a)$ in such a way that the tangent to the curve rotates uniformly; prove that the resultant acceleration of the particle varies as the square of the radius of curvature.** (*Gorakhpur* 93; *Purvanchal* 90)

Sol. The equation of the path is $y = a\log\sec(x/a)$ ...(i)

As the tangent to the curve rotates uniformly, so

$d\psi/dt = \text{constant} = \omega$ (say) ...(ii)

Differentiating (i), we get

$\dfrac{dy}{dx} = a\,\dfrac{1}{\sec(x/a)}\sec(x/a)\tan(x/a)\dfrac{1}{a}$ or $\dfrac{dy}{dx} = \tan\dfrac{x}{a}$ ...(iii)

Differentiating again $d^2y/dx^2 = (1/a)\sec^2(x/a)$,

The radius of curvature $= \dfrac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \dfrac{[1+\tan^2(x/a)]^{3/2}}{(1/a)\sec^2(x/a)}$

$= a\sec(x/a) = \rho$ ...(iv)

Also from (iii), we get $\tan\psi = dy/dx = \tan(x/a)$

or $\psi = x/a$ or $x = a\psi$.

Differentiating with respect to t, we get

$$dx/dt = a\,(d\psi/dt) = a\omega, \quad \text{...from (ii)}$$

$\therefore$ $d^2x/dt^2 = 0$, $\because$ $a\omega$ is constant.

Also $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \left(\tan\dfrac{x}{a}\right) \cdot a\omega$

$\therefore$ $\dfrac{d^2y}{dt^2} = a\omega\sec^2(x/a) \cdot \dfrac{1}{a} \cdot \dfrac{dx}{dt} = a\omega^2\sec^2\dfrac{x}{a}$.

Now resultant acceleration $= \sqrt{[(d^2x/dt^2)^2 + (d^2y/dt^2)^2]}$

$= \sqrt{[0 + a^2\omega^4\sec^4(x/a)]} = a\omega^2\sec^2(x/a) = (\omega^2/a)\,\rho^2$ .. from (iv)

Hence resultant acc. varies as square of radius of curvature.

Exercise on § 4

Ex. A particle moves in a plane with an acceleration which is always towards and perpendicular to a fixed straight line in a plane and varies inversely as the cube of the distance from it. Given the circumstances of projection, find the path.

***§ 5. Expressions for velocities and accelerations (Polar coordinates).**

Radial and Transverse Velocities. (*Avadh* 91; *Bundelkhand* 95, *Garhwal* 95; *Gorakhpur* 91; *Kanpur* 90, *Meerut* 90; *Purvanchal* 90; *Rohilkhand* 91)

If a particle moves in a plane curve and if at time t the position of particle be at $P(r, \theta)$, referred to O as pole and OX as initial line, then the resolved part of velocity at P along the radius vector OP in **the sense of r increasing is called the radial velocity and** the resolved part of the velocity at P along a line through P but at right angles to OP and in the sense in which θ increases is called the **Transverse velocity. Similarly Radial and Transverse Accelerations** are defined.

(Fig. 7 a)

Let $P(r, \theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be the positions of the particle at time t and $t+\delta t$ respectively.

The radial velocity at P

$$= \lim_{\delta t \to 0} \frac{\text{displacement along } OP \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{PN}{\delta t} = \lim_{\delta t \to 0} \frac{ON - OP}{\delta t} = \lim_{\delta t \to 0} \frac{(r + \delta r)\cos\delta\theta - r}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)[1 - (\delta\theta)^2/2! + \ldots] - r}{\delta t}, \text{ expanding } \cos\delta\theta$$

$$= \lim_{\delta t \to 0} \frac{(r+\delta r)(1) - r}{\delta t}, \text{ neglecting higher powers of } \delta\theta$$

$$= \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{dr}{dt} = \dot{r}, \text{ in the direction of } OP.$$

Ans transverse velocity at P $= \lim_{\delta t \to 0} \frac{QN}{\delta t}$

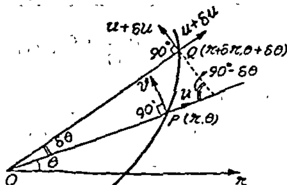
$$= \lim_{\delta t \to 0} \frac{(r+\delta r)\sin\delta\theta}{\delta t} = \lim_{\delta t \to 0} \frac{(r+\delta r)\left[\delta\theta - \frac{(\delta\theta)^3}{3!} + \ldots\right]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{r\,\delta\theta}{\delta t}, \text{ neglecting higher powers of } \delta\theta$$

$$= \frac{r d\theta}{dt} = r\dot{\theta} \text{ in the sense in which } \theta \text{ increases.}$$

****Radial and Transverse Accelerations.** (*Agra* 91; *Avadh* 92; *Bundelkhand* 95, 92, 90, *Gorakhpur* 95, 93, 91; *Kanpur* 94, 92, 90; *Lucknow* 92, 90; *Meerut* 97, 96, 90, *Purvanchal* 90, *Rohilkhand* 97, 91)

Let u and v be the radial and transverse velocities at (r, θ) and $(u + \delta u)$ and $(v + \delta v)$ be the radial and transverse velocities at



(fig. 7 b)

$Q(r + \delta r, \theta + \delta\theta)$. (In the figure write $v + \delta v$ in place of $u + \delta u$ in the direction perpendicular to OQ)

Then radial acceleration at P

$$= \lim_{\delta t \to 0} \frac{\text{change of velocity along } OP \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{[(u+\delta u)\cos\delta\theta - (v+\delta v)\cos(90^\circ - \delta\theta)] - u}{\delta t} \quad \text{(Note)}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)\cos\delta\theta - (v+\delta v)\sin\delta\theta - u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)(1) - (v+\delta v)\,\delta\theta - u}{\delta t},$$ expanding $\cos\delta\theta$, $\sin\delta\theta$ and neglecting higher powers of $\delta\theta$

$$= \lim_{\delta t \to 0} \frac{\delta u - v\delta\theta}{\delta t} = \frac{du}{dt} - v\frac{d\theta}{dt}, \text{ where } u = \frac{dr}{dt};\ v = r\frac{d\theta}{dt}$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - \left(r\frac{d\theta}{dt}\right)\cdot\frac{d\theta}{dt} = \frac{\mathbf{d^2r}}{\mathbf{dt^2}} - \mathbf{r}\left(\frac{\mathbf{d\theta}}{\mathbf{dt}}\right)^2 = \ddot{\mathbf{r}} - \mathbf{r}\dot{\theta}^2,$$

in the sense of OP *i.e.* in the sense in which r increases.

And transverse acceleration at P

$$= \lim_{\delta t \to 0} \frac{\text{change in velocity perpendicular to } OP \text{ in time } \delta t}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{[(u+\delta u)\sin\delta\theta + (v+\delta v)\sin(90^\circ - \delta\theta)] - v}{\delta t} \quad \text{(Note)}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)\sin\delta\theta + (v+\delta v)\cos\delta\theta - v}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{(u+\delta u)(\delta\theta) + (v+\delta v)(1) - v}{\delta t},$$ expanding $\sin\delta\theta$, $\cos\delta\theta$ and neglecting higher powers of $\delta\theta$

$$= \lim_{\delta t \to 0} \frac{u\,\delta\theta + \delta v}{\delta t} = \frac{u\,d\theta}{dt} + \frac{dv}{dt}, \text{ where } u = \frac{dr}{dt};\ v = r\frac{d\theta}{dt}$$

$$= \frac{dr}{dt}\cdot\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) = \frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\cdot\frac{dr}{dt}$$

$$= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt} = \frac{\mathbf{1}}{\mathbf{r}}\frac{\mathbf{d}}{\mathbf{dt}}\left(\mathbf{r}^2\frac{\mathbf{d\theta}}{\mathbf{dt}}\right) = \frac{\mathbf{1}}{\mathbf{r}}\frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{r}^2\dot{\theta})$$ in the sense *in* which θ increases.

In vector notations :

Refer Fig. 3 (b) Page 3 of this chapter.

As in § 3 Pages 2-3 of this chapter we can prove that if $\mathbf{v}$ be the linear velocity vector at P, θ be the angle which OP makes with the initial line OX and $\mathbf{r}$ be the position vector of P referred to O, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\,\hat{\mathbf{r}}),$$ where $r = |\mathbf{r}|$ and $\hat{\mathbf{r}}$ is the unit vector in the direction of OP.

or

$$\mathbf{v} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d}{dt}(\hat{\mathbf{r}})$$

$= \dot{r}\,\hat{\mathbf{r}} + r\,(\dot{\theta}\,\hat{\mathbf{n}})$, where $\hat{\mathbf{n}}$ is the unit vector perpendicular to OP.

or $\quad \mathbf{v} = (\dot{r})\,\hat{\mathbf{r}} + (r\,\dot{\theta})\,\hat{\mathbf{n}}$ ...(i)

From (i) we conclude that the components of the linear velocity $\mathbf{v}$ at P in the directions of OP and perpendicular to OP *i.e.* in the directions of $\hat{\mathbf{r}}$ and $\hat{\mathbf{n}}$ are $\dot{r}$ and $r\dot{\theta}$ respectively. *i.e.* the radial and transverse velocities of the particle at P are $\dot{r}$ and $r\dot{\theta}$ respectively. Again differentiating (i) with respect to t we get

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\dot{r}\,\hat{\mathbf{r}}) + \frac{d}{dt}(r\,\dot{\theta}\,\hat{\mathbf{n}})$$

or $$\mathbf{a} = [\ddot{r}\,(\hat{\mathbf{r}}) + \dot{r}\,\frac{d}{dt}(\hat{\mathbf{r}})] + [\dot{r}\,\dot{\theta}\hat{\mathbf{n}} + r\,\ddot{\theta}\,\hat{\mathbf{n}} + r\,\dot{\theta}\,\frac{d}{dt}(\hat{\mathbf{n}})],$$

where $\mathbf{a}$ is acceleration vector of the particle at P.

or $\quad \mathbf{a} = \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\dot{\theta}\,\hat{\mathbf{n}} + \dot{r}\,\dot{\theta}\,\hat{\mathbf{n}} + r\,\ddot{\theta}\,\hat{\mathbf{n}} + r\,\dot{\theta}\,(-\dot{\theta}\hat{\mathbf{r}})$, from § 2 Page 2

or $$\mathbf{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\hat{\mathbf{r}} + (2\dot{\theta} + r\,\ddot{\theta})\,\hat{\mathbf{n}} \quad \text{.. (ii)}$$

From (ii) we conclude that the components of acceleration vector $\mathbf{a}$ in the radial and transverse directions *i.e.* in the directions of $\hat{\mathbf{r}}$ and $\hat{\mathbf{n}}$ are $\ddot{r} - r\,\dot{\theta}^2$ and $2\dot{\theta} + r\,\ddot{\theta}$.

Solved Examples on § 5.

Ex. 1. (a). If the curve is equiangular spiral $r = ae^{\theta \cot \alpha}$, and if the radius vector to the particle has constant angular velocity, show that the resultant acceleration of the particle makes an angle 2α with the radius vector and is of magnitude v^2/r, where v is the speed of the particle. *(Lucknow 90)*

Sol. The curve is given by $r = ae^{\theta \cos \alpha}$. .. (i)

Also we are given angular velocity $\dot{\theta} = \text{constant} = \omega$ (say) ...(ii)

From (i), we get $\dot{r} = a \cot \alpha \,.\, e^{\theta \cot \alpha}\,\dot{\theta} = r\omega \cot \alpha$, from (ii) and (i)

Again differentiating, $\ddot{r} = \dot{r}\omega \cot \alpha = r\omega^2 \cot^2 \alpha$, $\because \dot{r} = r\omega \cot \alpha$

$\therefore$ Radial acceleration $= \ddot{r} - r\dot{\theta}^2 = (r\omega^2 \cot^2 \alpha) - r\omega^2$.

$$= r\omega^2 (\cot^2 \alpha - 1) \quad \text{..(iii)}$$

And transverse acceleration

$$= \frac{1}{r}\frac{d}{dt}(r^2\,\dot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2\omega), \text{ from (ii)}$$

$$= (1/r)\,(2r\,\omega)\,\dot{r} = 2\omega\dot{r} = 2\omega\,(r\omega \cot \alpha) = 2r\,\omega^2 \cot \alpha. \quad \text{.. (iv)}$$

If β be the angle which the resultant acceleration makes with the radius vector then

$$\tan \beta = \frac{\text{transverse acc.}}{\text{radial acc}} = \frac{2r\omega^2 \cot \alpha}{r\omega^2 (\cot^2 \alpha - 1)}, \text{ from (iii) and (iv)}$$

$$= \frac{(2 \cos \alpha/\sin \alpha) \sin^2 \alpha}{(\cos^2 \alpha - \sin^2 \alpha)} = \frac{2 \cos \alpha \sin \alpha}{\cos 2\alpha} = \frac{\sin 2\alpha}{\cos 2\alpha} = \tan 2\alpha$$

or $$\beta = 2\alpha$$ Hence proved.

Also speed of the particle $= \sqrt{[(\dot{r})^2 + (r\dot{\theta})^2]}$ ...(v)

$= \sqrt{[(r\omega \cot \alpha)^2 + (r\omega)^2]} = r\omega \operatorname{cosec} \alpha = v$ (given)

Again from (iii) and (iv), resultant acceleration

$= \sqrt{[(\text{radial acc.})^2 + (\text{transverse acc.})^2]}$

$= \sqrt{[\{r^2\omega^4 (\cot^2 \alpha - 1)^2\} + \{4r^2\omega^4 \cot^2 \alpha\}]}$

$= \sqrt{(r^2\omega^4 \operatorname{cosec}^4 \alpha)}$, on simplifying

$= r\omega^2 \operatorname{cosec}^2 \alpha = v^2/r$, from (v) Hence proved

Ex. 1 (b). A point P describes an equiangular spiral $r = ae^{\theta \cot \alpha}$ with constant angular velocity about the pole O. Find its acceleration and show its direction makes the same angle with the tangent at P as the radius vector OP makes with the tangent.

Hint. Proceed as in Ex. 1 (a) above and find angle β and acceleration. Then if ϕ be the angle which the radius vector OP makes with the tangent, then we know that $\tan \phi = r d\theta/dr$. (See **Author's Differential Calculus**)

$$\tan \phi = \frac{r\,(d\theta/dt)}{(dr/dt)} = \frac{r\dot{\theta}}{\dot{r}} = \frac{r\omega}{r\omega \cot \alpha} = \tan \alpha \text{ or } \phi = \alpha.$$

Also the angle which the direction of the acceleration makes with the radius vector $= 2\alpha$, as proved in Ex. 1. (a) above

$\therefore$ angle which direction of acceleration makes with the tangent at P.

$$= 2\alpha - \alpha = \alpha = \phi$$ Hence proved.

[Note. Students should draw the figure].

***Ex. 1 (c). A point P describes, with constant angular velocity about O, the equiangular spiral $r = ae^{\theta}$, O being the pole of the spiral. Obtain the radial and transverse accelerations of P.**

(Bundelkhand 92, 91; Kanpur 96)

Sol. The aquation of the curve is $r = ae^{\theta}$. ...(i)

Also angular velocity $= \dot{\theta} = \text{constant} = \omega$ (say) ...(ii)

From (i), we get $\dot{r} = ae^{\theta}\,\dot{\theta} = ae^{\theta}\,.\,\omega$ from (ii)

or $\dot{r} = r\omega$, $\because$ from (i) $r = ae^{\theta}$.

Differentiating again, we get $\ddot{r} = \omega\dot{r} = \omega\,(\omega r)$ ... $\because \dot{r} = \omega r$

$\therefore$ Radial acceleration $= \ddot{r} - r\dot{\theta}^2 = \omega^2 r - r\,(\omega^2) = 0.$ Ans.

And the transverse acceleration

$= \dfrac{1}{r}\dfrac{d}{dt}\left(r^2 \dfrac{d\theta}{dt}\right) = \dfrac{1}{r}\dfrac{d}{dt}(r^2\omega)$ ...from (ii)

$= (1/r)\, 2r\omega\, (dr/dt) = 2\omega\, (r\omega)$ ...$\because dr/dt = r\omega$

$= 2\omega^2 r$ which varies as radius vector r i.e. distance from the pole. Ans.

***Ex. 1 (d). A particle describes an equiangular spiral $r = ae^{n\theta}$ with constant angular velocity. Find its velocity and acceleration.**

(Kanpur 91)

Sol. Given curve is $r = ae^{n\theta}$ ...(i)

Also angular velocity $= \dot{\theta} =$ constant $= \omega$ (say) ...(ii)

From (i), we get $\dot{r} = ame^{m\theta}\,\dot{\theta} = mr\omega$, from (i), (ii)

$\therefore$ radial velocity $= \dot{r} = m\omega r$ ...(iii)

Also transverse velocity $= r\dot{\theta} = r\omega$, from (ii)

$\therefore$ Resultant velocity $= \sqrt{[(\dot{r})^2 + (r\dot{\theta})^2]} = \sqrt{[(m\omega r)^2 + (\omega r)^2]}$

$= [\sqrt{(m^2+1)}]\,\omega r.$ **Ans.**

Differentiating (iii), we get $\ddot{r} = m\omega\dot{r} = m\omega\,(m\omega r)$, from (iii)

$$\ddot{r} = m^2\omega^2 r$$

$\therefore$ Radial acceleration $= \ddot{r} - r\dot{\theta}^2 = m^2\omega^2 r - r\omega^2$

$= \omega^2 (m^2 - 1)\, r$ **Ans.**

And transverse acceleration

$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2\omega)$, from (ii)

$= (1/r)\,(2r\omega)\,\dot{r} = 2\omega\,(m\omega r)$. $\quad \because \dot{r} = mr\omega$

$= 2m\omega^2 r$

$\therefore$ Resultant acceleration

$= \sqrt{[\{\omega^2(m^2-1)\,r\}^2 + \{2m\omega^2 r\}^2]}$

$= \omega^2 r\sqrt{[(m^2-1)^2 + 4m^2]} = \omega^2 r\sqrt{(m^4 + 2m^2 + 1)}$

$= \omega^2 r\sqrt{[(m^2+1)^2]} = \omega^2 r\,(m^2+1)$ **Ans.**

Ex. 1 (e). A particle describes an equiangular spiral $r = ae^{\theta}$ in such a manner that its acceleration has no radial component. Prove that its angular velocity is constant and that magnitude to the velocity and acceleration is each proportional to r.

Sol. The given curve is $r = ae^{\theta}$. ...(i)

From (i), we get $\dot{r} = ae^{\theta}\,\dot{\theta} = r\,\dot{\theta}$, from (i)

And $\quad \ddot{r} = \dot{r}\dot{\theta} + r\ddot{\theta} = r\dot{\theta}^2 + r\ddot{\theta}$, $\quad ...\because \dot{r} = r\dot{\theta}$.

Also we are given radial acceleration $= 0$

or $\quad \ddot{r} - r\dot{\theta}^2 = 0 \quad$ or $\quad r\dot{\theta}^2 + r\ddot{\theta} - r\dot{\theta}^2 = 0$, from (ii)

or $\quad r\ddot{\theta} = 0 \quad$ or $\quad \ddot{\theta} = 0, \quad \because r \neq 0$

or $\quad \dot{\theta} =$ constant $= \omega$ (say). Hence proved.

Again transverse acceleration $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$, where $\dot{\theta} = \omega$

$= \frac{1}{r}\frac{d}{dt}(r^2\omega) = \frac{\omega}{r}\frac{d}{dt}(r^2) = \frac{\omega}{r}(2r\dot{r}) = 2\omega\dot{r}$

$= 2\omega\,(r\dot{\theta})$, $\quad \because \dot{r} = r\dot{\theta}$

$= 2\omega^2 r$, $\quad \because \dot{\theta} = \omega$.

$\therefore$ Resultant acceleration = Transverse acceleration, $\quad \because$ Radial acc. is zero.

$= 2\omega^2 r \propto r.$

Also resultant velocity $= \sqrt{[(\dot{r})^2 + (r\dot{\theta})^2]} = \sqrt{(r\dot{\theta})^2 + (r\dot{\theta})^2}$

$= \sqrt{[2r^2\omega^2]}$ as $\dot{\theta} = \omega$

$= (\sqrt{2})\,\omega r \propto r.$ Hence proved

Ex. 1 (f). A particle moves along a circle $r = 2a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Show that the transverse acceleration varies as the fifth power of cosec θ. (*Meerut* 94; *Rohilkhand* 95)

Sol. The given curve is $r = 2a\cos\theta$. ...(i)

Differentiating with respect to t, we get $\dot{r} = -2a\sin\theta\,\dot{\theta}$ ...(ii)

Again differentiating, we get

$\ddot{r} = -2a\,[(\sin\theta)\,\ddot{\theta} + (\cos\theta)\,\dot{\theta}^2]$ ...(iii)

If the acceleration towards the origin is always zero, then radial acceleration is zero. (Note)

i.e., $\ddot{r} - r\dot{\theta}^2 = 0$

or $[-2a(\sin\theta)\,\ddot{\theta} - 2a(\cos\theta)\,\dot{\theta}^2] - (2a\cos\theta)\,\dot{\theta}^2 = 0$, from (i), (iii) (Note)

or $(\sin\theta)\,\ddot{\theta} + (2\cos\theta)\,\dot{\theta}^2 = 0$ or $\dfrac{\ddot{\theta}}{\dot{\theta}} + \left(\dfrac{2\cos\theta}{\sin\theta}\right)\dot{\theta} = 0$

Integrating, $\log\dot{\theta} + 2\log(\sin\theta) = \log c$, where c is constant

or $\dot{\theta}(\sin\theta)^2 = c$ or $\dot{\theta} = c\,\mathrm{cosec}^2\,\theta$ ...(iv)

∴ From (i) and (iv) transverse acceleration of the particle

$= \dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = \dfrac{1}{2a\cos\theta}\dfrac{d}{dt}(4a^2\cos^2\theta \,.\, c\,\mathrm{cosec}^2\,\theta)$

$= 2ac\sec\theta\,\dfrac{d}{dt}(\cot^2\theta) = 2ac\sec\theta\,(-2\cot\theta\,\mathrm{cosec}^2\,\theta\,.\,\dot{\theta})$

$= -4ac\,(\mathrm{cosec}^3\,\theta)\,\dot{\theta} = -4ac\,\mathrm{cosec}^3\,\theta\,(c\,\mathrm{cosec}^2\,\theta)$, from (iv)

$= -4ac^2\,\mathrm{cosec}^5\,\theta \propto \mathrm{cosec}^5\,\theta$ Hence proved.

Ex. 1. (g). A particle moves along a circle $r = 2a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Prove that $\dfrac{d^2\theta}{dt^2} = -2\cot\theta\left(\dfrac{d\theta}{dt}\right)^2$. (*Avadh* 94; *Garhwal* 96, 93; *Meerut* 96 P; *Rohilkhand* 97, 93)

Sol. As in Ex. 1 (f) result (iv) above we can prove that the angular velocity $d\theta/dt$ is given by

$d\theta/dt = c\,\mathrm{cosec}^2\,\theta$ ...(A)

∴ $d^2\theta/dt^2 = -2c\,\mathrm{cosec}\,\theta\,(\mathrm{cosec}\,\theta\cot\theta)\,\dot{\theta} = -2\,(c\,\mathrm{cosec}^2\,\theta)^2\cot\theta$

$= -2\,(d\theta/dt)^2\cot\theta$, from (A) Hence proved

Ex. 2. The velocities of a particle along and perpendicular to the radius vector are λr and $\mu\theta$; find the path and show that the

ccelerations along and perpendicular to the radius vector are $\lambda^2 r - \mu^2\theta^2/r$ and $\mu\theta(\lambda + \mu/r)$. (*Bundelkhand* 96, 95, 90, *Meerut* 96 BP; *Gorakhpur* 97; *Rohilkhand* 93)

Sol. Given that $\dot{r} = \lambda r$ ...(i) and $r\dot{\theta} = \mu\theta$ ...(ii)

Differentiating (i) we get $\ddot{r} = \lambda\dot{r} = \lambda^2 r$. ...(iii)

Dividing (i) by (ii) we get $\dfrac{dr}{r\,d\theta} = \dfrac{\lambda r}{\mu\theta}$ or $\dfrac{\mu\,dr}{\lambda r^2} = \dfrac{d\theta}{\theta}$

Integrating, we get $-\mu/\lambda r = \log\theta - \log c$, where $\log c$ is constant of integration

or $-\mu/\lambda r = \log(\theta/c)$ or $\theta = ce^{-\mu/\lambda r}$, is the equation of the path.

Now radial acceleration $= \ddot{r} - r\dot{\theta}^2$.

$= \lambda^2 r - r(\mu\theta/r)^2$, from (ii) and (iii)

$= \lambda^2 r - (\mu^2\theta^2/r)$.

And transverse acceleration

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}\left(r^2 \cdot \frac{\mu\theta}{r}\right), \text{ from (ii)}$$

$$= \frac{1}{r}\frac{d}{dt}(\mu r\theta) = \frac{\mu}{r}\left[r\frac{d\theta}{dt} + \theta\frac{dr}{dt}\right]$$

$= (\mu/r)[\mu\theta + \theta\lambda r]$ ...from (i) and (ii)

$= \mu\theta[(\mu/r) + \lambda]$ Hence proved.

Ex. 3. The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$, where λ and μ are constants, find the polar equations of the path of the particle and also its radial and transverse accelerations in terms of r and θ only. (*Agra* 92, *Avadh* 92; *Bundelkhand* 91; *Garhwal* 90; *Kanpur* 95, 92; *Meerut* 95, 92)

Sol. Given $\dot{r} = \lambda r^2$ .. (i) and $r\dot{\theta} = \mu\theta^2$ ...(ii)

Differentiating (i), we get $\ddot{r} = 2\lambda r\dot{r} = 2\lambda r(\lambda r^2)$, from (i)

or $\ddot{r} = 2\lambda^2 r^3$ ...(iii)

$\therefore$ Radial acceleration of the particle $= \ddot{r} - r\dot{\theta}^2$

$= 2\lambda^2 r^3 - r(\mu\theta^2/r)^2$, from (ii) and (iii)

$= 2\lambda^2 r^3 - \mu^2\theta^4/r$. Ans.

And transverse acceleration of the particle

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{r}\frac{d}{dt}\left[r^2 \cdot \frac{\mu\theta^2}{r}\right], \text{ from (ii)}$$

$$= \frac{\mu}{r}\frac{d}{dt}[r\theta^2] = \frac{\mu}{r}\left[2r\theta\frac{d\theta}{dt} + \theta^2\frac{dr}{dt}\right]$$

$= (\mu/r)[2\mu\theta^3 + \lambda r^2\theta^2]$, from (i) and (ii)

$= (2\mu^2\theta^3/r) + \lambda\mu r\theta^2$. Ans.

Also dividing (i) by (ii), we get $\frac{dr}{r\,d\theta}=\frac{\lambda r^2}{\mu\theta^2}$ or $\frac{\mu\,dr}{r^3}=\lambda\frac{d\theta}{\theta^2}$.

Integrating, $-\mu/2r^2-c=-\lambda/\theta$, where c is const of integration

$\therefore$ The equation of the path of the particle is $\frac{\mu}{2r^2}+c=\frac{\lambda}{\theta}$.

***Ex. 4. If the angular velocity of a point moving in a plane curve be constant about a fixed origin, show that the transverse acceleration varies as its radial velocity.**

Sol. Given that angular velocity $=\dot{\theta}=$ cosntant $=\omega$ (say)

Now transverse acceleration of the point

$$=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{1}{r}\frac{d}{dt}(r^2\omega),\quad\because\frac{d\theta}{dt}=\omega$$

$$=(1/r)\,2r\dot{r}\omega=2\omega\,(\dot{r})=2\omega\text{ (radial velocity).}$$

Hence transverse acceleration varies as the radial velocity

****Ex. 5. If the radial and transvers velocities of a particle are always proportional to each other.**

(s) Show that the path is an equiangular spiral. *(Avadh 90, Gorakhpur 96)*

(b) If in addition, the radial and transverese accelerations are always proportional to each other; show that the velocity of the particle varies as some power of the radius vector. *(Avadh 95; Gorakhpur 90; Rohilkhand 94)*

Sol. (a) Given that $\dot{r}=k\,(r\dot{\theta})$, where k is some constant ...(i)

or $dr=k\,\dot{r}\,d\theta$ or $(1/r)\,dr=k\,d\theta$.

Integrating, $\log r-\log c=k\theta$, where $\log c$ is constant of integration or $\log(r/c)=k\theta$ or $r=ce^{k\theta}$, which is the equation of the path and also that of an equiangular spiral.

(b) If in addition to (i), we have

radial acceleration $=\lambda$ (transverse acceleration), where $\lambda=$ const

then $$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\lambda\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

or $$\frac{d^2r}{dt^2}-r\left[\frac{1}{kr}\frac{dr}{dt}\right]^2=\frac{\lambda}{r}\frac{d}{dt}\left[r^2\cdot\frac{1}{kr}\frac{dr}{dt}\right],\text{ from (i)}$$

or $$\frac{d^2r}{dt^2}-\frac{1}{k^2r}\left(\frac{dr}{dt}\right)^2=\frac{\lambda}{kr}\frac{d}{dt}\left[r\cdot\frac{dr}{dt}\right]=\frac{\lambda}{kr}\left[r\frac{d^2r}{dt^2}+\frac{dr}{dt}\cdot\frac{dr}{dt}\right]$$

or $$\left(1-\frac{\lambda}{k}\right)\frac{d^2r}{dt^2}=\frac{1}{r}\left(\frac{1}{k^2}+\frac{\lambda}{k}\right)\left(\frac{dr}{dt}\right)^2$$

or $$A\frac{d^2r}{dt^2}=\frac{B}{r}\left(\frac{dr}{dt}\right)^2,$$ where A and B are constants.

or $\frac{d^2r}{dt^2} = \frac{\mu}{r}\left(\frac{dr}{dt}\right)^2$, where $\mu = B/A$

or $\frac{d^2r}{dt^2} \Big/ \left(\frac{dr}{dt}\right) = \frac{\mu}{r}\frac{dr}{dt}$ (Note)

Integrating, $\log(dr/dt) = \mu \log(r) + \log D$, where D is constant

or $\dot{r} = dr/dt = r^{\mu} . D$

∴ Substituting this value of $\dot{r}$ in (i), we get $r^{\mu} . D = \lambda r \dot{\theta}$

or $r\dot{\theta} = \alpha . r^{\mu}$, where $\alpha = D/\lambda$.

∴ The resultant velocity

$= \sqrt{[(\dot{r})^2 + (r\dot{\theta})^2]} = \sqrt{[(r^{\mu} . D)^2 + (\alpha r^{\mu})^2]} = r^{\mu}\sqrt{(D^2 + \alpha^2)}$.

∴ Velocity varies as some power of radius vector.

***Ex. 6 (a). A particle P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as its distance from O; show that the curve is an equiangular spiral, and that the acceleration of P along the normal varies inversely as OP.**

Sol. Given that

Velocity of the particle is constant and equal to v (say)

or $\sqrt{[(\dot{r})^2 + (r\dot{\theta})^2]} = v$ or $(\dot{r})^2 + (r\dot{\theta})^2 = v^2$. ...(i)

And angular velocity $= \dot{\theta} = \lambda/r$, ...(ii)

where λ is some constant.

∴ From (i) and (ii), we get $(\dot{r})^2 + \lambda^2 = v^2$

or $(\dot{r})^2 = v^2 - \lambda^2 = \mu^2$ (constant), ∵ v and λ are constants

or $\dot{r} = \mu$. ...(iii)

Dividing (iii) by (ii), we get $\frac{dr}{d\theta} = \frac{\mu r}{\lambda} = kr$, where $k = \frac{\mu}{\lambda}$

or $(1/r)\,dr = k\,d\theta$

Integrating, $\log r = \log C + k\theta$, where C is any constant

or $\log(r/C) = k\theta$ or $r = Ce^{k\theta}$, which is the standard equation of an equiangular spiral.

(For the second part see Examples on next article)

****Ex. 6 (b). A particle describes the curve $r = ae^{m\theta}$ with a constant velocity. Find the components of velocity and acceleration along the radius vector and perpendicular to it.** (Meerut 92 P)

Sol. Given the path of the particle as $r = ae^{m\theta}$ ...(i)

Also velocity of the particle = constant = v (say)

i.e. $\sqrt{[(\dot{r})^2 + (r\dot{\theta})^2]} = v$ or $(\dot{r})^2 + (r\dot{\theta})^2 = v^2$ ...(ii)

From (i), we get $\dot{r} = ame^{m\theta}\dot{\theta}$, on differentiating

$= mr\dot{\theta}$, from (i)

or $r\dot{\theta} = (1/m)\dot{r}$ ...(iii)

$\therefore$ From (ii) and (iii), we get $(\dot{r})^2 + (\dot{r}/m)^2 = v^2$

or $\dot{r}^2 [1 + (1/m)^2] = v^2$ or $\dot{r}^2 = v^2 m^2/(m^2 + 1)$

or $\dot{r} = vm/\sqrt{(m^2 + 1)}$ ...(iv)

$\therefore$ From (iii) we get $r\dot{\theta} = v/\sqrt{(m^2 + 1)}$ ...(v)

$\therefore$ Component of velocity along and perpendicular to the radius vector viz. $\dot{r}$ and $r\dot{\theta}$ are given by (iv) and (v). Ans.

Again from (iv) on differentiating, we get ...(vi)

$\ddot{r} = 0$, since m, v are constants.

$\therefore$ Components of acceleration along the radius vector

$= "\ddot{r} - r\dot{\theta}^2" = -r\dot{\theta}^2$, from (vi)

$= -(1/r)\{v/\{\sqrt{(m^2 + 1)}\}\}^2$, from (v)

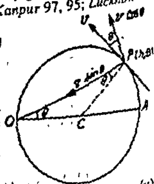
$= -v^2/\{r(m^2 + 1)\}$ Ans.

And component of acc. perpendicular to the radius vector

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{d}{dt}\left(\frac{vr}{\sqrt{(m^2 + 1)}}\right), \text{ from (v)}$$

$$= \frac{v\dot{r}}{r\sqrt{(m^2 + 1)}} = \frac{v^2 m}{r(m^2 + 1)}, \text{ from (iv)}$$ Ans.

****Ex. 7 (a). A point describes a circle of radius a with a uniform speed v; show that the radial and transverse accelerations are $-(v^2/a)\cos\theta$ and $-(v^2/a)\sin\theta$, if a diameter is taken as initial line and one end of this diameter as pole.** *(Agra 90, Gorakhpur 91; Kanpur 97, 95; Lucknow 92)*



(Fig. 8)

Sol. Let C be the centre of the circle.

Take O, any point on the circle, as pole and the diameter OA as initial line.

Then the polar equation of the circle is

$r = 2a\cos\theta$. ....(i)

Let $P(r, \theta)$ be any position of the particle. The velocity v of the particle acts along the tangent at P.

Its resolved parts along OP and perpendicular to OP are $-v\sin\theta$ and $v\cos\theta$ (as shown in the figure)

$\therefore$ Radial velocity $= \dot{r} = -v\sin\theta$ ...(ii)

And transverse velocity $= r\dot{\theta} = v\cos\theta$. ...(iii)

Differentiating (ii) we get $\ddot{r} = -(v\cos\theta)\dot{\theta}$

$= -v\cos\theta\,[(v\cos\theta)/r]$, from (iii)

or $\ddot{r} = -(v^2\cos^2\theta)/r$ ...(iv)

$\therefore$ Radial acceleration at $P = \ddot{r} - r\dot{\theta}^2$

$$= -\frac{v^2\cos^2\theta}{r} - r\left(\frac{v\cos\theta}{r}\right)^2, \text{ from (iii) and (iv)}$$

$= -\dfrac{2v^2\cos^2\theta}{r} = -\dfrac{2v^2\cos^2\theta}{2a\cos\theta}$, $\because$ from (i) $r = 2a\cos\theta$

$= -(v^2\cos\theta)/a$.

And transverse acceleration at P

$= \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) = \dfrac{1}{r}\dfrac{d}{dt}\left[r^2\left(\dfrac{v\cos\theta}{r}\right)\right]$, from (iii)

$= \dfrac{v}{r}\dfrac{d}{dt}[r\cos\theta] = \dfrac{v}{r}\left[\cos\theta\dfrac{dr}{dt} - r\sin\theta\dfrac{d\theta}{dt}\right]$

$= (v/r)\,[\cos\theta\,(-v\sin\theta) - r\sin\theta\,\{(v\cos\theta)/r\}]$, from (ii) and (iii)

$= -\dfrac{2v^2\sin\theta\cos\theta}{r} = -\dfrac{2v^2\sin\theta\cos\theta}{2a\cos\theta}$, from (i)

$= -(v^2\sin\theta)/a$. Hence proved

Ex. 7. (b). A particle moves in a circular path of radius a so that its angular velocity about a fixed point in the circumference is constant and equal to ω. Show that the resultant acceleration of the particle at every point is of constant magnitude $4a\omega^2$.

Sol. Refer Fig. 8 above.

Let C be the centre of the given circular path of radius a.

Take any point O on its circumference as pole and the line OC as initial line. Then the polar equation of the circle is $r = 2a\cos\theta$. .. (i)

Let $P(r, \theta)$ be the position of the particle at time t.

Given angular velocity about $O = \dot{\theta} = \omega$ (constant) ..(ii)

From (i), we get $\dot{r} = (-2a\sin\theta)\,\dot{\theta} = -2a\omega\sin\theta$, from (ii) .. (iii)

Again differentiating, $\ddot{r} = (-2a\omega\cos\theta)\,\dot{\theta}$

$= -2a\omega^2\cos\theta$, from (ii)

or $\ddot{r} = -\omega^2 r$, from (i)

$\therefore$ Radial acceleration of the particle at P

$= \ddot{r} - r\dot{\theta}^2 = (-\omega^2 r) - (r\omega^2) = -2\omega^2 r = -4a\omega^2\cos\theta$, from (i)

And transverse acceleration of the particle at P

$= \text{"}\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta})\text{"} = \dfrac{1}{r}\dfrac{d}{dt}(r^2\omega)$, from (ii)

$= (1/r)(2r\dot{r}\omega) = 2\omega\dot{r} = 2\omega(-2a\omega\sin\theta)$, from (iii)

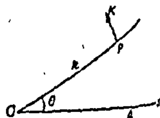
$= -4a\omega^2\sin\theta$

$\therefore$ Resultant acceleration at P

$= \sqrt{[(-4a\omega^2\cos\theta)^2 + (-4a\omega^2\sin\theta)^2]} = 4a\omega^2$.

****Ex. 8 (a). A ring which can slide on a thin long smooth rod rests at a distance d from one end O. The rod is then set revolving uniformly about O in a horizontal plane, show that in space the ring describes the curve $r = d\cosh\theta$.**

Sol. Let Ox be the initial position of the rod and A that of the ring Then $OA = d$



Let after time t, the position of the ring be P, such that $\angle POx = \theta$ and $OP = r$. The only force acting on the ring is the normal reaction between the rod and the ring acting in the transverse sense at P. The weight of the ring is inffective as the rod revolves in a horizontal plane.

(Fig. 9)

$\therefore$ There being no force in the radial sense, we get

$$\ddot{r} - r(\dot{\theta})^2 = 0. \quad ...(i)$$

Also as the rod revolves uniformly, so

$$\dot{\theta} = \text{constant} = \omega \text{ (say)} \quad ...(ii)$$

$\therefore$ From (i), $\ddot{r} = r\omega^2$

Multiplying both sides by $2\dot{r}$ and integrating, we have

$(\dot{r})^2 = r^2\omega^2 + C$, where C is constant of integration

Initially $r = d$ and $\dot{r} = 0$ so $0 = d^2\omega^2 + C$ or $C = -d^2\omega^2$.

$\therefore$ $(\dot{r})^2 = r^2\omega^2 - d^2\omega^2 = \omega^2(r^2 - d^2)$.

or $$\frac{dr}{\sqrt{(r^2 - d^2)}} = \omega\, dt = d\theta, \text{ from (ii)}$$

Integrating $\cosh^{-1}(r/d) = \theta + A$, where A is constant of integration

Initially $r = d$ and $\theta = 0$, $\therefore A = 0$.

Hence $\cosh^{-1}(r/d) = \theta$ or $r = d\cosh\theta$ Hence proved.

***Ex. 8 (b).** **A small ring is at rest on a smooth straight horizontal rod of length a at a distance b from one end O of the rod. The rod is then suddenly set rotating in a horizontal plane about the end O with constant angular velocity ω. *Prove that the ring* will leave the rod with velocity $\omega\sqrt{(2a^2 - b^2)}$ after a time $(1/\omega)\cosh^{-1}(a/b)$.** *(Meerut 95 P)*

Sol. Proceed as in Ex. 8 (a) above. Here 'd' $= b$ ...(i)

Then we can obtain $(\dot{r})^2 = \omega^2(r^2 - b^2)$ ...(ii)

Also $\dot{\theta} = \omega$, so $r\dot{\theta} = r\omega$.

$\therefore$ Velocity of the ring at $P = \sqrt{[(\dot{r})^2 + (r\dot{\theta})^2]}$

$= \sqrt{[\omega^2(r^2 - b^2) + r^2\omega^2]}$, from (i), (ii)

$= \omega\sqrt{(2r^2 - b^2)}$ ...(iii)

When the particle leaves the rod, $r = a$, the length of the rod (Note)

$\therefore$ From (iii), the velocity of the ring at that instant $= \omega\sqrt{(2a^2 - b^2)}$

Also from (i), we have $\dfrac{dr}{dt} = +\omega\sqrt{(r^2 - b^2)}$, since r increases as t increases

or $$\frac{dr}{\sqrt{(r^2 - b^2)}} = \omega\, dt$$

Integrating $\cosh^{-1}(r/b) = \omega t + C$, where C is an arbitrary const.
Initially $r = b, t = 0$, so $C = 0$

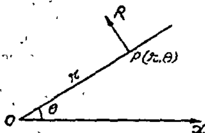
$\therefore \cosh^{-1}(r/b) = \omega t$ or $t = (1/\omega)\cosh^{-1}(r/b)$

$\therefore$ Required time $= (1/\omega)\cosh^{-1}(a/b)$,
since $r = a$ when the particle leaves the rod.

****Ex. 9. A straight smooth tube revolves with angular velocity in a horizontal plane about one extremity which is fixed, if at zero time a particle inside it be at a distance a from a fixed end and moving with velocity V along the tube, show that its distance at time t is**

$$a \cosh \omega t + (V/\omega) \sinh \omega t \qquad \textit{(Meerut 91 P)}$$

Sol. Let Ox be the initial position of the tube and A that of the particle inside it. O is the fixed end of the tube At any time t, let the position of the particle be at P, such that $OP = r$ and $\angle POx = \theta$.



(Fig. 10)

Since the particle is moving with constant velocity V along the tube i.e. along the radius vector, therefore radial acceleration of the particle is zero throughout the motion

Hence $\ddot{r} - r\dot{\theta}^2 = 0$. ...(i)

Its solution is $r = c_1 \cosh \omega t + c_2 \sinh \omega t$, ...(ii)

where c_1 and c_2 are the constants of integration and $d\theta/dt = \omega$

Initially $t = 0$, and $r = a$, $\therefore$ from (ii) $a = c_1$.

Differentiating (ii), $\dot{r} = c_1 \omega \sinh \omega t + c_2 \omega \cosh \omega t$.

Initially $t = 0$, $\dot{r} = V$, so $V = c_2 \omega$ or $c_2 = (V/\omega)$

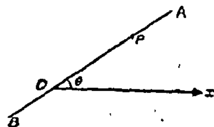
$\therefore$ From (ii), we get $r = a \cosh \omega t + (V/\omega) \sinh \omega t$, which gives the distance of the particle from O at time t. Hence proved.

Ex. 10. If a rod which always passes through the origin rotates with uniform angular velocity ω, while one end describes the curve $r = a + be^{\theta}$; show that the radial acceleration of any point of the rod is the same at every instant, and the radial velocity is the same at every point at a given instant.

Sol. Let AB be the rod which always passes through the origin O and whose end A describes the curve

$$r = a + be^{\theta} \qquad \text{...(i)}$$

Let P be a point on the rod, such that $AP = d$ Let at time t, the coordinates of A be (r, θ) i.e. $OA = r$ and $\angle AOx = \theta$. Then the coordinates of P are $(r - d, \theta)$.



(Fig. 11)

$\therefore$ Radial acceleration of A is $\ddot{r} - r\dot{\theta}^2$ and that of P is $\frac{d^2}{dt^2}(r-d)$ $-(r-d)\left(\frac{d\theta}{dt}\right)^2$, putting $(r-d)$ for r in the radial acceleration of A.

Hence radial acceleration of $P = \ddot{r} - (r-d)\dot{\theta}^2$ ...(i)

But the rod rotates with uniform angular velocity ω, so $\dot{\theta} = \omega$

Hence from (i), the radial acc. of $P = \ddot{r} - (r-d)\omega^2$ ...(ii)

Also $r = a + be^{\theta}$ ...(iii)

Differentiating, $\dot{r} = be^{\theta}\dot{\theta} = be^{\theta}\cdot\omega$, as $\dot{\theta} = \omega$

or $\ddot{r} = \omega be^{\theta}\cdot\dot{\theta} = b\omega^2 e^{\theta}$ $\because$ $\dot{\theta} = \omega$.

$= \omega^2(r-a)$, $\because$ from (iii) $r - a = be^{\theta}$

$\therefore$ from (ii) the radial acceleration of $P = (r-a)\omega^2 - (r-d)\omega^2$

$= (d-a)\omega^2 =$ constant and independent of t.

$\therefore$ Radial acc. of P is the same at every instant and P is any point on the rod.

$\therefore$ Radial acc. of any point of the rod is the same at every instant

Also the radial velocity of P

$= \frac{d}{dt}(r-d)$, putting $(r-d)$ for r

$= \dot{r} = be^{\theta}\cdot\omega$, (Proved above)

$= (r-a)\omega$, from (iii)

which being independent of d, the radial velocity of any point on the rod at given instant is the same. Also the radial velocity of A is $\dot{r} = (r-a)\omega$ (proved above).

Ex. 11. One end of a rod describes a plane curve and the rod always passes through a fixed point in the plane of the curve. If the angular velocity of the rod is constant; show that the transverse acceleration of every point of the rod is the same at the same instant. What curve must the end describe to make this acceleration the same at every instant.

Sol. See Figure 11 on Page 27 of this chapter

Let the coordinates of the end A be (r, θ) referred to fixed point O as pole

Let $AP = d$, where P is any point on the rod AB, so $OP = r - d$

The transverse acceleration of $P = \frac{1}{(r-d)}\frac{d}{dt}\left[(r-d)^2\cdot\frac{d\theta}{dt}\right]$, putting $(r-d)$ for r in the expression of transverse acceleration.

Also angular velocity of the rod $= d\theta/dt =$ constant $= \omega$ (say).

$\therefore$ The transverse acceleration of $P = \frac{1}{(r-d)}\frac{d}{dt}[(r-d)^2\omega]$

$= \frac{1}{(r-d)}2(r-d)\omega\cdot\frac{dr}{dt} = 2\omega\frac{dr}{dt}$, which being free from d is the same for every point of the rod at the same instant.

If this acceleration is the same at every instant also, then

$$2\omega\dot{r} = \text{constant or } \dot{r} = \lambda \text{ (say)}$$

∴ We have $\dot{r} = \lambda$ and $\dot{\theta} = \omega$,

whence dividing, we get $dr/d\theta = \lambda/\omega = k$ (say) or $dr = k\,d\theta$.

Integrating $r = k\theta + C$, which is the required equation of the curve

****Ex. 12. A point starts from the origin in the direction of the initial line with velocity f/ω and moves with constant angular velocity ω about the origin and with constant negative radial acceleration $-f$. Show that the rate of growth of the radial velocity is never positive, but tends to the limit zero and prove that the equation of the path is**

$$\omega^2 r = f(1 - e^{-\theta})$$ (*Agra* 91, *Meerut* 91 S)

Sol. Given that angular velocity $d\theta/dt = \omega$ ...(i)

and radial acceleration $= \ddot{r} - r\dot{\theta}^2 = -f$. ...(ii)

∴ from (i) and (ii), we have $\ddot{r} - r\omega^2 = -f$. ...(iii)

or $\ddot{r} = \omega^2 r - f$. Multiplying both sides by $2\dot{r}$ and integrating, we get

$$(\dot{r})^2 = \omega^2 r^2 - 2fr + C, \text{ where } C \text{ is any constant}$$

Initially, $r = 0$ and $\dot{r} = f/\omega$ (given), ∴ $f^2/\omega^2 = C$.

∴ Hence $(\dot{r})^2 = r^2\omega^2 - 2fr + (f^2/\omega^2) = [(f/\omega) - r\omega]^2$

or $\dfrac{dr}{dt} = \dfrac{f}{\omega} - r\omega$, the positive sign is due to the fact that the particle moves in the sense of r increasing.

or $$\frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{f}{\omega} - r\omega$$ (Note)

or $$\frac{dr}{d\theta}\,\omega = \frac{f}{\omega} - r\omega, \because \frac{d\theta}{dt} = \omega.$$

or $$\frac{dr}{d\theta} = \frac{f}{\omega^2} - r, \text{ or } \frac{dr}{[(f/\omega^2) - r]} = d\theta$$

Integrating, $-\log[(f/\omega^2) - r] + \log k = \theta$, where k is any constant.

Initially, $r = 0$ and $\theta = 0$, so $-\log(f/\omega^2) + \log k = 0$

or $$k = f/\omega^2.$$

$$\therefore -\log[(f/\omega^2) - r] + \log(f/\omega^2) = \theta$$

or $\log\{(f/\omega^2)/[(f/\omega^2) - r]\} = \theta$ or $f/(f - \omega^2 r) = e^{\theta}$

or $fe^{-\theta} = f - \omega^2 r$ or $\omega^2 r = f(1 - e^{-\theta})$, which is the required equation of the path.

Substituting this value of r in (iii), we get $\ddot{r} - f(1 - e^{-\theta}) = -f$

or $\ddot{r} = f(1 - e^{-\theta}) - f = -fe^{-\theta} = -f/e^{\theta}$, which is negative for all values of θ and tends to zero when θ is infinite.

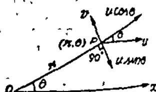
Also $\dfrac{d^2r}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dr}{dt}\right) =$ rate of growth of radial velocity $\dfrac{d\dot{r}}{dt}$.

Hence proved

Ex. 13. Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and the other is perpendicular to radius OP drawn from a fixed point O, is a conic whose focus is O and whose eccentricity is u/v.

(*Agra* 92; *Gorakhpur* 92, *Meerut* 95 P, 91 S)

Sol. Let the fixed direction be that of the initial line OX, drawn through the fixed point O, *i.e.* the line of action of u is parallel to OX.



(Fig. 12)

The resolved parts of u in the radial and transverse senses are $u\cos\theta$ and $u\sin\theta$ respectively. (see figure)

At P, the radial velocity $\dot{r}=u\cos\theta$ ...(i)

and transverse velocity $r\dot{\theta}=v-u\sin\theta$ ...(ii)

Dividing (i) by (ii), we get

$$\frac{dr}{r\,d\theta}=\frac{u\cos\theta}{v-u\sin\theta} \quad \text{or} \quad \frac{dr}{r}=\left(\frac{u\cos\theta}{v-u\sin\theta}\right)d\theta$$

Integrating, $\log r=-\log(v-u\sin\theta)+\log c$, where $\log c$ is constant of integration

or $\quad r=c/(v-u\sin\theta)$, or $\quad c/r=v-u\sin\theta$

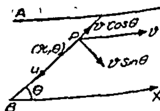
or $\quad \dfrac{(c/v)}{r}=1-(u/v)\sin\theta$, which being of the form $l/r=1-e\sin\theta$

represents a conic whose focus is O and eccentricity is (u/v)

Ex. 14. A boat which is rowed with constant velocity u, starts from a point A on the bank of a river which flows with a constant velocity v, and it points always towards a point B on the other bank exactly opposite to A; find the equation of the path of the boat.

If $v=u$, show that the path is a parabola whose focus is B.

Sol. Let B be taken as pole and the bank BX as initial line. Let at time t, the position of the boat be at (r,θ) referred to B as pole. The boat at P will have two velocities. u along PB and v parallel to BX



(Fig. 13)

At P, the radial velocity

$$dr/dt=v\cos\theta-u \quad \text{...(i)}$$

and the transverse velocity $r\,(d\theta/dt)$

$$=-v\sin\theta. \quad \text{...(ii)}$$

Dividing (i) by (ii), we get

$$\frac{dr}{r\,d\theta}=\frac{v\cos\theta-u}{-v\sin\theta} \quad \text{or} \quad \frac{dr}{r}=\left(\frac{u}{v}\operatorname{cosec}\theta-\cot\theta\right)d\theta$$

Integrating, $\log r = (u/v) \log \tan (\frac{1}{2}\theta) - \log \sin\theta + \log c$,
where $\log c$ is constant of integration

or $\log r + \log \sin\theta - \log c = (u/v) \log \tan (\frac{1}{2}\theta)$

or $[(r \sin\theta)/c] = (\tan \frac{1}{2}\theta)^{u/v}$, which is the required equation

If $v = u$, the above equation reduces to

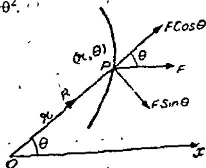
$$r \sin\theta = c \tan (\tfrac{1}{2}\theta)$$

or $\frac{c}{r} = \frac{\sin\theta}{\tan\frac{1}{2}\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{(\sin\frac{1}{2}\theta)/(\cos\frac{1}{2}\theta)} = 2\cos^2(\frac{1}{2}\theta) = 1 + \cos\theta$

or $c/r = 1 + \cos\theta$, which is a parabola whose focus is the pole B.

***Ex. 15. The acceleration of a point moving in a plane curve is resolved into two components, one parallel to the initial line and the other along the radius vector; prove that these components are**

$$\frac{-1}{r\sin\theta}\frac{d}{dt}(r^2\dot{\theta}) \text{ and } \frac{\cot\theta}{r}\frac{d}{dt}(r^2\dot{\theta}) + \ddot{r} - r\dot{\theta}^2.$$



(Fig. 14)

Sol. Let O be the pole and OX the initial line. Let $P(r, \theta)$ be any position of the moving particle. Let F and R be the components of acceleration of the particle parallel to OX and along the radius vector OP respectively. Then resolved parts of F along and perpendicular to OP are $F\cos\theta$ and $F\sin\theta$, as shown in the diagram.

Then in the radial and transverse senses, we get

$$\ddot{r} - r\dot{\theta}^2 = R + F\cos\theta \quad \text{...(i)}$$

and

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -F\sin\theta \quad \text{...(ii)}$$

From (ii), we have $F = -\frac{1}{r\sin\theta}\frac{d}{dt}(r^2\dot{\theta})$ ...(iii)

Also from (i), $R = \ddot{r} - r\dot{\theta}^2 - F\cos\theta$

$= \ddot{r} - r\dot{\theta}^2 + \frac{\cos\theta}{r\sin\theta}\frac{d}{dt}(r^2\dot{\theta})$ ...from (ii)

$= \ddot{r} - r\dot{\theta}^2 + \frac{\cot\theta}{r}\frac{d}{dt}(r^2\dot{\theta})$. Hence proved.

***Ex. 16. A point moves on a parabola $2a = r(1 + \cos\theta)$ in such a manner that the component of velocity at right angles to the radius vector from the focus is constant. Show that the acceleration of the point is constant in magnitude.** (Garhwal 94; Lucknow 91)

Sol. The equation of the parabola is $2a = r(1 + \cos\theta)$. ...(i)
whose focus is the pole.

Also we are given transverse velocity $= r\dot{\theta} =$ constant ...(ii)

i.e. $r\dot{\theta} = \lambda$ (say) or $\dot{\theta} = \lambda/r$

From (i), we get $r = \dfrac{2a}{1+\cos\theta} = \dfrac{2a}{2\cos^2\frac{1}{2}\theta} = a\sec^2\frac{1}{2}\theta.$ ...(iii)

$\therefore \quad \dot{r} = a \,.\, 2\sec\frac{1}{2}\theta \,.\, \sec\frac{1}{2}\theta\tan\frac{1}{2}\theta \,.\, \frac{1}{2}\dot{\theta}$

$= a\sec^2\frac{1}{2}\theta\tan\frac{1}{2}\theta \,.\, \dot{\theta} = r\tan\frac{1}{2}\theta \,.\, \dot{\theta}$, from (iii)

$= \lambda\tan\frac{1}{2}\theta$, from (ii).

$\therefore \quad \ddot{r} = \frac{1}{2}\lambda\sec^2\frac{1}{2}\theta \,.\, \dot{\theta} = \frac{1}{2}\lambda\sec^2\frac{1}{2}\theta\,(\lambda/r)$, from (ii)

$= \frac{1}{2}\lambda^2 (r/a)(1/r)$, from (iii)

$= \lambda^2/2a.$

Now radial acc. of the point $= \ddot{r} - r\dot{\theta}^2 = (\lambda^2/2a) - r(\lambda/r)^2$ (iv)

$= (\lambda^2/2a) - (\lambda^2/r) = \lambda^2(r-2a)/2ar.$

Ans transverse acc. of the point

$= \dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2 \,.\, \dfrac{\lambda}{r}\right)$, from (ii)

$= \dfrac{\lambda}{r}\dfrac{d}{dt}(r) = \dfrac{\lambda}{r}\dot{r} = \dfrac{\lambda^2}{r}\tan\frac{1}{2}\theta, \because \dot{r} = \lambda\tan\frac{1}{2}\theta$

$= \dfrac{\lambda^2}{r}(\sec^2\frac{1}{2}\theta - 1)^{1/2} = \dfrac{\lambda^2}{r}\left(\dfrac{r}{a} - 1\right)^{1/2}$, from (iii)

$\therefore$ Resultant acceleration of the particle

$$= \sqrt{\left[\left\{\frac{\lambda^2(r-2a)}{2ar}\right\}^2 + \left\{\frac{\lambda^2}{r}\frac{(r-a)^{1/2}}{\sqrt{a}}\right\}^2\right]}$$ [Note]

$$= \frac{\lambda^2}{r}\sqrt{\left[\left(\frac{r-2a}{2a}\right)^2 + \left(\frac{r-a}{a}\right)\right]} = \frac{\lambda^2}{r}\sqrt{\left[\left(\frac{r}{2a}-1\right)^2 + \left(\frac{r}{a}-1\right)\right]}$$

$$= \frac{\lambda^2}{r}\sqrt{\left[\frac{r^2}{4a^2} - \frac{r}{a} + 1 + \frac{r}{a} - 1\right]} = \frac{\lambda^2}{r}\sqrt{\left(\frac{r^2}{4a^2}\right)} = \frac{\lambda^2}{2a},$$

which is constant. Hence proved.

*Ex. 17. A small bead slides with a constant speed v on a smooth wire in the shape of the cardioid $r = a(1+\cos\theta)$. Show that the value of $\dot{\theta}$ is $(v/2a)\sec\frac{1}{2}\theta$, and that the radial component of the acceleration is constant. (Meerut 93, 9?)

Sol. The path of the bead is $r = a(1+\cos\theta)$. ...(i)

$\therefore \quad \dot{r} = a(-\sin\theta)\dot{\theta}$ ...(ii)

and $\quad \ddot{r} = (-a\cos\theta)\dot{\theta}^2 - (a\sin\theta)\ddot{\theta}$ ...(iii)

$\therefore$ The speed of the bead $= \sqrt{[(\dot{r})^2 + (r\dot{\theta})^2]}$

$= \sqrt{[\{(-a\sin\theta)\dot{\theta}\}^2 + \{a(1+\cos\theta)\dot{\theta}\}^2]}$

$= a\sqrt{[\sin^2\theta + (1+\cos\theta)^2]}\,\dot\theta = a\sqrt{[2(1+\cos\theta)]}\,\dot\theta$

$= a\sqrt{[2(2\cos^2\frac{1}{2}\theta)]}\,\dot\theta = 2a(\cos\frac{1}{2}\theta)\,\dot\theta = v$ (given constnat)

i.e. $\dot\theta = (v/2a)\sec\frac{1}{2}\theta$ ...(iv)

Hence proved

Again radial acceleration of the bead = " $\ddot r - r\dot\theta^2$ "

$= [(-a\cos\theta)\dot\theta^2 - (a\sin\theta)\ddot\theta] - a(1+\cos\theta)\dot\theta^2$, from (i), (ii)

$= -a(1+2\cos\theta)\dot\theta^2 - a(\sin\theta)\ddot\theta$, where $\dot\theta$ is given by (iv)

$= -a(1+2\cos\theta)\cdot\left[\frac{v}{2a}\sec\frac{1}{2}\theta\right]^2 - a(\sin\theta)\left[\frac{v}{2a}\frac{1}{2}\sec\frac{1}{2}\theta\tan\frac{1}{2}\theta\,\dot\theta\right]$, from (iv)

$= -a(1+2\cos\theta)\cdot\left[\frac{v}{2a}\sec\frac{1}{2}\theta\right]^2 - \frac{1}{2}a(\tan\frac{1}{2}\theta\sin\theta)\left[\frac{v}{2a}\sec\frac{1}{2}\theta\right]^2$, from (iv)

$= -a(v^2/4a^2)(\sec^2\frac{1}{2}\theta)[(1+2\cos\theta)+\frac{1}{2}\tan\frac{1}{2}\theta\sin\theta]$

$= -\frac{1}{4}(v^2/a)(\sec^2\frac{1}{2}\theta)[(1+2\cos\theta)+\sin^2\frac{1}{2}\theta]$

$= -\frac{1}{4}(v^2/a)(\sec^2\frac{1}{2}\theta)[(1+2\cos\theta)+\frac{1}{2}(1-\cos\theta)]$

$= -\frac{1}{4}(v^2/a)(\sec^2\frac{1}{2}\theta)[(3/2)(1+\cos\theta)]$

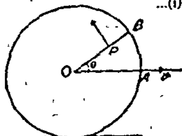
$= -\frac{3}{4}(v^2/a)(\sec^2\frac{1}{2}\theta)(\cos^2\frac{1}{2}\theta) = -(3/4)(v^2/a) =$ constant.

Hence proved

****Ex. 18. An insect crawls at a consant rate u along the spoke of a cart wheel of radius a, the cart is moving with velocity v, find the acceleration along and perpendicular to the spoke.**

Sol. Let the initial position of the insect be at centre O and the spoke be OA. After a time t, let the spoke turn through an angle θ and take the position OB. Let the position of the insect be at P after time t. Since the insect is crawling at a constant rate u along the spoke, so if $OP = r$, then

$r = ut$. ...(i)



(Fig. 15)

The equation of the circle (cart wheel) referred to centre O as pole is $r = a$.

If p be the length of the perpendicular from the pole to tangent at any point on the circle, then $p = a$.

∴ Angular velocity

$= \frac{d\theta}{dt} = \frac{vp}{r^2} = \frac{v\,a}{a^2} = \frac{v}{a} = \omega$ (say) ...(ii)

∴ Acceleration along the spoke = radial acceleration

$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2}{dt^2}(ut) - ut(\omega^2)$ from (i) and (ii)

$= 0 - ut\,\omega^2 = -ut\,(v^2/a^2)$, from (ii).

Acceleration perpendicular to the spoke

$= $ transverse acceleration

$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{ut}\frac{d}{dt}(u^2t^2\omega)$, from (i) and (ii)

$= (1/ut)\,2u^2t\omega = 2u\omega = 2\,(uv/a)$, from (ii). **Ans.**

***Ex. 19. A particle describes a plane curve with an acceleration which is always directed to a fixed point O in the plane. Show that equal areas are swept by this radius vector in equal times.**

Sol. Refer Fig. 1 (a) Page 1 of this chapter.

Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring positions of the particle at times t and $(t + \delta t)$ respectively referred to O as pole and OX as initial line. The acceleration of the particle is always directed towards O and hence the particle has only radial acceleration towards O and no transverse acceleration, (Note)

so we have $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$ or $\frac{d}{dt}(r^2\dot{\theta}) = 0$

Integrating, we get $r^2\dot{\theta} = \text{constant} = h$ (say). ...(i)

Again in time δt, the radius vector OP sweeps out the sectorial area OPQ and its magnitude

$= \frac{1}{2}OP \cdot OQ \sin\delta\theta$ (Note)

$= \frac{1}{2}r(r + \delta r)\sin\delta\theta = \frac{1}{2}r(r + \delta r)[\delta\theta - \frac{1}{3!}(\delta\theta)^3 + \dots]$

$= \frac{1}{2}r^2\,\delta\theta$, to a first approximation.

$\therefore$ Rate of description of the sectorial area by the radius vector as it passes through OP

$= \lim_{\delta t \to 0} \frac{\frac{1}{2}r^2\,\delta\theta}{\delta t} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h$, from (i)

$=$ constant, as h is constant.

Hence equal areas are swept out by the radius vector in equal times

Exercises on § 5

***Ex. 1.** A particle moves along the curve $r = Ae^{k\theta}$, $\theta = Bt$. Prove that its acceleration is proportional to r.

Ex. 2. If the radial and transverse velocities of a particle are always equal, prove that the particle describes an equiangular spiral

[Hint. Put $k = 1$ in Ex. 5 (a) Page 22].

Ex. 3. If the path of a particle is $r = a\tan\theta$ and acceleration is directed towards the origin, show that the acceleration in terms of r is

$$\frac{h^2}{r^2}\left[3 + \frac{2a^2}{r^2}\right], \text{ where } h = r^2\frac{d\theta}{dt}.$$

*Ex. 4. If the angular velocity ω of a particle about the origin is constant, and the rate of change of acceleration is directed wholly along the radius vector, prove that $(d^2r/dt^2)=(\omega^2 r)/3$.

(*Lucknow* 91; *Rohilkhand*91)

*Ex. 5. A vessel streams at a constant speed v along a straight line whilst another streaming at a constant speed V, always moves at right angles to the radius vector to the former. Show that the path of either vessel relative to the other is a conic section of eccentricity v/V

[Hint. See Ex. 13 Page 30].

Ex. 6. If a particle $P(r, \theta)$ describes a plane curve, then the transverse acceleration of P at any time t is given by

(i) $\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$, (ii) $\frac{d}{dt}\left(r\frac{d\theta}{dt}\right)$,

(iii) $r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}$, (iv) $r\frac{d^2\theta}{dt^2}+\frac{dr}{dt}\frac{d\theta}{dt}$ Ans. (iii)

Ex. 7. A particle is moving in a plane under the attraction of magnitude $2/r^3$ per unit of mass towards a fixed point in the plane, r being the distance of the particle from the fixed point. At $t=0$, $r=2$ and the radial and transverse components of velocity are $\sqrt{(3/2)}$ and 1 respectively. Show that $\ddot{r}=2r^{-3}$.

Ex. 8. If a particle $P(r, \theta)$ is describing a circle of radius a with centre at the pole, the radial acceleration is

(i) $-a\dot{\theta}^2$, (ii) $a\dot{\theta}$, (iii) $a\ddot{\theta}$, (iv) 0.

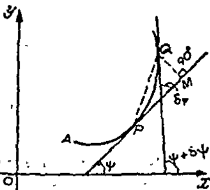
§ 6. Expression for velocities and accelerations (Intrinsic form) Tangential and Normal velocities. (*Bun. Khand* 96; *Garhwal* 96, 94, 91; *Gorakhpur* 96, 94; *Kanpur* 91, *Lucknow* 90)

Let the positions of the particle at time t and $t+\delta t$ be P and Q. Let arc $AP=s$ and $AQ=s+\delta s$. Let the tangents at P and Q make angles ψ and $\psi+\delta\psi$ with x-axis. Let $\angle QPM=\alpha$. When $Q\to P, \alpha\to 0$ From Q draw QM perpendicular to the tangent at P. Let chord $PQ=\delta c$.

Then as $Q\to P$,

$$\frac{\text{chord } PQ}{\text{arc } PQ}\to 1$$

i.e. $\frac{\delta c}{\delta s}\to 1$ as $Q\to P$



(Fig. 16)

∴ **Tangential velocity at P**

$$=\lim_{\delta t\to 0}\frac{\text{displacement along the tangent at } P}{\delta t}$$

$= \lim_{\delta t \to 0} \frac{PM}{\delta t} = \lim_{\delta t \to 0} \frac{(\text{chord } PQ) \cdot \cos\alpha}{\delta t}$ (Note)

$= \lim_{\delta t \to 0} \frac{\delta c \cos\alpha}{\delta t}$ $\because$ chord $PQ = \delta c$.

$= \lim_{\delta t \to 0} \frac{\delta c}{\delta s} \cdot \frac{\delta s}{\delta t} \cdot \cos\alpha = 1 \cdot \frac{ds}{dt} \cdot \cos 0,$

$\because \alpha \to 0$ as $Q \to P$ i.e. $\delta t \to 0$

$= \frac{ds}{dt}$

Normal velocity at P

$= \lim_{\delta t \to 0} \frac{\text{displacement along the normal at } P}{\delta t}$

$= \lim_{\delta t \to 0} \frac{QM}{\delta t} = \lim_{\delta t \to 0} \frac{(\text{chord } PQ) \cdot \sin\alpha}{\delta t} = \lim_{\delta t \to 0} \frac{\delta c \sin\alpha}{\delta t}$

$= \lim_{\delta t \to 0} \frac{\delta c}{\delta s} \cdot \frac{\delta s}{\delta t} \cdot \sin\alpha = 1 \cdot \frac{ds}{dt} \cdot 0$; $\because \alpha \to 0$ as $Q \to P$ i.e. $\delta t \to 0$

$= 0$

Hence for a particle moving in a plane curve, the tangential velocity is given by ds/dt and normal velocity is zero and so its direction of motion is always along the tangent. (Lucknow 90)

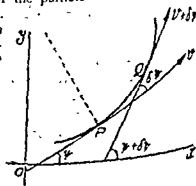
****Tangential and Normal Accelerations.**

(*Avadh* 95, 93; *Bundelkhand* 96, 94, 92, 91; *Garhwal* 96, 94, 91; *Gorakhpur* 96, 92, 90; *Kanpur* 96, 93, 91; *Lucknow* 92; *Meerut* 96 P, *Purvanchal* 91; *Rohilkhand* 94, 90)

Let P and Q be the positions of the particle at times t and $t+\delta t$ respectively. Let the tangents at P and Q make angles ψ and $\psi+\delta\psi$ with x-axis. Let v and $v+\delta v$ be the velocities of the particle at P and Q respectively. Since the velocity of the particle moving in a plane curve is always tangential, so v and $v+\delta v$ are acting along the tangents at P and Q respectively. Also we have proved above that

tangential velocity at P

$= v = ds/dt.$ ...(ii)



(Fig. 17)

$\therefore$ **Tangential acceleration at P**

$= \lim_{\delta t \to 0} \frac{\text{change in velocity along the tangent at } P \text{ in time } \delta t}{\delta t}$

$= \lim_{\delta t \to 0} \frac{(v+\delta v)\cos\delta\psi - v}{\delta t}$ (Note)

$= \lim_{\delta \to 0} \frac{(v+\delta v)\cdot 1 - v}{\delta t}$, $\because \cos \delta\psi = 1$, neglecting powers of $\delta\psi$

$= \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$

Also $\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v = v\frac{dv}{ds}$

$\therefore$ **Tangential acceleration at P** $= \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}$ **(Remember)**

Normal acceleration at P.

$= \lim_{\delta t \to 0} \frac{\text{change in velociy along the normal at } P \text{ in time } \delta t}{\delta t}$

$= \lim_{\delta t \to 0} \frac{(v+\delta v)\sin \delta\psi - 0}{\delta t}$ **(Note)**

$= \lim_{\delta t \to 0} \frac{(v+\delta v)\,\delta\psi}{\delta t}$, $\because \sin \delta\psi = \delta\psi$, neglecting higher powers of $\delta\psi$

$= \lim_{\delta t \to 0} v\frac{\delta\psi}{\delta t}$, neglecting powers of small quantities higher than first

$= v\frac{d\psi}{dt} = v\frac{d\psi}{ds} \cdot \frac{ds}{dt} = v \cdot \frac{1}{\rho} \cdot v$, since $\rho = \frac{ds}{d\psi}$

$= v^2/\rho$.

In vector notations. (Refer fig. 16, Page 35 of this chapter)

Let the particle be at P at time t and at Q at time $t + \delta t$, where arc $PQ = \delta s$.

Let $\hat{t}$ and $\hat{n}$ be unit vectors along the tangent and *inward drawn* normal to the path of the particle at P. Then the velocity vector **v** of the particle, whose direction is that of tangent to the path of the particle at P, is given by $\mathbf{v} = v\,\hat{t}$, where $v = |\mathbf{v}|$

or $\mathbf{v} = (ds/dt)\,\hat{t}$, as $v = ds/dt$.

$\therefore$ The magnitude of the tangential velocity at P is ds/dt

Also the rate of change of unit vector $\hat{t}$ is given by

$$\frac{d}{dt}(\hat{t}) = \dot{\psi}\hat{n}, \quad \text{..(i)}$$

where ψ is the angle which the tangent at P to the path of the particle makes with fixed tangent to the curve, say at A .. See § 2 Page 2 of this chapter

$\therefore$ The acceleration vector **a** of the particle at P is given by

$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\,\hat{t}) = \frac{dv}{dt}\,\hat{t} + v\frac{d}{dt}(\hat{t})$

$= \left(\frac{dv}{dt}\right)\hat{t} + v\,\dot{\psi}\,\hat{n}$, from (i)

$= \left(\frac{dv}{dt}\right)\hat{t} + v\frac{d\psi}{ds} \cdot \frac{ds}{dt}\,\hat{n}$, where $\dot{\psi} = \frac{d\psi}{dt} = \frac{d\psi}{ds}\frac{ds}{dt}$

$$=\left(\frac{dv}{dt}\right)\hat{\mathbf{t}}+v\cdot\frac{1}{\rho}\cdot v\,\hat{\mathbf{n}}, \quad \because \rho=\frac{ds}{d\psi}, v=\frac{ds}{dt}$$

or $\mathbf{a}=\left(\frac{dv}{dt}\right)\hat{\mathbf{t}}+\left(\frac{v^2}{\rho}\right)\hat{\mathbf{n}}.$

From here we conclude that tangential and normal (inward drawn) components of acceleration vector **a** at P are (dv/dt) and (v^2/ρ) respectively.

Solved Examples on § 6.

Ex. 1. A particle describes the parabola $x=64t, y=96t-16t^2$. Compute the tangential and normal accelerations at $t=3$ secs.

Sol. Given $x=64t$ and $y=96t-16t^2$. ...(i)

$\therefore$ $\dot{x}=64$ and $\dot{y}=96-32t$. Also $\ddot{x}=0, \ddot{y}=-32$

$\therefore$ If v be the magnitude of the velocity at time t, then

$$v=\sqrt{(\dot{x}^2+\dot{y}^2)}=\sqrt{[(64)^2+(96-32t)^2]}=32\sqrt{[(2)^2+(3-t)^2]}$$

$$=32\sqrt{(4+9-6t+t^2)}=32\sqrt{(13-6t+t^2)}. \quad \text{...(ii)}$$

Also ρ, the radius of curvature is given by

$$\rho=\frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{(\dot{x}\ddot{y}-\dot{y}\ddot{x})}, \text{ see Author's Diff. Calculus}$$

$$=\frac{[v^2]^{3/2}}{[64(-32)-0]}, \text{ from (i) and } v^2=\dot{x}^2+\dot{y}^2 \quad \text{...(iii)}$$

$$=-v^3/(32\times 64).$$

$\therefore$ Tangential acc. $=(dv/dt)=32\cdot\frac{1}{2}(13-6t+t^2)^{-1/2}(-6+2t)$

$$=32(13-6t+t^2)^{-1/2}(t-3)=0, \text{ when } t=3$$

And normal acceleration $=v^2/\rho=-v^2(32\times 64)/v^3$, from (iii)

$=-(32\times 64)/v=-(32\times 64)/(64)$, from (ii) at $t=3$

$=-32$ unit/sec^2. Ans.

***Ex. 2. The rate of change of direction of velocity of a particle moving in a cycloid is constant. Prove that acceleration must be constant in magnitude.**

Or A point moves along the arc of a cycloid in such a manner that tangent at it rotates with a constant angular velocity; show that the acceleration of the moving point is constant in magnitude.

Sol. The equation of the cycloid is $s=4a\sin\psi$ ...(i)

$\because$ The rate of change of direction of velocity of the particle is constant, so $d\psi/dt=$ constant $=\omega$ (say) ...(ii)

Differentiating (i), we have $ds/dt=4a\cos\psi\,(d\psi/dt)$

or $\qquad ds/dt=4a\omega\cos\psi$, from (ii)

Differentiating again we get $d^2s/dt^2=-4a\omega\sin\psi\,(d\psi/dt)$

or $\qquad d^2s/dt^2=-4a\omega\sin\psi\cdot\omega$, from (ii)

or $\qquad d^2s/dt^2=-4a\omega^2\sin\psi$. ...(iii)

Also from (i), we have $ds/d\psi=4a\cos\psi$.

$\therefore$ radius of curvature $\rho = ds/d\psi = 4a \cos \psi$.

$\therefore$ Normal acceleration $= \dfrac{v^2}{\rho} = \dfrac{(ds/dt)^2}{\rho} = \dfrac{16a^2 \omega^2 \cos^2 \psi}{4a \cos \psi}$

$= 4a\omega^2 \cos \psi$

$\therefore$ Resultant acceleration of the particle

$= \sqrt{[(d^2s/dt^2)^2 + (v^2/\rho)^2]}$

$= \sqrt{[(-4a\omega^2 \sin \psi)^2 + (4a\omega^2 \cos \psi)^2]}$

$= \sqrt{(16a^2\omega^4)} = 4a\omega^2 =$ constant.

Ex. 3 (a). A particle describes a circle of radius r with a uniform peed v, show that its acceleration at any point of this path is v^2/r and s directed towards the centre of the circle.

Sol. Given that $v =$ constant. ...(i)

$\therefore$ Tangential acceleration $= dv/dt = 0$, from (i)

Also normal acceleration $= v^2/\rho = v^2/r$,

$\because$ in the case of circle $\rho =$ radius of the circle $= r$.

$\therefore$ At any point of the path, the resultant acceleration

$= \sqrt{[(dv/dt)^2 + (v^2/\rho)^2]} = \sqrt{[0 + (v^2/r)^2]} = v^2/r$

i.e. the resultant acceleration consists of v^2/r only which is the ormal acceleration at the point.

We know that the normal acceleration at a point is directed towards he inward drawn sense. Hence in the case of a circle it is towards the entre. Hence proved.

***Ex. 3. (b). A point describes the cycloid $s = 4a \sin \psi$ with unirom speed v. Find its acceleration at any point.**

(*Agra 90, Meerut 96 BP*)

Sol. The equation of the cycloid is $s = 4a \sin \psi$. ...(i)

Also given that $ds/dt = v =$ constant.

$\therefore$ $d^2s/dt^2 = 0$ *i.e.* the tangential acceleration $= 0$

Differentiating both sides of (i) with respect to ψ, we get

$$ds/d\psi = 4a \cos \psi.$$

e. radius of curvature $\rho = 4a \cos \psi$.

$\therefore$ The resultant acceleration

$= \sqrt{[(d^2s/dt^2)^2 + (v^2/\rho)^2]} = [0 + (v^2/\rho)^2] = v^2/\rho$

$= \dfrac{v^2}{4a \cos \psi} = \dfrac{v^2}{4a\sqrt{(1 - \sin^2 \psi)}} = \dfrac{v^2}{4a\sqrt{[1 - (s^2/16a^2)]}}$, from (i)

$= v^2/\sqrt{(16a^2 - s^2)}$. **Ans.**

Ex. 4. If the tangential and normal acceleration of a particle lescribing a plane curve be constant throughout, prove that the radius of curvature at any point t is given by $\rho = (at + b)^2$

Sol. Given $dv/dt = \lambda$ ...(i) and $v^2/\rho = \mu$, ...(ii)

where λ and μ are constants

Integrating (i), we get $v=\lambda t+c$, ...(iii)

where c is constant of integration.

$\therefore$ from (ii), $\rho=v^2/\mu=(\lambda t+c)^2/\mu=(at+b)^2$,

where $a=\lambda/\sqrt{\mu}$ and $b=c/\sqrt{\mu}$. *Hence proved.*

***Ex. 5. A particle describes a cycloid with uniform speed. Prove that the normal acceleration at any point varies inversely as the square root of the distance from the base of the cycloid.**

Sol. As in Ex. 3 (b) Page 39, we can find that the normal acceleration

$$=v^2/\rho=v^2/\sqrt{(16a^2-s^2)}, \quad ...(i)$$

where v is the uniform speed of the particle.

Also we know that if the distance of any point on the cycloid from the tangent at the vertex be y, then the arcural distance of that point measured from the vertex is given by $s^2=8ay$. ...(ii)

Also the distance of this point from the base of the cycloid "$s=4a\sin\psi$" is $2a-y$. (Note)

$\therefore$ From (i) and (ii), we get the normal acceleration

$$=\frac{v^2}{\sqrt{(16a^2-8ay)}}=\frac{v^2}{\sqrt{(8a)}\sqrt{(2a-y)}}=\frac{\text{constant}}{\sqrt{(2a-y)}}$$

$$=\frac{\text{constant}}{\sqrt{\text{(distance from the base)}}}. \quad \textit{Hence proved}$$

****Ex. 6 (a). Prove that the acceleration of a point moving in a curve with uniform speed is $\rho\dot{\psi}^2$.** *(Meerut 96 BP)*

Sol. Given that speed is unifrom *i.e.* $ds/dt=$ constant.

Differentiating, $d^2s/dt^2=0$ *i.e.* tangential acceleration is zero.

$\therefore$ Resultant acceleration of the particle ...(i)

$$=\sqrt{[(d^2s/dt^2)^2+(v^2/\rho)^2]}=\sqrt{[0+(v^2/\rho)^2]}=v^2/\rho.$$

Now $\dot{\psi}=\frac{d\psi}{dt}=\frac{d\psi}{ds}\frac{ds}{dt}=\frac{1}{\rho}\ v$ or $v=\rho\dot{\psi}$.

$\therefore$ From (i), resultant acceleration of the particle

$$=\frac{v^2}{\rho}=\frac{(\rho\dot{\psi})^2}{\rho}=\rho\dot{\psi}^2 \quad \textit{Hence proved}$$

***Ex. 6 (b). If the velocity at a point moving in a plane curve varies as the radius of curvature, show that the direction of motion revolves with constant angular velocity.** *(Purvanchal 91)*

Sol. Given that $v=k\rho$

or $\frac{ds}{dt}=k\frac{ds}{d\psi}$ or $d\psi=k\,dt$ or $\frac{d\psi}{dt}=k=$ constant.

Hence the direction of motion revolves with constant angular velocity.

****Ex. 7. A point moves in a plane curve, so that its tangential and normal accelerations are equal and the angular velocity of the**

tangent is constant. Find the curve. (*Agra* 91; *Bundelkhand* 95, 92, 90, *Garhwal* 91; *Gorakhpur* 95, 93, 91, *Kanpur* 97, 91; *Meerut* 92 P; *Purvanchal* 90)

Sol. Given $\dfrac{v\,dv}{ds}=\dfrac{v^2}{\rho}$ ...(i) and $\dfrac{d\psi}{dt}=\omega=\text{constant}$. (ii)

From (i), $\dfrac{dv}{ds}=\dfrac{v}{\rho}=\dfrac{v}{(ds/d\psi)}$

or $dv=v\,d\psi$ or $(1/v)\,dv=d\psi$

Integrating it we get, $\log v=\psi+\log c$,

where $\log c$ is constant of integration

or $\log(v/c)=\psi$ or $v=ce^{\psi}$

or $\dfrac{ds}{dt}=ce^{\psi}$ or $\dfrac{ds}{d\psi}\dfrac{d\psi}{dt}=ce^{\psi}$ or $\dfrac{ds}{d\psi}\omega=ce^{\psi}$, from (ii)

or $ds=(c/\omega)\,e^{\psi}\,d\psi$.

Integrating, $s=(c/\omega)\,e^{\psi}+k$, where k is constant of integration

or $s=Ae^{\psi}+B$, where A and B are arbitrary constants

This is the required intrinsic equation of the curve **Ans.**

Ex. 7 (a). **If tangential and normal acceleration components be equal, prove that the velocity varies as e^{ψ}.**

Sol. Given $\dfrac{v\,dv}{ds}=\dfrac{v^2}{\rho}$ or $\dfrac{dv}{ds}=\dfrac{v\,d\psi}{ds}$, $\because\ \rho=\dfrac{ds}{d\psi}$

or $(1/v)\,dv=d\psi$.

Integrating, $\log v=\psi+\log c$, where $\log c$ is constant of integration

or $\log(v/c)=\psi$ or $v=ce^{\psi}$ *i.e.* v varies as e^{ψ}. Hence proved

Ex. 8 (a). A particle describes a curve (for which s and ψ vanish simultaneously) with uniform speed v. If the acceleration at any point s be $v^2c/(s^2+c^2)$ find the intrinsic equation of the curve.

(*Avadh* 94; *Gorakhpur* 97, 94, 90; *Kumaun* 92; *Rohilkhand* 95, 90)

Sol. It is given that $ds/dt=v$ (constant), so $d^2s/dt^2=0$.

$\therefore$ Acceleration at any point

$=\sqrt{[(d^2s/dt^2)^2+(v^2/\rho)^2]}=\sqrt{[0+(v^2/\rho)^2]}=v^2/\rho$

Now it is given that acceleration at any point $=v^2c/(s^2+c^2)$

or $$\frac{v^2}{\rho}=\frac{v^2c}{s^2+c^2}\quad\text{or}\quad\frac{1}{\rho}=\frac{c}{s^2+c^2}$$

or $$\frac{d\psi}{ds}=\frac{c}{s^2+c^2}\quad\text{or}\quad\frac{1}{c}\,d\psi=\frac{ds}{s^2+c^2}$$

Integrating, we get $(1/c)\,\psi+A=(1/c)\tan^{-1}(s/c)$,

where A is constant of integration.

When $\psi=0$, $s=0$ (given), $\therefore\ A=\tan^{-1}0=0$

Hence $(1/c)\psi = (1/c)\tan^{-1}(s/c)$ or $s = c \tan \psi$ is the required intrinsic equation of the curve, which is a catenery. **Ans.**

Ex. 8 (b). A particle describes a plane curve with a constant speed and its acceleration is constant in magnitude. Prove that the path is a circle.

Sol. Let v be the constant speed of the particle.

The tangential acceleration of the particle at any instant

$$= dv/dt = 0, \text{ as } v \text{ is constant.}$$

And normal acceleration of the particle at that instant $= v^2/\rho$.

$\therefore$ Resultant acc. of the particle $= [(dv/dt)^2 + (v^2/\rho)^2]$

$$= \sqrt{[0 + (v^2/\rho)^2]} = v^2/\rho = \text{ constant (given).}$$

$\therefore \quad v^2/\rho =$ constant $= k$ (say) or $\rho =$ constant, $\because v =$ const.

i.e., the path of the particle is such that its radius of curvature is const. and such a curve is circle. Hence prov

****Ex. 9. A particle is describing a plane curve. If the tangenti and normal accelerations are each constant throughout the motio prove that the angle ψ, through which the direction of motion turns time t is given by $\psi = A \log (1 + Bt)$.** (*Avadh* 93, 90, *Bundelkhand* 96, 9 *Gorakhpur* 94; *Lucknow* 92; *Meerut* 91 P, *Purvanchal* 9

Sol. Given that

$$d^2s/dt^2 = k \quad ...(i) \quad \text{and} \quad v^2/\rho = \lambda, \quad ...(i$$

where k and λ are constants.

Integrating (i), we get $ds/dt = kt + c$, ...(ii

where c is constant of integration.

From (ii), we get $\dfrac{v^2}{\rho} = \lambda$ or $\dfrac{v^2}{(ds/d\psi)} = \lambda$, where $v = \dfrac{ds}{dt}$

or $$\frac{(ds/dt)^2}{(ds/d\psi)} = \lambda \text{ or } \frac{ds}{dt} \cdot \frac{d\psi}{dt} = \lambda$$

or $$(kt + c)\frac{d\psi}{dt} = \lambda, \text{ from (iii) or } d\psi = \frac{\lambda}{kt + c}\, dt$$

Integrating, $\psi = (\lambda/k) \log (kt + c) + \log \mu$,

were $\log \mu$ is a constant of integration.

Let $\psi = 0$ when $t = 0$, then $0 = (\lambda/k) \log c + \log \mu$.

$$\therefore \quad \psi = \frac{\lambda}{k} \log (kt + c) - \frac{\lambda}{k} \log c = \frac{\lambda}{k} \log \left(\frac{kt + c}{c}\right)$$

or $\quad \psi = (\lambda/k) \log [1 + (kt/c)]$

or $\quad \psi = A \log (1 + Bt)$, where $A = (\lambda/k)$ an

***Ex. 10. A particle moves in a p. ... that its tangential and normal accelerations are ... its velocity varies as $\exp(\tan^{-1} s/c)$, ... length ... measured from a fixed point on ... the**

(A ... pur 97.

Sol. Given $\frac{d^2s}{dt^2}=\frac{v^2}{\rho}$ ...(i) and $v=k\,e^{\tan^{-1}(s/c)}$ (ii)

From (i), we get $v\frac{dv}{ds}=\frac{v^2}{(ds/d\psi)}$, $\because v\frac{dv}{ds}=\frac{d^2s}{dt^2}$ and $\rho=\frac{ds}{d\psi}$,

or $(1/v)\,dv=d\psi$.

Integrating, we get $\log v=\psi+A$,

where A is constant of integration $\therefore$ from (ii), we get

$\log(ke^{\tan^{-1}s/c})=\psi+A$ or $\log k+\tan^{-1}(s/c)=\psi+A$

or $\tan^{-1}(s/c)=\psi+B$, where $B=A-\log k$

Let $s=0$ when $\psi=0$, then $B=0$.

$\therefore \tan^{-1}(s/c)=\psi$ or $s=c\tan\psi$ is the required path **Ans.**

***Ex. 11. A point moves in a plane curve so that its tangential acceleration is constant, and the magnitudes of the tangential velocity and normal acceleration are in a constant ratio; find the intrinsic equation of the curve.** *(Bundelkhand 94, Garhwal 95, 93, 90)*

Sol. Given that $v\,(dv/ds)=k$ ...(i) and $v=\lambda\,(v^2/\rho)$. (ii)

From (i), we get $v\,dv=k\,ds$,

Integrating we get, $v^2=2ks+c$, where c is constant of integration

$\therefore$ $v=\sqrt{(2ks+c)}$. (iii)

From (ii), $\rho=v\lambda$ or $ds/d\psi=\lambda\sqrt{(2ks+c)}$, from (iii)

or $(2ks+c)^{-1/2}\,ds=\lambda\,d\psi$.

Integrating, we get $[2\sqrt{(2ks+c)}]/2k=\lambda\psi+\alpha$,

where α is constant of integration

or $\sqrt{(2ks+c)}=k\lambda\psi+\beta$, where $\beta=\alpha k$

or $2ks+c=(k\lambda\psi+\beta)^2=k^2\lambda^2\psi^2+2k\lambda\beta\psi+\beta^2$

or $s=A\psi^2+B\psi+D$, where $A=\frac{1}{2}k\lambda^2$, $B=\lambda\beta$, $D=(\beta^2-c)/2k$.

This is the required equation of the curve. **Ans.**

****Ex. 12. A particle is moving in a parabola ($p^2=ar$) with uniform angular velocity about the focus, prove that its normal acceleration at any point is proportional to the radius of curvature of its path at that point.** *(Agra 91; Gorakhpur 96, Kanpur 93, 90, Meerut 96, 93, 92, 91)*

Sol. The equation of the parabola is $p^2=ar$ .. (i)

[Students should commit this equation to memory, as it is sometimes not given in the problem.]

Differentiating (i), we get $2p\frac{dp}{dr}=a$ or $\frac{dp}{dr}=\frac{a}{2p}$

$\therefore$ Radius or curvature $\rho=r\frac{dr}{dp}=\frac{r}{(a/2p)}=\frac{2pr}{a}$. ...(ii)

Angular velocity $=d\theta/dt=$ constant $=\omega$ (say) ...(iii)

Also $\frac{d\theta}{dt}=\frac{"vp"}{r^2}$ or $\omega=\frac{vp}{r^2}$, from (iii)

or $$v=\omega r^2/p \quad \text{...(iv)}$$

Also normal acceleration $=v^2/\rho$.

$\therefore$ $\frac{\text{normal acceleration}}{\rho}=\frac{v^2}{\rho^2}=\frac{\omega^2 r^4/p^2}{4p^2r^2/a^2}$, from (ii) and (iv)

$=r^2\omega^2a^2/(4p^4)=r^2\omega^2a^2/4(a^2r^2)$, from (i)

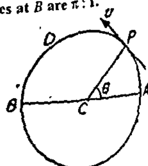
$=\omega^2/4=\text{constant}=k$ (say),

or normal acceleration $=k\times\rho$.

i.e. normal acceleration $\propto\rho$. Hence proved

Ex. 13. One point describes the distance AB of a circle with constant velocity, and another the semi-circumference AB from rest with constant tangential acceleration. They start together from A and arrive together at B. Show that the velocities at B are $\pi:1$.

Sol. Let V be the constant velocity with which the point describes the diameter $AB=2a$, where a is the radius of the circle with centre C.



(Fig. 18)

Then if t_1 be the time taken by this point in moving from A to B, then

$$Vt_1=2a \quad \text{or} \quad t_1=2a/V \quad \text{...(i)}$$

The second point describes the semi-circumference ABD of this circle with centre C and radius a

Let P be the position of this particle after time t. Let v be its velocity at P.

Given tangential acc. of this particle = constant.

$\therefore$ $dv/dt=\text{constant}=\lambda$ (say) or $dv=\lambda\,dt$.

Integrating, we have $v=\lambda t+C_1$, where C_1 is constant.

Initially at A, $v=0$, $t=0$, so we get $C_1=0$

$$\therefore \quad v=\lambda t \quad \text{...(ii)}$$

or $$ds/dt=\lambda t \quad \text{or} \quad ds=\lambda t\,dt.$$

Integrating, we have $s=\frac{1}{2}\lambda t^2+C_2$, where C_2 is constant. ...(iii)

Initially at A, $s=0$, $t=0$, so we get $C_2=0\Rightarrow s=\frac{1}{2}\lambda t^2$.

When the particle arrives at B, $s=\text{arc } ADB=\pi a$ and $t=t_1$, as both particles start from A together and reach B at the same time.

$\therefore$ From (iii), we get $\pi a=\frac{1}{2}\lambda t_1^2=\frac{1}{2}\lambda(2a/V)^2$, from (i)

or $$\pi a=2a^2\lambda/V^2 \quad \text{or} \quad \lambda=(\pi V^2/2a) \quad \text{...(iv)}$$

$\therefore$ From (ii), we get $v=(\pi V^2/2a)\,t$.

$\therefore$ If V_1 be the velocity of the second particle when it reaches B, then from (iv), we get $V_1=(\pi V^2/2a)\,t_1$. (Note)

or $V_1 = \frac{\pi V^2}{2a} \cdot \frac{2a}{V}$, from (i) or $V_1 = \pi V$.

$\therefore$ Velocities at B are $v_1 : V = \pi : 1$. Hence proved.

****Ex. 14. A particle moves in a catenary $s = c \tan \psi$, the direction of its acceleration at any point makes equal angles with the tangent and the normal to the path at the point. If the speed at the vertex (where $\psi = 0$) be u, show that the velocity and acceleration at any point ψ are ue^{ψ} and $(\sqrt{2}/c)\, u^2 e^{2\psi} \cos^2 \psi$.** *(Meerut 97; Rohilkhand 92)*

Sol. Since the direction of the resultant acceleration makes equal angles with the tangent and the normal, so the tangential and normal accelerations are equal

i.e. $$v \frac{dv}{ds} = \frac{v^2}{\rho} \quad ...(i)$$

or $$\frac{dv}{v} = \frac{1}{\rho} ds = \frac{d\psi}{ds} ds = d\psi.$$

Integrating we have, $\log v = \psi + \log k$, where $\log k$ is constant of integration.

When $\psi = 0$, $v = u$, so we have $\log u = 0 + \log k$ or $u = k$

$\therefore$ $\log v = \psi + \log u$ or $\log (v/u) = \psi$ or $v = ue^{\psi}$, ...(ii)

which gives velocity at any point.

Now the equation of the catenary is $s = c \tan \psi$

$\therefore$ The radius of curvature $\rho = ds/d\psi = c \sec^2 \psi$ ...(iii)

Now resultant acceleration

$= \sqrt{[(v\, dv/ds)^2 + (v^2/\rho)^2]} = \sqrt{[(v^2/\rho)^2 + (v^2/\rho)^2]}$, from (i).

$= \frac{v^2}{\rho} \sqrt{2} = \frac{u^2 e^{2\psi}}{c \sec^2 \psi} \cdot \sqrt{2}$, from (ii) and (iii)

$= (\sqrt{2}/c)\, u^2 e^{2\psi} \cos^2 \psi$. Hence proved

***Ex. 15. A particle projected with a velocity u is acted on by a force which produces a constant acceleration f in the plane of the motion inclined at a constant angle α with the direction of motion. Obtain the intrinsic equation of the curve described, and show that the particle will be moving in the opposite direction to that of projection at time $\frac{u}{f \cos \alpha} (e^{\pi \cot \alpha} - 1)$.**

Sol. Since the acceleration f is inclined at an angle α with the direction of motion i.e. the tangential sense.

$\therefore$ The equation of motion along tangential and normal directions are given by $v \frac{dv}{ds} = f \cos \alpha$ ...(i) and $\frac{v^2}{\rho} = f \sin \alpha$. ...(ii)

Integrating (i), we get $\frac{1}{2} v^2 = fs \cos \alpha + C$.

Let $s = 0$ initially i.e. when $v = u$.

Then $\frac{1}{2}u^2 = 0 + C$ or $C = \frac{1}{2}u^2$.

$\therefore \quad v^2 = 2fs \cos\alpha + u^2$. ...(iii)

Substituting this value of v^2 in (ii), we get

$2fs\cos\alpha + u^2 = f\rho \sin\alpha = f.(ds/d\psi)\sin\alpha$, $\quad \because \rho = ds/d\psi$

or $\dfrac{f\,ds}{2fs\cos\alpha + u^2} = \dfrac{d\psi}{\sin\alpha}$ or $\dfrac{2f\cos\alpha\,ds}{2fs\cos\alpha + u^2} = 2\cot\alpha\,d\psi$,

multiplying both sides by $2\cos\alpha$

Integrating, $\log(2fs\cos\alpha + u^2) = 2\psi\cot\alpha + \log A$.

Let $\psi = 0$ when $s = 0$, then $\log u^2 = \log A$,

$\therefore \log(2fs\cos\alpha + u^2) = 2\psi\cot\alpha + \log u^2$. ...(iv)

or $$\frac{u^2 + 2fs\cos\alpha}{u^2} = e^{2\psi\cot\alpha}$$ (Note)

or $$[(2f\cos\alpha)/u^2]\,s = e^{2\psi\cot\alpha} - 1$$

or $$s = [u^2/(2f\cos\alpha)]\,(e^{2\psi\cot\alpha} - 1),$$

is the required intrinsic equation of the path

Again from (i), we get $d^2s/dt^2 = f\cos\alpha$.

Integrating, $ds/dt = ft\cos\alpha + K$.

When $t = 0$, $ds/dt = u$, $\therefore u = 0 + K$ ...(v)

Hence $\quad ds/dt = ft\cos\alpha + u$.

Also from (iii), we have $v = \sqrt{(2fs\cos\alpha + u^2)}$

or $\quad ds/dt = \sqrt{(u^2 e^{2\psi\cot\alpha})}$, from (iv)

or $\quad ds/dt = u\,e^{\psi\cot\alpha}$

$\therefore$ From (v), we get $ue^{\psi\cot\alpha} = ft\cos\alpha + u$ ...(vi)

or $\quad t = u\,(e^{\psi\cot\alpha} - 1)/(f\cos\alpha)$.

When the particle is moving in a direction opposite to that of the projection (i.e. to opposite to $\psi = 0$), we have $\psi = \pi$ and hence putting $\psi = \pi$ in (vi), we have the required time

$$u\,(e^{\pi\cot\alpha} - 1)\,(f\cos\alpha).$$

Hence proved

****Ex. 16. The tangential acceleration of a particle moving along a circle of radius a is λ times the normal acceleration. If the speed at a certain time is u, prove that it will return to the same point after a time $(a/\lambda u)(1 - e^{-2\pi\lambda})$.** (Agra 92; Bundelkhand 92; Gorakhpur 92)

Sol. Given that tangential acceleration $= \lambda$ (normal acceleration),

$$v\frac{dv}{ds} = \lambda\frac{v^2}{\rho} \quad \text{or} \quad \frac{v\,dv}{ds} = \frac{\lambda v^2}{(ds/d\psi)}, \quad \because \rho = \frac{ds}{d\psi}$$

or $\quad dv = \lambda v\,d\psi$ or $(1/v)\,dv = \lambda\,d\psi$

or $\quad \log v - \log c = \lambda\psi$ or $v/c = e^{\lambda\psi}$ or $v = ce^{\lambda\psi}$ ...(i)

Also the equation of the path of the particle is given to be a circle of radius a so its equation is $s = a\psi$.

Now let velocity v be u when $\psi = 0$, then from (i), we get

$$u = ce^0 \quad \text{or} \quad c = u.$$

$\therefore$ From (i) we get, $v = ue^{\lambda\psi} = ue^{\lambda s/a}$

or $\quad ds/dt = ue^{\mu s}$, where $\mu = \lambda/a$ ...(iii)

or $\quad dt = (1/u)\, e^{-\mu s}\, ds$

Required time = time taken in moving from $s = 0$ to $s = 2\pi a$, the circumference of the circle. **(Note)**

$$= \frac{1}{u}\int_{s=0}^{2\pi a} e^{-\mu s}\, ds = \left[-\frac{1}{u\mu} e^{-\mu s}\right]_0^{2\pi a} = \frac{1}{u\mu}\left[1 - e^{-2\pi a\mu}\right]$$

$= (a/\lambda\mu)\,(1 - e^{-2\pi\lambda})$, from (iii). Hence proved

****Ex. 17. The direction of the acceleration of a particle moving in a cycloid makes with the normal an angle equal to that which the tangent to the cycloid at the point makes with tangent at the vertex and is in the same sense. Prove that the tangent at the point turns uniformly, and that the magnitude of the acceleration is constant.**

Sol. Let the tangent at any point P on the cycloid make an angle ψ with the tangent at the vertex (measured in anti-clockwise direction) Then this tangent makes an angle $\pi - \psi$ with other side of the tangent. Hence the direction of acceleration makes an angle $\pi - \psi$ with the normal at P (measured in clockwise direction)

Hence $\tan(\pi - \psi) = v\dfrac{dv}{ds}\Big/\dfrac{v^2}{\rho} = v\dfrac{dv}{ds}\Big/ v^2\dfrac{d\psi}{ds}$

or $$-\tan\psi = \frac{dv}{v\, d\psi} \quad \text{or} \quad \frac{dv}{v} = -\tan\psi\, d\psi$$

Integrating, $\log v = \log\cos\psi + \log A$

or $\quad v = A\cos\psi$, where A is constant

or $\quad \dfrac{ds}{dt} = A\cos\psi \quad \text{or} \quad \dfrac{ds}{d\psi}\cdot\dfrac{d\psi}{dt} = A\cos\psi.$ ...(i)

Also the equation of the cycloid is $s = 4a\sin\psi$

or $\quad ds/d\psi = 4a\cos\psi = \rho$ (ii)

$\therefore$ From (i), we get $4a\cos\psi\,(d\psi/dt) = A\cos\psi$

or $\quad d\psi/dt = A/4a$, which is constant

Hence the tangent at the point turns uniformly.

From (i), $\quad v = ds/dt = A\cos\psi.$ (iii)

$\therefore$ Normal acceleration $= \dfrac{\text{“}v^2\text{”}}{\rho} = \dfrac{A^2\cos^2\psi}{4a\cos\psi}$, from (ii) and (iii)

$$= (A^2/4a)\cos\psi.$$

And tangential acceleration $= \dfrac{\text{“}dv\text{”}}{dt} = \dfrac{d}{dt}(A\cos\psi)$, from (iii)

$= -A\sin\psi\,(d\psi/dt) = -(A\sin\psi)\,(A/4a) = -(A^2/4a)\sin\psi$

$\therefore$ Resultant acceleration $= \sqrt{\left\{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right\}}$

$$= \sqrt{\left\{\left(-\frac{A^2}{4a}\sin\psi\right)^2 + \left(\frac{A^2}{4a}\cos\psi\right)^2\right\}} = \frac{A^2}{4a} = \text{constant}$$

Ex. 19. A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent. Show that the curve is an equiangular spiral. (Agra 92)

Sol. Let the particle, at any instant be at P, the directions of tangential and normal accelerations at P are as shown in the figure 19 below The resultant acceleration makes a constant angle α (say) with the tangent at P, so

$$\tan\alpha = \frac{(v^2/\rho)}{(v\,dv/ds)}$$

or $\quad v = (\rho \tan\alpha)\frac{dv}{ds} = \tan\alpha \frac{ds}{d\psi}\cdot\frac{dv}{ds}$, $\quad \because \rho = \frac{ds}{d\psi}$

or $\quad (1/v)\,dv = \cot\alpha\, d\psi.$ ...(i)

Integrating, $\log v = (\cot\alpha)\,\psi + \log c$, where c is constant

$\therefore \quad v = ce^{\psi\cot\alpha}$ ...(ii)

Also given that acceleration is constant, therefore, we have

$\sqrt{[(v^2/\rho)^2 + (v\,dv/ds)^2]} = k$

$\therefore$ from (i) and (iii), we get

or $\quad (v\,dv/ds)^2 \sec^2\alpha = k^2$

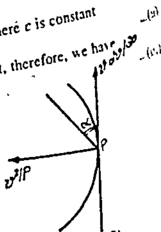
or $\quad (v\,dv/ds)^2 = k^2\cos^2\alpha$

or $\quad v\,dv/ds = k\cos\alpha$

or $\quad v\,dv = (k\cos\alpha)\,ds$

Integrating, we have

$v^2 = (2k\cos\alpha)\,s + c_1$,



(Fig. 19)

where c_1 is constant of integration

or $\quad v = \sqrt{(2ks\cos\alpha + c_1)}$ ...(iv)

$\therefore$ From (ii) and (iv) we get $\sqrt{(2ks\cos\alpha + c_1)} = ce^{\psi\cot\alpha}$

or $\quad 2ks\cos\alpha + c_1 = c^2 e^{2\psi\cot\alpha}$,

where is the intrinsic form of the equation of the equiangular spiral (See Author's Integral Calculus).

Exercises on § 6

Ex. 1. A particle is moving on a smooth curve under gravity $s = 4a$ ... velocity varies as the arcual distance from the highest point. Prove that the curve is a cycloid

Ex. 2. The normal acceleration at P is given by

(i) $v^2\rho$, (ii) v^2/ρ, (iii) ρ/v^2, (iv) v/ρ Ans. (ii)

Projectiles

§ 1. In the present chapter we shall consider the motion of a particle in a plane curve which it describes when projected in any direction in a vertical plane through the point of projection. We shall suppose that the resistance due to air and the slight variation in the forces of gravity to be negligible. The particle which is projected is called projectile and the path (curve) traced out by it is called its trajectory.

§ 2. A particle of mass m is projected from a fixed point into the air with velocity u, in a direction making an angle α with the horizontal. To find its motion and path described.

(Allahabad 80 ; Berahampur 81 ; Bhopal 83 ; Calcutta 81 ; Garhwal 79 ; Gorakhpur 84; 83 ; Indore 79; Lucknow 84, 80 ; Meerut 86, 83, 81)

Take O, the point of projection, as the origin. Take the vertical line through O as y-axis and the horizontal line through O and lying in the plane in which the particle is projected as x-axis.

Hence x and y-axes lie in a plane in which the particle moves and this plane is known as **"plane of flight"**.

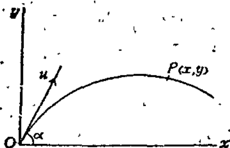


Fig. 1. (a)

Let the particle be at $P(x, y)$ after time t. Then d^2x/dt^2 and d^2y/dt^2 will be the accelerations of the particle in the directions in which x and y increase. Also the only force acting on the particle at P is its weight mg, acting vertically downwards.

Then the equations of motion in the horizontal and vertical directions are
$$m\frac{d^2x}{dt^2}=0 \quad \text{or} \quad \ddot{x}=0 \qquad \text{...(i)}$$
and
$$m\frac{d^2y}{dt^2}=-mg \quad \text{or} \quad \ddot{y}=-g. \qquad \text{...(ii)}$$

Integrating (i) we have $\dot{x}=$constant *i.e.* the horizontal component of velocity of the particle is constant.

Initially *i.e.* at O, horizontal component of velocity $=u\cos\alpha$
$$\therefore \quad \dot{x}=u\cos\alpha. \qquad \text{...(iii)}$$

Hence *the horizontal component of velocity will remain constant and equal to $u\cos\alpha$ throughout the motion.* (Remember) 39/P/1

Integrating (ii), we have

$\dot{y}=-gt+A$, where A is constant of integration.

At the origin O, $t=0$ and $\dot{y}=u\sin\alpha$.

$\therefore\ u\sin\alpha=-g.0+A$ or $A=u\sin\alpha$.

$\therefore\ \dot{y}=gt+u\sin\alpha$ ...(iv)

Results (iii) and (iv) give us the horizontal and vertical components of velocity at $P(x, y)$ i.e. at any time t.

From (iii) integrating we have $x=(u\cos\alpha)t+B$, where B is constant of integration.

Initially, i.e. at O, $x=0$, $t=0$.

$\therefore\ 0=u\cos\alpha\ 0+B$ or $B=0$.

Hence $x=(u\cos\alpha)t$. ...(v)

Also from (iv) integrating, we have

$y=(u\sin\alpha)t-\frac{1}{2}gt^2+C$, where C is constant of integration.

Initially i.e. at O, $y=0$, $t=0$.

$\therefore\ 0=0+C$ or $C=0$.

Hence $y=(u\sin\alpha)t-\frac{1}{2}gt^2$. ...(vi)

Results (v) and (vi) give us the position of the particle after time t, i.e. the coordinates of P in terms of t.

The equation of the path traced out by the particle is obtained by eliminating the variable t between (v) and (vi)

From (v), we have $t=\dfrac{x}{u\cos\alpha}$.

Substituting in (vi), we get $y=u\sin\alpha\left(\dfrac{x}{u\cos\alpha}\right)-\frac{1}{2}g\left(\dfrac{x}{u\cos\alpha}\right)^2$

or $y=x\tan\alpha-\frac{1}{2}g\dfrac{x^2}{u^2\cos^2\alpha}$, ...(A)

is the required equation of the path of the particle.

By vector method :—

Take O, the point of projection as the origin. Take the vertical line through O as y-axis and the horizontal line through O and lying in the plane in which the particle is projected as x-axis.

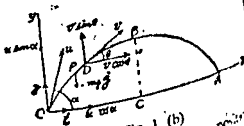


Fig. 1. (b)

Let the particle be at P after time t. Let $\mathbf{r}$ be the position vector of the point P and $\mathbf{v}$ the velocity vector of the particle at P. Let $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors in the directions of Ox and Oy. Then at P the only force acting on the particle is its weight mg acting vertically downwards and so in the vector notations it can be expressed as $-mg\,\mathbf{j}$. Therefore the equation of motion of the particle is

$$m\frac{d^2\mathbf{r}}{dt^2}=-mg\,\mathbf{j}\quad\text{or}\quad\frac{d^2\mathbf{r}}{dt^2}=-g\mathbf{j}$$

Also we know $\mathbf{v}=\frac{d\mathbf{r}}{dt}$, so from (i), $\frac{d\mathbf{v}}{dt}=-g\mathbf{j}$. ...(ii)

Integrating (ii), we get $\mathbf{v}=-gt\,\mathbf{j}+\mathbf{C}_1$, (iii)
where $\mathbf{C}_1$ is a constant vector.

Let $\mathbf{u}$ be the initial velocity vector of the particle and its magnitude be u.

Then from figure we find $\mathbf{u}=(u\cos\alpha)\,\mathbf{i}+(u\sin\alpha)\mathbf{j}$. ...(iv)

Initially *i.e.* at O, we have $\mathbf{u}=\mathbf{v}$ and $t=0$

$\therefore$ From (iii), we get $\mathbf{u}=0+\mathbf{C}_1$ or $\mathbf{C}_1=\mathbf{u}$.

$\therefore$ From (iii), we get $\mathbf{v}=-gt\,\mathbf{j}+\mathbf{u}$

or $\mathbf{v}=-gt\,\mathbf{j}+[(u\cos\alpha)\,\mathbf{i}+(u\sin\alpha\,\mathbf{j}]$, from (iv)

or $\mathbf{v}=-(u\cos\alpha)\,\mathbf{i}+(u\sin\alpha-gt)\,\mathbf{j}$. ...(v)

From this result we conclude that the components of velocity $\mathbf{v}$ at P in the directions of x and y axes are $u\cos\alpha$ and $u\sin\alpha-gt$ respectively, *i.e.*

$$\dot{x}=u\cos\alpha \text{ and } \dot{y}=u\sin\alpha-gt, \qquad ...(vi)$$

where P is (x, y) and $\dot{x}$, $\dot{y}$ have their usual meanings.

Again from (v), we get

$$\frac{d\mathbf{r}}{dt}=(u\cos\alpha)\,\mathbf{i}+(u\sin\alpha-gt)\mathbf{j}. \quad \because \mathbf{v}=\frac{d\mathbf{r}}{dt}$$

Integrating, we have

$$\mathbf{r}=[(u\cos\alpha)\,t]\,\mathbf{i}+[(u\sin\alpha)\,t-\tfrac{1}{2}gt^2]\,\mathbf{j}+\mathbf{C}_2.$$

Initially, $\mathbf{r}=0$, $t=0$, $\therefore \mathbf{C}_2=0$.

$\therefore$ We have $\mathbf{r}=[(u\cos\alpha)\,t]\,\mathbf{i}+[(u\sin\alpha)\,t-\frac{1}{2}gt^2]\,\mathbf{j}$

or $x\mathbf{i}+y\mathbf{j}=[(u\cos\alpha)\,t]\,\mathbf{i}+[(u\sin\alpha)\,t-gt^2]\,\mathbf{j}$. since $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$.

Equating the coefficients of $\mathbf{i}$ and $\mathbf{j}$ on both sides, we get

$$x=(u\cos\alpha)\,t \text{ and } y=(u\sin\alpha)\,t-\tfrac{1}{2}gt^2. \qquad ...(vii)$$

Eliminating t between the two results given by (vii), we get

$$y=(u\sin\alpha)\left[\frac{x}{u\cos\alpha}\right]-\tfrac{1}{2}g\left[\frac{x}{u\cos\alpha}\right]^2, \because t=\frac{x}{u\cos\alpha}$$

or
$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}, \qquad ...(B)$$

which is the required equation of the path of the projectile.

**Note. Since the second degree term in equation (A) or (B) forms a perfect square [after multiplying each term by $(2u^2\cos^2\alpha)/g$], hence each of these equations represents a parabola.

(Burdwan 79)

***§ 3. To find the latus rectum, the vertex, the focus and the directrix of the parabola traced out by the projectile.**

(Gorakhpur 83 ; Indore 79; Meerut 86)

From § 2 equation (A), the equation of the path of the projectile is

$$y=x\tan\alpha-\tfrac{1}{2}\frac{x^2}{u^2\cos^2\alpha} \text{ or } \frac{2u^2\cos^2\alpha}{g}\,y=\frac{2u^2\cos^2\alpha}{g}\,x\tan\alpha-x^2$$

$$x^2-\frac{2u^2 \sin\alpha \cos\alpha}{g}x=-\frac{2u^2\cos^2\alpha}{g}y$$

or $$x^2-\frac{2u^2 \sin\alpha\cos\alpha}{g}x+\frac{u^4\sin^2\alpha\cos^2\alpha}{g^2}$$

or. $$=\frac{u^4\sin^2\alpha\cos^2\alpha}{g^2}-\frac{2u^2\cos^2\alpha}{g}y,$$ (Note)

adding $(u^4\sin^2\alpha\cos^2\alpha)/g^2$ to both sides.

or $$\left[x-\frac{u^2\sin\alpha\cos\alpha}{g}\right]^2=-\frac{2u^2\cos^2\alpha}{g}\left[y-\frac{u^2\sin^2\alpha}{2g}\right].$$

Shift the origin to $(\bar{x}, \bar{y})$, where $\bar{x}=(u^2\sin\alpha\cos\alpha)/g$ and $\bar{y}=(u^2\sin^2\alpha)/2g$.

Then the equation of the path of the projectile becomes $x^2=-[(2u^2\cos^2\alpha)/g]\,y$, ...(i) which is of the standard form $x^2=-4ay$.

Therefore the new origin is the vertex of the parabola.

Hence the coordinates of the vertex of the parabola are

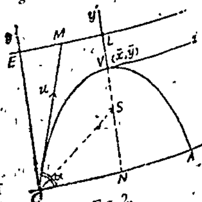


Fig. 2.

$\bar{x}=\text{ON}=(u^2\sin\alpha\cos\alpha)/g$. ...(ii)

and $\bar{y}=\text{VN}=(u^2\sin^2\alpha)/2g$ ...(iii)

Also latus rectum of the parabola $=(2u^2\cos^2\alpha)/g$
$=(2/g)$ [Horizontal component of velocity]^2

Let S be the focus and EL the directrix of the parabola.

Then $VS=VL=\frac{1}{4}\times$latus rectum$=\frac{1}{4}\left(\frac{2u^2\cos^2\alpha}{g}\right)=\frac{u^2\cos^2\alpha}{2g}$

$\therefore\ LN=VN+VL=[(u^2\sin^2\alpha)/2g]+[(u^2\cos^2\alpha)/2g]$, from (iii)
$=u^2/2g$.

$\therefore$ The equation of the directrix EL is $y=u^2/2g$ ...(iv)

From here we conclude that if a number of particles are projected from O, with the same velocity u and in the same plane though in different directions, the equation of the directrix is the same
$y=u^2/2g$.

Again the abscissae of the x-coordinates of the focus and the vertex are the same *viz.* $ON=(u^2\sin\alpha\cos\alpha)/g$, (ii).

But the ordinate of the focus $S=SN=VN-VS$
$$=\frac{u^2\sin^2\alpha}{2g}-\frac{u^2\cos^2\alpha}{2g}=-\frac{u^2}{2g}(\cos^2\alpha-\sin^2\alpha)$$ ...(v)
$$=-(u^2/2g)\cos 2\alpha.$$ ...(vi)

$\therefore$ **Coordinates of the focus S are**
$$\left(\frac{u^2\sin\alpha\cos\alpha}{g}, -\frac{u^2}{2g}\cos 2\alpha\right)$$

The focus S lies above, on or below the x-axis according as the ordinate of $S >, =$ or < 0

i.e. $(-u^2/2g)\cos 2\alpha >, =\text{or} < 0$

i.e. $\cos 2\alpha <, =\text{or} > 0$

i.e. $2\alpha >, =\text{or} < \frac{1}{2}\pi$ (Notie)

i.e. $\alpha >, =\text{or} < \frac{1}{4}\pi$

Polar coordinate of the focus.

Referred to O as pole and OX as initial line, the polar coordinates of S are $(OS, \angle SOX)$ *i.e.* (r, θ).

Now we now for a parabola the focal distance (distance from the focus) of any point on it is equal to its distance from directrix.

$\therefore$ $OS=OE=LN=u^2/2g$, from (iv)

Also for a parabola, the tangent at any point bisects the angle between the line joining that point to the focus and perpendicular from the point on the directrix.

$\therefore$ If the tangent at O to the parabola, meets the directrix in M, then $\angle EOM=\angle MOS$.

But $\angle BOM=\frac{1}{2}\pi-\alpha$

$\therefore$ $\angle MOS=\frac{1}{2}\pi-\alpha$. Also $\angle MOx=\alpha$

$\therefore$ $\angle SOx=\angle MOx-\angle MOS=\alpha-(\frac{1}{2}\pi-\alpha)=2\alpha-\frac{1}{2}\pi$.

Hence the polar coordinate of the focus S are

$$[u^2/2g, (2\alpha-\tfrac{1}{2}\pi)] \quad \text{...(vii)}$$

§ 4. **Velocity at any point.** The velocity at any point of the patricle is the resultant of the horizontal and vertical components of velocity at that point.

We have found in § 2 page 1 that at time t the horizontal and vertical components of velocity are given by

$\dot{x}=u\cos\alpha$ and $\dot{y}=u\sin\alpha-gt$ [see results (iii), (iv)]

$\therefore$ The magnitude v of the resultant velocity at time t

$$=\sqrt{(\dot{x}^2+\dot{y}^2)}$$

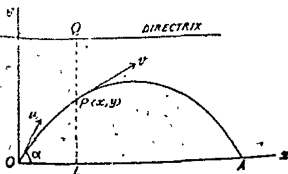
$=\sqrt{\{(u\cos\alpha)^2+(u\sin\alpha-gt)^2\}}=\sqrt{(u^2-2gt\sin\alpha+g^2t^2)}$ and its direction is inclined to horizontal at an angle

$$=\tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)=\tan^{-1}\left\{\frac{u\sin\alpha-gt}{u\cos\alpha}\right\}$$

§ 5. To prove that the velocity of projectile at any point of its path is that due to a fall from the level of the directrix.

(Agra 82, 81; Allahabad 80; Kanpur 82)

Let the particle be projected from O with velocity u making an angle α with the horizonal. Let the particle come at P (x, y) after time t. Now if $\dot{x}$ and $\dot{y}$ be the horizontal and vertical components of velocity at P, then we know that



$\dot{x}=u\cos\alpha$ and $\dot{y}=u\sin\alpha-gt$

Fig. 3.

$\therefore$ If v be the magnitude of the velocity v at P, then we have

$v^2=(\dot{x}^2+\dot{y}^2)=u^2-2ugt\sin\alpha+g^2t^2$ ...See § 4 Page 5

$=u^2-2g\,(u\sin\alpha\;t-\frac{1}{2}gt^2)$

$=u^2-2gy,\quad\because\quad y=u\sin\alpha.t-\frac{1}{2}gt^2$ ...See § 2 Pages 1-3

$=2g\left(\dfrac{u^2}{2g}-y\right)=2g\,(QL-PL),\quad\because$ directrix is given by $y=\dfrac{u^2}{2g}$

or $v^2=2g\,(PQ)=2g$ [the depth of P below the directrix]

Now suppose a particle falls freely from Q to P.

Then from "$v^2=u^2+2gx$" we have

(velocity at $P)^2=0+2g\,(PQ)=2g\,(PQ)$, which is the same as proved above.

Hence the square of velocity at any point on the path is the same as the square of velocity acquired by a particle in falling freely from rest from the level of the directrix to the point under consideration.

***§ 6. A particle is projected with velocity u making an angle α with the horizontal, to find (a) the time of flight, (b) the horizontal range (c) the greatest height attained and (d) the time for a given height.** (Agra 85)

(a) Time of flight.

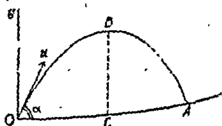


Fig. 4

The time taken by the particle in moving from the point of projection O to A, the point where the horizontal line through O (in the plane xOy) intersects the path of the flight, is known as the time of flight.

Let T be the time of flight. Then in time T, the vertical distance moved by the particle is zero.

Also from § 2 equation (vi) Page 2 the vertical distance moved by the particle in time t is given by $y=u\sin\alpha.t-\frac{1}{2}gt^2$.

At A, $\quad y=0$ and $t=T$.

$\therefore\quad 0=u\sin\alpha\,T-\frac{1}{2}gT^2\quad$ or $\quad T=(2u\sin\alpha)/g$

$\therefore$ Time of flight $=\dfrac{2u\sin\alpha}{g}=\dfrac{2}{g}$ (initial vertical velocity) (Agra 85)

(b) Horizontal Range.

OA is called the horizontal range and is generally denoted by R. Hence horizontal range is the horizontal distance described by the particle in time $T=\{(2u\sin\alpha)/g\}$ with a constant horizontal velocity $u\cos\alpha$:

$\therefore$ Horizontal range $=R=(u\cos\alpha)\,T=u\cos\alpha\,\{(2u\sin\alpha)/g\}$

$$=\frac{2u^2\sin\alpha\cos\alpha}{g}=\frac{2}{g}\,(u\cos\alpha\times u\sin\alpha)$$

$=(2/g)$ (horizontal velocity) $\times$ (initial vertical velocity)

$=(u^2/g)\sin 2\alpha$ ...(ii)

$=2$ (abscissa of the vertex) (See § 3 Page 3)

Maximum Horizontal Range.

If u remains same, α only varying, then the maximum value of the range $R=u^2/g$, since maximum value of $\sin 2\alpha=1$.

Also $\sin 2\alpha=1$, when $2\alpha=\frac{1}{2}\pi$ or $\alpha=\frac{1}{4}\pi$.

Hence the horizontal range is maximum when the particle is projected at an angle $\pi/4$ to the horizontal and maximum value of horizontal range $=u^2/g$. ...(iii)

Also for $\alpha=\frac{1}{4}\pi$ we have proved in § 3 Page 3 that focus lies on the horizontal line Ox.

Directions for Range. *(Gorakhpur 79)*

We know horizontal range $R=(u^2/g)\sin 2\alpha=(u^2/g)\sin(\pi-2\alpha)$ $=(u^2/g)\sin 2\,(\frac{1}{2}\pi-\alpha)=(u^2/g)\sin 2\beta$, where $\beta=(\frac{1}{2}\pi-\alpha)$.

Hence the range remains the same, when the particle is projected at an angle α or $(\frac{1}{2}\pi-\alpha)$ with the x-axis. Thus there are two directions in which a particle may be projected with a given velocity u so as to have a given horizontal range. These two directions are equally inclined to the horizontal and vertical directions Ox and Oy respectively and hence equally inclined to the direction of maximum range which is the bisector of Ox and Oy ($\because$ $\alpha=\frac{1}{4}\pi$).

(c) Greatest Height attained.

(Agra 86; Maharishi Dayanand 79; Meerut 81)

We have proved in § 2 Page 1 that if $\dot{x}$ and $\dot{y}$ be the horizontal and vertical components of the velocity v of the particle at time t, then
$$\dot{x}=u\cos\alpha \text{ and } \dot{y}=u\sin\alpha-gt;$$
whence we conclude that the horizontal component of velocity remains constant throughout the motion whereas the vertical component of velocity changes with the time.

At the highest point the vertical velocity should be zero because if it is not so then there is a tendency of the particle to rise still further.

Let T_1 be the time taken by the particle in reaching the highest point B. Then from $y=u\sin\alpha-gt$ we have

$0=u\sin\alpha-gT_1$ ...(iv)

or $T_1=(u\sin\alpha)/g=\frac{1}{2}$ (time of flight)

$\therefore$ Greatest height attained $=CB=$ "$u\sin\alpha.t-\frac{1}{2}gt^2$"

$=u\sin\alpha\,[(u\sin\alpha)/g]-\frac{1}{2}g\,[(u\sin\alpha)/g]^2$

$$=\frac{u^2\sin^2\alpha}{g}-\frac{u^2\sin^2\alpha}{2g}=\frac{u^2\sin^2\alpha}{2g} \quad ...(v)$$

= ordinate of the vertex. ...See § 3 Page 3 (Agra 8?)

(b) Time for a given height.

Let h be the given height. Then during the motion of the projectile the particle will be at a height h above the horizontal level through the point of projection O at two different positions P_1 and P_2 as shown in the adjoining Fig. 5.

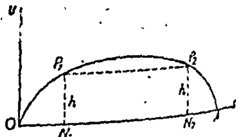


Fig 5

Then from "$y=u \sin\alpha . t-\frac{1}{2}gt^2$" we have

$h=u\sin\alpha . t-\frac{1}{2}gt^2$ or $gt^2-2u\sin\alpha . t+2h=0$,

which being a quadratic equation in t, gives two values of t corresponding to two positions P_1 and P_2 of the projectile at the same height h.

§ 7. Properties of Parabola.

The following properties of parabola should be borne in mind as they facilitate in working out problems on projectiles:—

(a) The distance of any point on the parabola from its focus is always equal to its distance from the directrix.

(b) Tangents at the extremities of any focal chord cut at right angles.

(c) The tangent at any point bisects the angle between the perpendicular from the point on the directrix and the focal distance of the point.

(d) The line joining the middle point of any chord to the parabola and the point of intersection of the tangent at the extremities of the chord is parallel to the axis of the parabola.

Solved Examples on § 2 to § 7.

Ex. 1 (a). A cricket ball is thrown with a velocity of 49m sec., find the greatest range on the horizontal plane and the two directions in which the ball may be thrown so as to give a range of 122.5 m.

Solution. Given that $u=49$m/sec.

$\therefore$ Greatest range $=\frac{u^2}{g}=\frac{49\times 49}{9.8}=245$ m. Ans

Now it is given that range $=122.5$ *metres*

$\therefore \frac{u^2\sin 2\alpha}{g}=122.5$ or $\frac{49\times 49\sin 2\alpha}{9.8}=122.5$

or $\sin 2\alpha=\frac{1}{2}$ or $2\alpha=30°$ or $\alpha=15°$.

$\therefore$ The required two directions which will give the range of 122·5 metres are 15° and 90 −15° *i.e.* 15° and 75°. **Ans.**

Ex. 1 (b). If a particle is projected inside a horizontal tunnel which is 4 meters high with a velocity of 50 meters per second, find the greatest possible range.

Solution. Let u metres/sec be the velocity and α be the angle of projection respectively.

Then we are given $u=50$ m/sec.

and max. height $=(u^2 \sin^2 \alpha)/2g=4$ meters.

i.e. $(50)^2 \sin^2 \alpha=8g=8\times 9{\cdot}8=78{\cdot}4$

or $\sin^2 \alpha=(78{\cdot}4)/(50\times 50)=0{\cdot}03$

or $\sin^2 \alpha=0{\cdot}17$ nearly and $\cos \alpha=0{\cdot}9$ nearly

$\therefore$ The greatest range $=\dfrac{2u^2 \sin \alpha \cos \alpha}{g}=\dfrac{2\times (50)^2}{9{\cdot}8}\times 0{\cdot}17\times 0{\cdot}9$

$=78$ m. nearly. **Ans.**

Ex. 1 (c). What is the least velocity with which a cricket ball can be thrown 80 meters ? (Note)

Solution. With the least velocity u (say), the maximum range will be 80 metres.

i.e. $u^2/g=80$ or $u^2=80\times 9{\cdot}8$ or $u=\sqrt{(784)}=28$m/sec. **Ans.**

***Ex. 2 (a). A particle is projected with a velocity v so that its range on a horizontal plane is twice the greatest height attained. Show that the range is $4v^2/5g$.**

Solution. Let α be the angle of projection.

Given range on horizontal plane $=2$ (greatest height attained)

i.e. $\dfrac{2v^2 \sin \alpha \cos \alpha}{g}=2\cdot\dfrac{v^2 \sin^2 \alpha}{2g}$ .. See §·6 (b) and (c) Pages 6–7

or $\tan \alpha=2$, $\therefore$ $\sin \alpha=2/\sqrt{5}$ and $\cos \alpha=1/\sqrt{5}$

Hence the required range $=\dfrac{2v^2 \sin \alpha \cos \alpha}{g}=\dfrac{2v^2}{g}\cdot\dfrac{2}{\sqrt{5}}\cdot\dfrac{1}{\sqrt{5}}$

$=4v^2/5g$. Hence, proved.

Ex. 2 (b). Prove that if the time of flight of a bullet over a horizontal range R is T seconds, the inclination of the direction of projection to the horizontal is $\tan^{-1}[gT^2/2R]$. *(Berahampur 81)*

Solution. If u and α be the velocity and angle of projection of the bullet, then we have

$R=(2u^2 \sin \alpha \cos \alpha)/g$ and $T=(2u \sin \alpha)/g$

$\therefore$ $\dfrac{gT^2}{2R}=\dfrac{g\,[(2u \sin \alpha)/g]^2}{2\,[(2u^2 \sin \alpha \cos \alpha)/g]}=\tan \alpha$, on simplifying.

or $\alpha=\tan^{-1}[gT^2/2R]$. Hence proved.

Solution. Let u_1 and u_2 be the velocities of projection of the bodies. Since the horizontal ranges of the bodies are the same, so

$$(2u_1^2 \sin\alpha_1 \cos\alpha_1)/g=(2u_2^2 \sin\alpha_2 \cos\alpha_2)/g$$

whence $\quad u_1^2/u_2^2=\sin\alpha_2 \cos\alpha_2/\sin\alpha_1 \cos\alpha_1$ $\quad$ ...(i)

Also their times of flight are given by

$$t_1=(2u_1 \sin\alpha_1)/g \text{ and } t_2=(2u_2 \sin\alpha_2)/g.$$

$$\therefore \quad \frac{t_1^2-t_2^2}{t_1^2+t_2^2}=\frac{(4/g^2)(u_1^2 \sin^2\alpha_1-u_2^2 \sin^2\alpha_2)}{(4/g^2)(u_1^2 \sin^2\alpha_1+u_2^2 \sin^2\alpha_2)}$$

$$=\frac{(u_1^2/u_2^2)\sin^2\alpha_1-\sin^2\alpha_2}{(u_1^2/u_2^2)\sin^2\alpha_1+\sin^2\alpha_2}$$

$$=\frac{\left(\dfrac{\sin\alpha_2 \cos\alpha_2}{\sin\alpha_1 \cos\alpha_1}\right)\sin^2\alpha_1-\sin^2\alpha_2}{\left(\dfrac{\sin\alpha_2 \cos\alpha_2}{\sin\alpha_1 \cos\alpha_1}\right)\sin^2\alpha_1+\sin^2\alpha_2}, \text{ from (i)}$$

$$=\frac{\sin\alpha_2 \cos\alpha_2 \sin\alpha_1-\sin^2\alpha_2 \cos\alpha_1}{\sin\alpha_2 \cos\alpha_2 \sin\alpha_1+\sin^2\alpha_2 \cos\alpha_1}$$

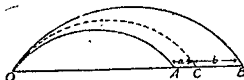
$$=\frac{\sin\alpha_2(\sin\alpha_1 \cos\alpha_2-\sin\alpha_2 \cos\alpha_1)}{\sin\alpha_2(\sin\alpha_1 \cos\alpha_2+\sin\alpha_2 \cos\alpha_1)}=\frac{\sin(\alpha_1-\alpha_2)}{\sin(\alpha_1+\alpha_2)}$$

****Ex 6 (a).** **A projectile aimed at a mark which is in a horizontal plane through the point of projection, falls a m. short of it when the elevation is α and goes bm. too far when the elevation is β. Show that if the velocity of projection be the same in all cases, the proper elevation is $\frac{1}{2}\sin^{-1}[(a \sin 2\beta+b \sin 2\alpha)/(a+b)]$.**

(Bhopal 83; Gorakhpur 82; Indore 79; Meerut 86, 85, 82)

Solution. Let O be the point of projection of the projectile which is aimed at C, such that $OC=c$ (say).

Let θ be the required proper elevation and u the velocity of projection which is the same in all cases.



(Fig. 7)

When angle of projection is α, the range is $OA=c-a$, $\quad \because AC=a$

$$\therefore \quad c-a=(u^2/g)\sin 2\alpha \quad \text{...(i)}$$

When angle of projection is β, the range is $OB=c+b$, $\quad \because CB=b$

$$\therefore \quad c+b=(u^2/g)\sin 2\beta \quad \text{...(ii)}$$

When angle of projection is θ, the range is $OC=c$

$$\therefore \quad c=(u^2/g)\sin 2\theta \quad \text{...(iii)}$$

Multiplying (i) by b and (ii) by a, and adding we have

$$c\,(b+a)=(u^2/g)\,(b \sin 2\alpha+a \sin 2\beta) \qquad \text{(iv)}$$

Substituting value of c from (iii) in (iv), we get

$$(u^2/g)\,(\sin 2\theta)\,(b+a)=(u^2/g)\,(b \sin 2\alpha+a \sin 2\beta)$$

or $\sin 2\theta=(b \sin 2\alpha+a \sin 2\beta)/(b+a)$

or $\theta=\frac{1}{2}\sin^{-1}[(b \sin 2\alpha+a \sin 2\beta)/(b+a)]$ Hence proved.

Ex. 6 (b). Two bodies are projected from the same point with the same velocity but in different directions. If the range in each case be R and the times of flight be t and t', prove that

$$R=\tfrac{1}{2}\,gtt'.$$

Solution. If for one body u be the velocity of projection and α the angle of projection, then

range $R=\dfrac{2u^2 \sin \alpha \cos \alpha}{g}$ and time of flight $=\dfrac{2u \sin \alpha}{g}$.

Then for the other body the velocity of projection remaining the same *i.e.* u but angle of projection should be $(\frac{1}{2}\pi-\alpha)$ to give the same range R. Therefore time of flight

$$t'=\frac{2u \sin(\frac{1}{2}\pi-\alpha)}{g}=\frac{2u \cos \alpha}{g}.$$

$$\therefore\ \tfrac{1}{2}gtt'=\tfrac{1}{2}g\times\frac{2u \sin \alpha}{g}\times\frac{2u \cos \alpha}{g}=\frac{2u^2 \sin \alpha \cos \alpha}{g}=R$$

, Hence proved.

****Ex. 6 (c). If h and h' be the greatest heights in the two paths of a projectile with a given velocity for a given range R, prove that $R=4\sqrt{(h\,h')}$.** (*Burdwan 79*)

Sol. Let u be the velocity of projection of the particle. Then for a given range R, there can be two possible paths one with angle of projection α (say) and the other with angle of projection $(90^\circ-\alpha)$.

$\therefore$ As h and h' are given as the greatest heights in these two paths, so we have

$$h=\frac{u^2 \sin^2 \alpha}{2g}\ ;\ h'=\frac{u^2 \sin^2(90^\circ-\alpha)}{2g}=\frac{u^2 \cos^2 \alpha}{2g}.$$

Also $R=$ horizontal range $=(2u^2 \sin \alpha \cos \alpha)/g$ (Note)

Now $h\,h'=\dfrac{u^2 \sin^2 \alpha}{2g}\cdot\dfrac{u^2 \cos^2 \alpha}{2g}=\left(\dfrac{u^2 \sin \alpha \cos \alpha}{2g}\right)^2$

or $16h\,h'=[(2u^2 \sin \alpha \cos \alpha)/g]^2=R^2$

or $R=4\sqrt{(h\,h')}$. Hence proved.

Ex. 7 (a). If R be the range of projectile on a horizontal plane and h its maximum height for a given angle of projection, show that the maximum horizontal range with the same velocity of projection is $2h+R^2/8h$.

Solution. If u be the velocity of projection and α the angle of projection, then $R=(u^2 \sin 2\alpha)/g$ and $h=(u^2 \sin^2 \alpha)/2g$

$$\therefore \quad 2h+\frac{R^2}{8h}=\frac{2u^2 \sin^2 \alpha}{2g}+\frac{u^4 \sin^2 2\alpha}{g^2}\cdot\frac{2g}{8u^2 \sin^2 \alpha}$$

$$=\frac{u^2 \sin^2 \alpha}{g}+\frac{u^2\, 4 \sin^2 \alpha \cos^2 \alpha}{4g \sin^2 \alpha}=\frac{u^2 \sin^2 \alpha}{g}+\frac{u^2 \cos^2 \alpha}{g}$$

$=u^2/g=$ max. horizontal range. Hence proved.

Ex. 7 (b). A particle is projected with initial velocity u. If h is the maximum height, r is the horizontal range from the point of projection and t is the time of flight, show that

$$h=\tfrac{1}{8} gt^2 \text{ and } (u^2/2g)=h+\tfrac{1}{4}(r^2/4h).$$

Solution. We are given that

$$h=\frac{u^2 \sin^2 \alpha}{2g} \quad \ldots(\text{i}), \qquad r=\frac{2u^2 \sin \alpha \cos \alpha}{g} \quad \ldots(\text{ii})$$

and $t=(2u \sin \alpha)/g$ ...(iii)

From (iii) we get $\sin \alpha=gt/(2u)$

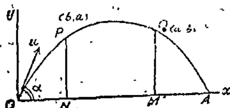
$\therefore$ From (i) we get $h=\dfrac{u^2}{2g}\left(\dfrac{gt}{2u}\right)^2=\dfrac{gt^2}{8}$ Hence proved.

Also $\dfrac{r^2}{h}=\dfrac{4u^4 \sin^2 \alpha \cos^2 \alpha}{g^2}\cdot\dfrac{2g}{u^2 \sin^2 \alpha}=\dfrac{8u^2 \cos^2 \alpha}{g}$

$\therefore$ $h+\tfrac{1}{4}\left(\dfrac{r^2}{4h}\right)=\dfrac{u^2 \sin^2 \alpha}{2g}+\dfrac{u^2 \cos^2 \alpha}{2g}=\dfrac{u^2}{2g}$ Hence proved.

Ex. 8. A particle is projected from a point on the level of the ground and its height is h when it is at horizontal distances a and $2a$ from its point of projection. Prove that the velocity of projection u is given by $u^2=\dfrac{g}{4}\left[\dfrac{4a^2}{h}+9h\right]$

Solution. Let α be the angle of projection and O the point of projection. Take the upward drawn vertical line through O as y-axis and the horizontal line through O (lying in the plane of flight) as x-axis. Then the coordinates of the points P and Q which are at height h above O are (a, h) and $(2a, h)$ respectively (correct the figure).



(Fig. 8)

Also the equation of the path of the particle is

$$y=x \tan \alpha-\frac{gx^2}{2u^2 \cos^2 \alpha}$$

As $P(a, h)$ and $Q(2a, h)$, lie on it, so we have

$h=a \tan \alpha-\frac{ga^2}{2u^2 \cos^2 \alpha}$ ...(i) and $h=2a \tan \alpha-\frac{4ga^2}{2u^2 \cos^2 \alpha}$ ...(ii)

Subtracting (i) from (ii) we get $0=a \tan \alpha-\frac{3ga^2}{2u^2 \cos^2 \alpha}$

or $a \tan \alpha=3ga^2/2u^2 \cos^2 \alpha$

$\therefore$ From (i), $h=\frac{3ga^2}{2u^2 \cos^2 \alpha}-\frac{ga^2}{2u^2 \cos^2 \alpha}=\frac{ga^2}{u^2 \cos^2 \alpha}$. (iii)

or $\sec^2 \alpha=u^2h/ga^2$

Substituting this value in (i) we have $h=a \tan \alpha-\frac{1}{2}h$

or $a \tan \alpha=\frac{3}{2}h$ or $\tan \alpha=\frac{3}{2}(h/a)$ .. or $\tan^2 \alpha=9h^2/4a^2$

or $\sec^2 \alpha-1=\frac{9h^2}{4a^2}$ or $\frac{u^2h}{ga^2}-1=\frac{9h^2}{4a^2}$, from (iii)

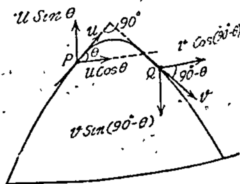
or $\frac{u^2}{g}=\frac{a^2}{h}\left[\frac{9h^2}{4a^2}+1\right]=\frac{1}{4h}\left[9h^2+4a^2\right]$ or $u^2=\frac{g}{4}\left(9h+\frac{4a^2}{h}\right)$

Hence proved

****Ex. 9. If at any instant the velocity of a particle be u, and its direction of motion θ to the horizontal, then show that it will be moving at right angles to the direction after time $(u/g) \operatorname{cosec} \theta$.**

(Avadh 81; Gorakhpur 86; Kanpur 83)

Solution. Let at P the velocity of the particle be u, making an angle θ with the horizontal. Let v be the velocity of the particle at Q, when it is moving at right angles to its direction at P. $\therefore$ At Q its direction of motion is inclined to the horizontal at an angle $(90°-\theta)$ as shown in the diagram.



(Fig. 9)

$\therefore$ Horizontal component of velocity remains constant throughout the motion.

$\therefore$ $u \cos \theta=v \cos (90°-\theta)$ or $u \cos \theta=v \sin \theta$ ..(i)

Also for the vertical component of velocity from "$v=u+ft$" (Note we have $-v \sin (90°-\theta)=u \sin \theta-gt$,

where t is the time taken in moving from P to Q.

or $-v \cos \theta=u \sin \theta-gt$,

or $gt=u \sin \theta+v \cos \theta=u \sin \theta+[u \cos \theta/\sin \theta] \cos \theta$, from (i)

$=(u/\sin \theta)[\sin^2 \theta+\cos^2 \theta]=u \operatorname{cosec} \theta$

or $t=(u/g) \operatorname{cosec} \theta$. Hence proved

***Ex. 10. If α be the angle between the tangents at the extremities of any arc of a parabolic path, v and v' the velocities at these extremities and u the velocity at the vertex of the path, show that the time of describing the arc is $(vv' \sin \alpha)/(ug)$.** *(Bhopal 81)*

Solution. Let PQ be the arc of the parabolic path under consideration. Let θ be the angle which direction of motion of the particle at P makes with the horizontal, then the direction of motion at Q makes an angle $(\pi-\theta-\alpha)$ with the horizontal, as shown in the diagram.

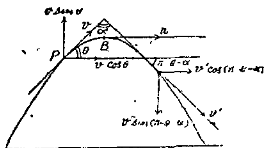


Fig. 10.

$\because$ Horizontal component of velocity remains constant throughout the motion.

$\therefore$ $v \cos \theta = u = v' \cos (\pi-\theta-\alpha)$

or $v \cos \theta = u = -v' \cos (\theta+\alpha)$ ...(1)

Also for the vertical component of velocity from "$v=u+ft$" for the motion from P to Q, we have

$-v' \sin (\pi-\theta-\alpha) = v \sin \theta - gt$ (Note)

or $gt = v \sin \theta + v' \sin (\theta+\alpha)$

$$= v \sin \theta \left[\frac{-v' \cos (\theta+\alpha)}{u}\right] + v' \sin (\theta+\alpha)\left[\frac{v \cos \theta}{u}\right], \text{ from (1)}$$ (Note)

or $t = \dfrac{vv'}{gu}[\sin (\theta+\alpha) \cos \theta - \cos (\theta+\alpha) \sin \theta]$

$= (vv'/ug)\sin [(\theta+\alpha)-\theta] = (vv'/ug) \sin \alpha.$ Hence proved

****Ex. 11. If v_1, v_2 be the velocities at the ends of a focal chord of a projectile's path and u the velocity at the vertex of the path, then show that $\dfrac{1}{v_1^2}+\dfrac{1}{v_2^2}=\dfrac{1}{u^2}$** *(Gorakhpur 81 ; Kanpur 82)*

Solution. Let PQ be the given focal chord. Let the direction of motion of the projectile at P make an angle θ with the horizontal, then the direction of motion of the projectile at Q makes an angle $(90°-\theta)$ with the horizontal (as shown in the diagram).

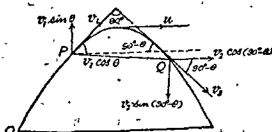


Fig. 11.

Here we should remember that the tangents at the ends of a focal chord of a parabola intersect each other at right angles.

∵ Horizontal component of the velocity remains constant thoughout the motion.

∴ $v_1 \cos \theta = v_2 \cos (90° - \theta) = u$ or $v_1 \cos \theta = v_2 \sin \theta = u$

or $\cos \theta = u/v_1$ and $\sin \theta = u/v_2$.

Squaring and adding, we get $1 = \frac{u^2}{v_1^2} + \frac{u^2}{v_2^2}$ or $\frac{1}{v_1^2} + \frac{1}{v_2^2} = \frac{1}{u^2}$.

Ex. 12. If v_1, v_2 are the velocities at two points P and Q on a parabolic trajectory, and PT and QT the corresponding tangents, prove that $v_1 : v_2 = PT : QT$

Solution. If K be the mid-point of the chord PQ, then KT is parallel to the axis of a parabola, which is vertical. Hence KT is a vertical line.

Let PT and QT be inclined at angles β and α respectively to the vertical.

∵ Horizontal component of velocity remains constant throughout the motion,

∴ $v_1 \sin \beta = v_2 \sin \alpha$

or $\frac{v_1}{v_2} = \frac{\sin \alpha}{\sin \beta}$ ...(i)

Also from $\triangle QKT$,

$\frac{QT}{\sin \theta} = \frac{KQ}{\sin \alpha}$,

where $\angle QKT = \theta$ (say)

or $QT \sin \alpha = KQ \sin \theta$. ...(ii)

Similarly from $\triangle PTK$,

$\frac{PT}{\sin (\pi - \theta)} = \frac{PK}{\sin \beta}$ ...(iii)

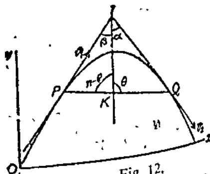


Fig. 12.

or $PT \sin \beta = PK \sin \theta$.

Also K being the mid-point of PQ, $PK = KQ$.

∴ From (ii) and (iii), we get $QT \sin \alpha = PT \sin \beta$

or $\sin \alpha / \sin \beta = PT/QT$. Hence proved

∴ From (i) we get $v_1/v_2 = PT/QT$.

Ex. 13. (a) A particle is projected under gravity with a velocity u in a direction making an angle α with the horizon. Show that the amount of deviation D in the direction of motion of the particle is given by $\tan D = (gt \cos \alpha)/(u - gt \sin \alpha)$.

Solution. After time t, let the particle be at P. Let the direction of motion of the particle at P be inclined at an angle β to the horizontal and v be the velocity thereat.

Then deviation D in the direction of motion of the particle during this time t = angle between the tangents at O and $P = \alpha - \beta$. ...(i)

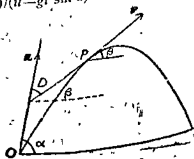


Fig. 13.

∵ Horizontal component of a velocity remains constant throughout the motion, so

$$v \cos \beta = u \cos \alpha. \quad ...(ii)$$

Also for the vertical motion from O to P, we have

$$v \sin \beta = u \sin \alpha - gt. \quad ...(iii)$$

Dividing (iii) by (ii), we get $\tan \beta = \dfrac{u \sin \alpha - gt}{u \cos \alpha}$ (iv)

or $\tan \beta = \tan \alpha - \dfrac{gt}{u \cos \alpha}$ or $\tan \alpha - \tan \beta = \dfrac{gt}{u \cos \alpha}$ ...(v)

Now $\tan D = \tan (\alpha - \beta)$, from (i)

$$= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{gt/(u \cos \alpha)}{1 + \tan \alpha \, [(u \sin \alpha - gt)/u \cos \alpha]},$$

from (iv) and (v)

$$= \frac{gt}{u \cos \alpha + u \sin \alpha \tan \alpha - gt \tan \alpha} = \frac{gt \cos \alpha}{u - gt \sin \alpha}.$$

Ex. 13. (b) If a particle be projected from the point O at an elevation α and after time t it be at P, then prove that

$$\tan \theta = \tfrac{1}{2} (\tan \alpha + \tan \beta),$$

where θ and β are the angles which OP and the direction of the particle at P make with the horizontal. (*Bhopal 82*)

Solution. Refer Fig. 13 Page 16.

If v be velocity of the particle at P, then as the horizonal component of velocity remains constant throughout the motion, so we have

$$v \cos \beta = u \cos \alpha \quad (i)$$

And for the vertical motion from O to P, we have from "$v = u + ft$",

$$v \sin \beta = u \sin \alpha - gt \quad ..(ii)$$

and from "$s = ut + \frac{1}{2} ft^2$",

$$PN = OP \sin \theta = (u \sin \alpha)\, t - \tfrac{1}{2} gt^2 \quad ...(iii)$$

(Students are requested to join OP and from P draw PN perpendicular to Ox.)

Also $ON = OP \cos \theta$ = horizontal distance moved by the particle in time t

$$= (u \cos \alpha)\, t \quad ...(iv)$$

Dividing (ii) by (i) we get $\tan \beta = \dfrac{u \sin \alpha - gt}{u \cos \alpha}$

or $\tan \beta = \tan \alpha - [gt/(u \cos \alpha)]$

or $[gt/(u \cos \alpha)] = \tan \alpha - \tan \beta$ ...(v)

Dividing (iii) by (iv) we get

$$\tan \theta = \frac{(u \sin \alpha)\, t - \frac{1}{2} gt^2}{(u \cos \alpha)\, t} = \tan \alpha - \frac{gt}{2 (u \cos \alpha)}$$

or $\tan\theta = \tan\alpha - \frac{1}{2}(\tan\alpha - \tan\beta)$, from (v)

or $\tan\theta = \frac{1}{2}(\tan\alpha + \tan\beta)$. Hence proved

****Ex. 13.** A particle is projected with a velocity $2\sqrt{(ag)}$ so that it just clears two walls of equal height a which are at a distance $2a$ from each other. Show that the latus rectum of the path is equal to $2a$ and the time of passing between the walls is $2\sqrt{(a/g)}$. *(Kanpur 85)*

Solution. Let the angle of projection be α. Let PN and QM be two walls of equal height a at a distance $2a$ apart i.e. $NM = 2a$.

Here the velocity of projection $= "u" = 2\sqrt{(ag)}$.

Now the height of the directrix above the point of projection

$$= \frac{"u^2"}{2g} = \frac{4ag}{2g} = 2a.$$

Hence the height of the directrix above PQ

$$= 2a - a = a.$$

Also we know that in the case of a parabola, the distances of any point on the parabola from the directrices and the focus must be equal.

(Fig. 14)

$\therefore$ The distance of P from the focus S (say) $=$ the distance of P from the directrix $= a$ (proved)

$\therefore$ S must be the mid point of PQ which is equal to $NM = 2a$

$\therefore$ PQ is the latus rectum of the parabola and equal to $2a$ ...(i)

Also latus rectum $= "(2u^2 \cos^2\alpha)/g"$.

$\therefore$ $(2u^2\cos^2\alpha/g) = 2a$ or $u\cos\alpha = \sqrt{(ag)}$

$\therefore$ If t be the time of passing between the walls, then

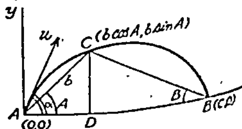
$$(u\cos\alpha).t = 2a \text{ or } t = \frac{2a}{u\cos\alpha} = \frac{2a}{\sqrt{(ag)}}, \text{ from (i)}$$

$$= 2\sqrt{(a/g)}.$$ Hence proved

***Ex. 15. A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection, prove that**

$$\tan\alpha = \tan A + \tan B$$

(Agra 85 ; Gorakhpur 85)



(Fig. 15)

Solution. Let u be the velocity of projection and A the point of projection. Take AB as x-axis and upward drawn vertical line through A as y-axis Then coordinates of the vertex C and the point B are $(b \cos A, b \sin A)$ and $(c, 0)$, where $AC=b$ and $AB=c$.

The equation of the path of the particle is

$y=x \tan \alpha-\dfrac{gx^2}{2u^2 \cos^2 \alpha}$, where α is the angle of projection.

As $C(b \cos A, b \sin A)$ and $B(c, 0)$ lie on it,

$$\therefore\ b \sin A=b \cos A \tan \alpha-\frac{gb^2 \cos^2 A}{2u^2 \cos^2 \alpha} \qquad \ldots\text{(i)}$$

and $$0=c \tan \alpha-\frac{gc^2}{2u^2 \cos^2 \alpha} \qquad \ldots\text{(ii)}$$

From (ii) we get $\dfrac{2u^2 \cos^2 \alpha}{g}=\dfrac{c}{\tan \alpha}$

$\therefore$ From (i), $b \sin A=b \cos A \tan \alpha-\dfrac{b^2 \cos^2 A \tan \alpha}{c}$

or $$c \sin A=(c \cos A-b \cos^2 A) \tan \alpha$$

or $$\tan \alpha=\frac{c \sin A}{c \cos A-b \cos^2 A}$$

$$=\frac{\sin C \sin A}{\sin C \cos A-\sin B \cos^2 A}, \quad \because \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$=\frac{\sin (A+B) \sin A}{\sin (A+B) \cos A-\sin B \cos^2 A}, \quad \because C=\pi-(A+B)$$

$$=\frac{\sin (A+B) \sin A}{\sin A \cos B \cos A}=\frac{\sin A \cos B+\cos A \sin B}{\cos A \cos B}$$

$$=\tan A+\tan B.$$ Hence proved.

Ex. 16. A shot fired at an elevation α is observed to strike the foot of a tower which rises above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is $\frac{1}{2}[\theta+\sin^{-1}(\sin \theta+\sin 2\alpha \cos \theta)]$.

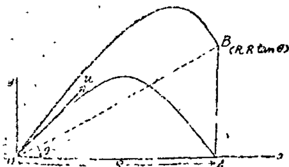


Fig. 16

Solution. Let AB be the tower and O the point of projection of the shot. Let $OA=R$, then $AB=R \tan \theta$.

Referred to horizontal line (lying in the plane of flight) and upward drawn vertical line through O as axes the cordinates of the point B are $(R, R \tan \theta)$

Also if u be the velocity of projection when angle of projection is α, we have $R=(u^2 \sin 2\alpha)/g$. ..(i)

Let β be the required angle of projection to strike the top, then the equation of the trajectory is $y=x \tan \beta-\dfrac{gx^2}{2u^2 \cos^2 \beta}$.

Since the point $B\ (R, R \tan \theta)$ lies on it, so

$$R \tan \theta=R \tan \beta-\frac{gR^2}{2u^2 \cos^2 \beta} \quad \text{or} \quad \tan \theta=\tan \beta-\frac{gR}{2u^2 \cos^2 \beta}$$

or $\dfrac{\sin \theta}{\cos \theta}=\dfrac{\sin \beta}{\cos \beta}-\dfrac{g}{2u^2 \cos^2 \beta}\left(\dfrac{u^2 \sin 2\alpha}{g}\right)$, from (i)

or $2 \sin \theta \cos^2 \beta=2 \sin \beta \cos \beta \cos \theta-\sin 2\alpha \cos \theta$

or $\sin 2\alpha \cos \theta=2 \cos \beta\ [\sin \beta \cos \theta-\sin \theta \cos \beta]$

$=2 \cos \beta \sin (\beta-\theta)=\sin (2\beta-\theta)-\sin \theta$

or $\sin (2\beta-\theta)=\sin 2\alpha \cos \theta+\sin \theta$

or $(2\beta-\theta)=\sin^{-1} [\sin \theta+\sin 2\alpha \cos \theta]$

or $\beta=\frac{1}{2}\ [\theta+\sin^{-1} (\sin \theta+\sin 2\alpha \cos \theta)]$. Hence proved

***Ex. 17.** Shots are fired simultaneously from the top and bottom of a vertical cliff with elevation α and β respectively strike an object simultaneously. Show that if a be the horizontal distance of the object from the cliff, the height of the cliff is $a (\tan \beta-\tan \alpha)$.

Solution. OO' is the cliff of height h (say). P is the object Let u_1 and u_2 be the velocities of the shots fired from the top and bottom of the cliff respectively.

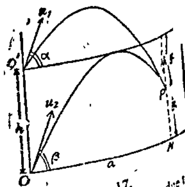


Fig. 17.

Since the shots are fired simultaneously and strikes the object P simultaneously.

$\therefore$ time taken by each shot in reaching P is the same.

Let this time be t, then as the horizontal component of velocity remains constant throughout the motion and the horizontal distance travelled is due to this component of velocity, so we have

$$a=u_1 \cos \alpha . t_1=u_2 \cos \beta\ t_1 \quad ..(i)$$

or $$u_1 \cos \alpha=u_2 \cos \beta.$$

Also referred to horizontal and vertical lines through the points of projection as coordinate axes, the equations of the paths traced out by the shots projected from O' and O are

$$y=x\tan\alpha-\frac{gx^2}{2u_1^2\cos^2\alpha}\quad\text{...(ii) and}\quad y=x\tan\beta-\frac{gx^2}{2u_1^2\cos^2\beta}\quad\text{...(iii)}$$

Let the height of P above O be y_1, then as shown in the diagram the depth of P below O' is $(h-y_1)$,

$\therefore$ The coordinates of P referred to axes through O are (a, y_1) and through O' are $(a, -h-y_1)$.

Hence from (ii) and (iii) we have

$$-(h-y_1)=a\tan\alpha-\frac{ga^2}{2u_1^2\cos^2\alpha}\text{ and } y_1=a\tan\beta-\frac{ga^2}{2u_2^2\cos^2\beta}$$

Subtracting, $h=a(\tan\beta-\tan\alpha)-\dfrac{ga^2}{2}\left(\dfrac{1}{u_2^2\cos^2\beta}-\dfrac{1}{u_1^2\cos^2\alpha}\right)$

$=a(\tan\beta-\tan\alpha)$, from (i), Hence proved.

*Ex. 13. A gun is firing from the sea level out to sea. It is then [illegible] d fired at the same [illegible] of itself, [illegible] u being the velocity of projection.

(*Avadh 80; Gorakhpur 79; Meerut 81*)

Solution. Let R_1 be the range when the gun is firing from the sea level. Then we have $R_1=(2u^2\sin\alpha\cos\alpha)/g$. ...(i)

Now let the gun be mounted on the battery at a height h feet above the sea level.

Then referred to O, the point of projection, as origin and Ox and Oy (as shown in the fig.) as coordinate axes, the coordinates of A, where the shot strikes the water are $(R_2, -h)$, where R_2 is the range in this case *i.e.*, $O'A=R_2$, where O' is vertically below O at a depth h.

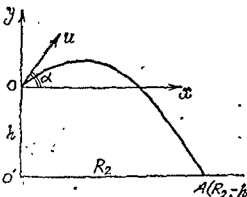


Fig. 8.

The equation of the path of this shot is

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}$$

$\because$ $A(R_2, -h)$ lies on it, so $-h=R_2\tan\alpha-\dfrac{gR_2^2}{2u^2\cos^2\alpha}$

or $gR_2^2-2u^2\sin\alpha\cos\alpha\, R_2-2u^2h\cos^2\alpha=0$

or $gR_2^2-gR_1R_2-2u^2h\cos^2\alpha=0$, from (i)

or $gR_2^2-gR_1R_2-2u^2h\left(\frac{gR_1}{2u^2\sin\alpha}\right)^2=0$, from (i)

or $R_2^2-R_1R_2-\frac{hgR_1^2}{2u^2\sin^2\alpha}=0$

or $\left(R_2-\frac{R_1}{2}\right)^2=\frac{R_1^2}{4}+\frac{hgR_1^2}{2u^2\sin^2\alpha}=\frac{R_1^2}{4}\left[1+\frac{2gh}{u^2\sin^2\alpha}\right]$

or $R_2-\frac{R_1}{2}=\frac{R_1}{2}\left[1+\frac{2gh}{u^2\sin^2\alpha}\right]^{1/2}$

or $R_2-R_1=\frac{R_1}{2}\left[1+\frac{2gh}{u^2\sin^2\alpha}\right]^{1/2}-\frac{R_1}{2}$, (Note) subtracting $\frac{1}{2}R_1$ from both sides

or $R_2-R_1=\frac{R_1}{2}\left[\left(1+\frac{2gh}{u^2\sin^2\alpha}\right)^{1/2}-1\right]$. Hence proved.

Ex. 19. A cirkcet ball is thrown from a height of 3 metres at anangle of 30° to the horizontal, with a speed of 35 meters/sec. It is caught by another fieldsmen at a height of 1 metre from the ground. How far apart were the two fieldsmen ?

Solution. Let O be the point of projection of the cricket ball. Take the horizonal and upward drawn vertical lines through O as x and y-axes repectively (as shown in the figure). Then for the ball, we are given 'u'=35 metres/sec. and $\alpha=30°$.

Then equation of the path of the ball-referred to Ox and Oy as axes is

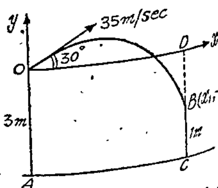


Fig. 19

"$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}$"

. see§2 Page 1 of this chapter

or $y=x\tan 30°$.

$-[(9.8)\,x^2/\{2(35)^2\cos^2 30°\}]$

$=(x/\sqrt{3})$

$-[(9.8)\,x^2/\{2(35)^2(3/4)\}]$

$=(x/\sqrt{3})-2x^2/225$

or $225y=75\sqrt{3}x-2x^2$...(i)

Let B the point at which the other fieldsman catches the ball. Then from figure it is evident that the two fieldsmen are at A and C, such that $OA=3\,m$ and $BC=1\,m$ (given). ∴ If $AC=x_1$, then the coordinates of B referred to Ox and Oy are $[x_1, -(3-1)]$ or $(x_1, -2)$ (Note)

Hence the y-coordinate of B will be negative as it is below x-axis, (see fig. 19 above).

∵ $B(x_1, -2)$ lie on the parabocic path of the ball, so its coordinates must satisfy (i) and we have

$225(-2)=75\sqrt{3}\,x_1-2x_1^2$ or $2x_1^2-75\sqrt{3}\,x_1-450=0$

$x_1=\frac{1}{4}[75\sqrt{3}\pm\sqrt{\{(75\sqrt{3})^2+4\times2\times450\}}]$
$=\frac{1}{4}[75\sqrt{3}\pm5\sqrt{\{(9\times75)+144\}}]$
$=\frac{1}{4}[75\sqrt{3}\pm15\sqrt{(91)}]$, $\because x_1$ is positive

$\therefore$ Required distance $=\frac{1}{4}[75\sqrt{3}+15\sqrt{(91)}]$ metres
$=\frac{1}{4}[75(1\cdot732)+15(9\cdot539)]$ m.
$=68\cdot246$ metres. **Ans.**

Ex. 20 An aeroplane moving horizontally at 396 kilometres per hour drops a bomb from a height of 1960 metres. How far should it be horizontally from the object to be hit?

Solution. Let O be the point of projection and B the object to be hit.

Initially horizontal velocity
$=396$ km./hour
$=\frac{396\times1000}{3600}$ m/sec.
$=110$ m./sec.

Initially vertical velocity $=0$

Let t be the time taken by the bomb in moving from O to B and the height of B above O
$=-1960$ m. (Note)

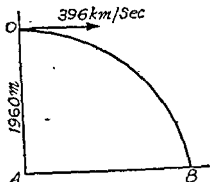


Fig. 20.

Consider the vertical motion from O to B. From "$s=ut+\frac{1}{2}ft^2$" we get $-1960=0+\frac{1}{2}(-g)t^2$

or $t^2=\frac{1960\times2}{9\cdot8}=\frac{39200}{98}=400$ or $t=20$ sec.

$\therefore$ Required distance $=AB=$ (horizontal velocity) $\times t$
$=110\times20=2200$ m. $=2\cdot2$ kms. **Ans.**

Ex. 21. A ball is thrown from the top of a tower 200 feet high with a velocity of 80 feet per second at an elevation of 30° above the horizon. Find the horizontal distance from the foot of the tower to the point where it hits the ground.

Solution. Refer Fig. 18 Page 21.

Here $u=80$ ft./sec; $\alpha=30°$; $h=200$ feet and $g=32$ ft/sec²

Referred to O as origin and Ox and Oy as shown in the figure as axes, the equation of the path of the ball is

$$y=x\tan30°-\frac{gx^2}{2(80)^2\cos^2 30°} \quad \text{(Note)}$$

or $$y=\frac{x}{\sqrt{3}}-\frac{32x^2\times4}{2\times6400\times3} \text{ or } y=\frac{x}{\sqrt{3}}-\frac{x^2}{300} \quad ...(i)$$

Also the co-ordinates of the point where the ball hits the ground is $(R, -200)$, where R is the required distance.

$\therefore$ From (i) as this point lies on the path, so we have

$$-20=\frac{R}{\sqrt{3}}-\frac{R^2}{300} \text{ or } \sqrt{3}R^2-300R-6000\sqrt{3}=0.$$

or $R=200\sqrt{3}$ ft. (solving the quadratic equation)

**Ex. 22. Two particles are projected simultaneously in the same vertical plane from the same point with velocities u and v at angles α and β with the horizontal. Show that the interval between their transits through the other point common to their path is

$$\frac{2uv \sin(\alpha-\beta)}{g\,(u\cos\alpha+v\cos\beta)}.$$

(Lucknow 84)

Solution. Let the point of projection be taken as origin, the horizontal line (lying in the plane of flight) and the upward vertical line through the point of projection as x and y-axes respectively.

Then the equations of the paths of the two particles are

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha} \quad ...(i) \quad \text{and} \quad y=x\tan\beta-\frac{gx^2}{2v^2\cos^2\beta} \quad ...(ii)$$

Let (h, k) be the other common point of their paths, then as (h, k) lies on both (i) and (ii), we get

$$k=h\tan\alpha-\frac{gh^2}{2u^2\cos^2\alpha} \quad \text{and} \quad k=h\tan\beta-\frac{gh^2}{2v^2\cos^2\beta}$$

Subtracting, $0=h(\tan\alpha-\tan\beta)-\frac{gh^2}{2}\left(\frac{1}{u^2\cos^2\alpha}-\frac{1}{v^2\cos^2\beta}\right)$

or $(\tan\alpha-\tan\beta)=\frac{gh}{2}\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)\left(\frac{1}{u\cos\alpha}+\frac{1}{v\cos\beta}\right)$

or $\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\frac{gh}{2}\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)\left(\frac{v\cos\beta+u\cos\alpha}{uv\cos\alpha\cos\beta}\right)$ (Note)

or $h\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)=\frac{2}{g}\,\frac{uv\sin(\alpha-\beta)}{g\,(u\cos\alpha+v\cos\beta)}$. ...(iii)

Also if t_1 and t_2 be times taken by the particles in reaching (h, k) from the point of projection, then considering their motion in the horizontal sense, we get $h=u\cos\alpha.\,t_1=v\cos\beta.t_2$. ...(iv)

$\therefore$ Required time $=t_1-t_2=\frac{h}{u\cos\alpha}-\frac{h}{v\cos\beta}$, from (iv)

$=h\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)=\frac{2uv\sin(\alpha-\beta)}{g\,(u\cos\alpha+v\cos\beta)}$, from (iii)

Ex. 23. In the above example, prove that the line joining them moves parallel to itself and the time that elapses when their velocities are parallel is $\frac{uv\sin(\alpha-\beta)}{g\,(v\cos\beta-u\cos\alpha)}$.

Solution. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the positions of the particles after time t then we have

$x_1=(u\cos\alpha)\,t$ and $y_1=(u\sin\alpha)\,t-\frac{1}{2}gt^2$

and $x_2=(v\cos\beta)\,t$ and $y_2=(v\sin\beta)\,t-\frac{1}{2}gt^2$

$\therefore$ Slope of $P_1P_2=\frac{y_2-y_1}{x_2-x_1}$. (See Coordinate Geometry)

$=\frac{v\sin\beta-u\sin\alpha}{v\cos\beta-u\cos\alpha}$, which being free from t is constant

Hence the line joining them (*i.e.*, P_1P_2) moves parallel to itself.

Let θ_1 and θ_2 be the angles which the directions of motion of the particles at P_1 and P_2 make with the horizontal, then

$$\tan\theta_1 = \frac{\text{vertical component of velocity}}{\text{horizontal component of velocity}} = \frac{u\sin\alpha - gt}{u\cos\alpha}$$

Similarly $\tan\theta_2 = \dfrac{v\sin\beta - gt}{v\cos\beta}$.

When the velocities of the particles are parallel, $\theta_1 = \theta_2$

i.e. $$\frac{u\sin\alpha - gt}{u\cos\alpha} = \frac{v\sin\beta - gt}{v\cos\beta}$$

or $$gt\,(u\cos\alpha - v\cos\beta) = uv\,(\sin\beta\cos\alpha - \sin\alpha\cos\beta)$$

or $$t = \frac{uv\sin(\alpha-\beta)}{g(v\cos\beta - u\cos\alpha)}.$$

Hence proved.

*Ex. 24. **From a tower an object was observed on the ground at a depression ϕ below the horizon. A gun was fired at an elevation α, but the shot missing the object struck the ground at a point whose depression was ψ. Prove that the correct elevation θ of the gun is given by** $$\frac{\cos\theta\sin(\theta+\phi)}{\cos\alpha\sin(\alpha+\psi)} = \frac{\cos^2\phi\sin\psi}{\cos^2\psi\sin\phi}.$$

Solution Let P and Q be the points whose angles of depression are ϕ and ψ as observed from the top of the tower. Let h be the height of the tower. Take top of the tower as origin and the horizontal line (lying in the plane of flight) and the vertical line through the top of the tower as x and y axes respectively. Then referred to these axes, the coordinates of P and Q are $(h\cot\phi, -h)$ and $(h\cot\psi, -h)$ respectively.

The equation of the path of the shot, which is projected with a velocity u making an angle α with the horizon, is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$\because$ It passes through Q $(h\cot\psi, -h)$,

$$\therefore\quad -h = h\cot\psi\tan\alpha - \frac{gh^2\cot^2\psi}{2u^2\cos^2\alpha}$$

or $$1+\cot\psi\tan\alpha = \frac{gh}{2u^2}\cdot\frac{\cot^2\psi}{\cot^2\alpha} \quad\text{or}\quad \frac{\sin(\alpha+\psi)}{\sin\psi\cos\alpha} = \frac{gh\cot^2\psi}{2u^2\cos^2\alpha} \qquad \text{.. (i)}$$

If θ be the correct elevation of the gun to strike P $(h\cot\phi, -h)$, then the equation of the path of the shot is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$ and P $(h\cot\phi, -h)$ lies on it.

$$\therefore \text{ we have } -h = h\cot\phi\tan\theta - \frac{gh^2\cot^2\phi}{2u^2\cos^2\theta}$$

simplifying as in the above case, we have

or $$\frac{\sin(\theta+\phi)}{\sin\phi\cos\theta}=\frac{gh\cot^2\phi}{2u^2\cos^2\theta} \quad ..(ii)$$

Dividing (ii) by (i) we get

$$\frac{\sin(\theta+\phi)\sin\psi\cos\alpha}{\sin(\alpha+\psi)\sin\phi\cos\theta}=\frac{\cot^2\phi\cos^2\alpha}{\cot^2\psi\cos^2\theta}$$

or $$\frac{\sin(\theta+\phi)\cos\theta}{\sin(\alpha+\psi)\cos\alpha}=\frac{\cos^2\phi\sin\psi}{\cos^2\psi\sin\phi}.$$ Hence proved.

***Ex. 25 Two particles are projected from the same point in the same vertical plane with equal velocities. If t, t' be the times taken to reach the other common point of their paths and T, T' the times to the highest points, show that $(tT+t'T')$ is independent of the directions of projection** (*Gorakhpur 81, Lucknow 81*)

Solution. Let u be the velocity of projection and α, β the angles of projection of the two particles. Then referred to horizontal and upward drawn vertical lines through the point of projection (and lying in the plane of flight) as coordinate axes, the equations of the paths of the particles are

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha} \text{ and } y=x\tan\beta-\frac{gx^2}{2u^2\cos^2\beta}$$

Let (h, k) be the other common point of their path,

then $$k=h\tan\alpha-\frac{gh^2}{2u^2\cos^2\alpha}=h\tan\beta-\frac{gh^2}{2u^2\cos^2\beta}$$

or $$h(\tan\alpha-\tan\beta)=(gh^2/2u^2)(\sec^2\alpha-\sec^2\beta)$$

or $$\tan\alpha-\tan\beta=(gh/2u^2)(\tan^2\alpha-\tan^2\beta)$$

or $$\tan\alpha+\tan\beta=2u^2/gh$$

Also we know that the time to reach the highest point is given by $(u\sin\alpha)/g$.

$$\therefore \quad T=(u\sin\alpha)/g \quad \text{and} \quad T'=(u\sin\beta)/g \quad ..(ii)$$

Again considering the motion of the particles in the horizontal direction, we have $h=(u\cos\alpha).t=(u\cos\beta)t'$

whence $$t=\frac{h}{u\cos\alpha} \text{ and } t'=\frac{h}{u\cos\beta} \quad ..(iii)$$

$$\therefore \quad tT+t'T'=\frac{h}{u\cos\alpha}\cdot\frac{u\sin\alpha}{g}+\frac{h}{u\cos\beta}\cdot\frac{u\sin\beta}{g}, \text{ from (ii) and (iii)}$$

$$=\frac{h}{g}\left(\tan\alpha+\tan\beta\right)=\frac{h}{g}\left(\frac{2u^2}{gh}\right), \text{ from (i)}$$

$=2u^2/g^2$, which is independent of α and β.

*Ex. 26. **A body is projected at an angle α to the horizon, so as to clear two walls of equal height a at a distance $2a$ from each other. Show that the range is equal to $2a \cot \frac{1}{2}\alpha$.**

Solution. Let u be the velocity of projection and R the required range of the particle.

Then $$R=(2u^2 \sin \alpha \cos \alpha)/g. \qquad ...(i)$$

Also referred to horizontal and upward drawn vertical lines through the point of projection (and lying in the plane of flight) as coordinate axes, the equation of the path is

$$y=x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}.$$

As the top of the walls of height a lie on it.

∴ The distance of the walls from the point of projection are given by $$a=x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$$

or $$gx^2-2u^2x \sin \alpha \cos \alpha+2au^2 \cos^2 \alpha=0. \qquad ...(ii)$$

Let the distances of the walls from the point of projection of the particle be x_1 and x_2. Then x_1 and x_2 are the roots of the equation (ii).

∴ $$x_1+x_2=(2u^2 \sin \alpha \cos \alpha)/g=R, \text{ from (i)} \qquad ...(iii)$$

and $$x_1 x_2=(2au^2 \cos^2 \alpha)/g \qquad ...(iv)$$

Now distance between the walls $=2a=x_2-x_1$.

Squaring $4a^2=(x_2-x_1)^2=(x_2+x_1)^2-4x_1x_2$

or $$4a^2=R^2-\frac{8au^2 \cos^2 \alpha}{g}, \text{ from (iii) and (iv)}$$

or $$4a^2=R^2-4a \cos^2 \alpha\, [R/(\sin \alpha \cos \alpha)], \text{ from (i)}$$

$$=R^2-4aR \cot \alpha$$

or $$R^2-4aR \cot \alpha-4a^2=0$$

or $$R=\tfrac{1}{2}\,[4a \cot \alpha+\sqrt{\{16a^2 \cot^2 \alpha+16a^2\}}]$$

(negative sign is inadmissible as R is positive)

or $$R=2a \cot \alpha+2a \operatorname{cosec} \alpha=2a\,[(\cos \alpha+1)/\sin \alpha]$$

$$=2a\left[\frac{2 \cos^2 \frac{1}{2}\alpha}{2 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha}\right]=2a \cot \tfrac{1}{2}\alpha.$$

Ex. 27. **A shell bursts on contact with the ground and pieces from it fly in all directions with velocities upto 49 metres per second. Show that a man 122·5 metres away is in danger for $5\sqrt{2}$ seconds.**

Solution. Given that $R=122·5$ metres and $u=49·5$ m/sec.

But $$R=(u^2 \sin 2\alpha)/g$$

∴ $$122·5=[(49)^2 \sin 2\alpha]/9·8 \quad \text{or} \quad \sin 2\alpha=\tfrac{1}{2}$$

or $$2\alpha=30° \quad \text{or} \quad \alpha=15°.$$

As there are always two directions of projections α and $(\frac{1}{2}\pi - \alpha)$ for a given range, hence for the range of 122·5 metres there will be two directions of projection *i.e.* 15° and 90°−15° *i.e.* 15° and 75°.

Let t_1 and t_2 be the times of flight in the two cases, then

$$t_1=\frac{2u\sin 15°}{g} \text{ and } t_2=\frac{2u\sin 75°}{g}.$$

The man is in danger for a time $=t_2-t_1$

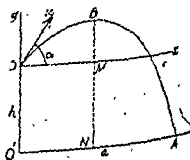
$$=\frac{2u}{g}(\sin 75°-\sin 15°)=\frac{2u}{g}(2\cos 45°\sin 30°)$$

$$=\frac{2\times 49}{9\cdot 8}\times 2\times\frac{1}{\sqrt{2}}\times\frac{1}{2}=5\sqrt{2} \text{ seconds. Hence proved.}$$

Ex. 28. **A gun is fired from the top of a cliff of height h and the shot attains a maximum height of $(h+b)$ above sea level and strikes the sea at a distance a from the foot of the cliff. Prove that angle of elevation of the gun is given by the equation**

$$a^2\tan^2\alpha-4ab\tan\alpha-bh=0,$$

Solution. O and O' are the top and foot of the cliff respectively. $BN=(h+b)$ is the max. height above the sea level attained by the shot. The shot strikes the sea at A, such that $O'A=a$.



(Fig. 2

Let u and α be the velocity and angle of projection of the shot from O. The coordinates of A referred to the axes through O (as shown in the figure) are $(a, -h)$.

Then equation of the path of the shot is

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}.$$

$\because$ $A(a, -h)$ lies on it, so $-h=a\tan\alpha-\dfrac{ga^2}{2u^2\cos^2\alpha}$ ...(i)

Also $BH=\dfrac{u^2\sin^2\alpha}{2g}$ ...see § 6 (c) Page 7 of this Chapter.

or $BN-MN=\dfrac{u^2\sin^2\alpha}{2g}$ or $(h+b)-h=\dfrac{u^2\sin^2\alpha}{2g}$

or $u^2/2g=b/\sin^2\alpha.$ ...(ii)

Eliminating (u^2/g) from (i) and (ii), we get

$$-h=a\tan\alpha-\frac{a^2}{2\cos^2\alpha}\left[\frac{\sin^2\alpha}{2b}\right]$$

or $a^2\tan^2\alpha-4ab\tan\alpha-4bh=0.$ Hence proved.

Exercises § 1 to § 7

Ex. 1. If R be the horizontal range and h the greatest height of a projectile, prove that the initial velocity is $[2g\{h+(R^2/16h)\}]^{1/2}$.

Ex. 2. A stone thrown at an elevation of 45° from the top of a tower reaches the ground in 4 seconds at a distance of 60 metres from the base of the tower. How long would the stone take to reach the ground, if it were projected with the same velocity in a horizontal direction ? *(Burdwan 80)*

Ex. 3. If a particle is projected from O with a given velocity so as to pass through a given point (h, k); show that there are two angles of projection possible. *(Gorakhpur 80)*

Ex. 4. If the maximum horizontal range of a given gun is R, determine the firing angle which should be used to hit a target located at a distance $\frac{1}{2}R$ on the same level.

Ex. 5. A particle is projected from a point at a height $3h$ above a horizontal plane, the direction of projection making an angle α with the horizon. Show that if the greatest height above the point of projection is h, the horizontal distance travelled before striking the plane is $6h \cot \alpha$.

Ex. 6. A particle is to be projected with a given speed u so that its horizontal range is $R\ (< u^2/g)$. Prove that there are two possible values for the angle of projection. Further show that if t, t' are the times of flight corresponding to the two angles of projection, then $t\,t'=2R/g$.

[Hint : See § 6 (b) Pages 6-7, Ex 6 (b) Page 12].

*Ex. 7. If v_1 and v_2 be the velocities at the ends of a focal chord of projectile's path and u be the horizontal component of the velocity, then show that $1/v_1^2+1/v_2^2=1/u^2$.

[Hint : See Ex. 11 Page 15].

Ex. 8. Two particles are projected from the same point at the same instant with equal velocities V at elevations α and α'. Prove that the time that elapses between their transit through the point where the paths intersect, is $\frac{2V}{g} \cdot \frac{\sin \frac{1}{2}(\alpha-\alpha')}{\cos \frac{1}{2}(\alpha+\alpha')}$.

[Hint : Put $u=v=V$ and in $\beta=\alpha'$ in Ex. 22 Page 24].

Ex. 9. A shell bursts on contact with the ground and pieces from it fly in all directions with velocities upto $24\frac{1}{2}$ m. per second. Show that a man $30\frac{5}{8}$ metres away is in danger for $\frac{5}{8}\sqrt{3}$ seconds. (Here $g=9{\cdot}8$ m./sec²). [Hint : See Ex. 27 Page 27].

**§ 8. Locus of the vertices and the focii of the trajectory.

To determine the locus of the vertices and the focii of the paths of the particles projected in the same vertical plane from the same point with same velocity but in different directions.

Locus of the vertices.

If (x_1, y_1) be the coordinates of the vertex for any trajectory where u and α are the velocity and angle of projection respectively then $x_1=(u^2 \sin\alpha \cos\alpha)/g$ and $y_1=(u^2 \sin^2\alpha)/2g$

...See § 3 Page 3 of this chapter.

From these eliminating α we have

$$x_1^2=\frac{u^4 \sin^2\alpha \cos^2\alpha}{g}=\frac{u^2}{g^2}\left(\frac{2gy_1}{u^2}\right)\left(1-\frac{2gy_1}{u^2}\right),$$

$\because \cos^2\alpha=1-\sin^2\alpha$

or $gx_1^2=2(u^2-2gy_1)\,y_1$ or $x_1^2+4y_1^2=(2u^2/g)\,y_1$.

Generalising (x_1, y_1) we have the required locus of the vertex as $x^2+4y^2=(2u^2/g)\,y$.

Locus of the focii :

If (x_2, y_2) be the coordinates of the focus of any trajectory, where u and α are velocity and angle of projection respectively,

then $$x_2=\frac{u^2 \sin\alpha \cos\alpha}{g}=\frac{u^2 \sin 2\alpha}{2g};\ y=-\frac{u^2 \cos 2\alpha}{2g}$$

.. See § 3 Page 3 of this chapter.

Squaring and adding, α is eliminated and we get

$$x_2^2+y_2^2=(u^4/4g^2).$$

Generalising (x_2, y_2) the required locus of the focus is $x^2+y^2=u^4/4g^2$, which is a circle of radius $u^2/2g$ and the point of projection as its centre.

Solved Examples on § 8.

Ex. 1 (a). Particles are projected from the same point in a vertical plane with velocity $\sqrt{(2kg)}$; prove that the locus of the vertices of their paths is the ellipse $x^2+4y(y-k)=0$.

Solution. If α be the angle of projection and (x_1, y_1) be the coordinates of the vertex of one of the trajectories, then

$$x_1=\frac{\text{“}u^2 \sin\alpha \cos\alpha\text{”}}{g}=\frac{2gk \sin\alpha \cos\alpha}{g},\quad \because u=\sqrt{(2gk)}$$

$$=2k \sin\alpha \cos\alpha$$

and $$y_1=\frac{\text{“}u^2 \sin^2\alpha\text{”}}{2g}=\frac{2gk \sin^2\alpha}{2g}=k \sin^2\alpha$$

Eliminating α from these we have,

$$x_1^2=4k^2 \sin^2\alpha \cos^2\alpha=4k^2\left(\frac{y_1}{k}\right)\left(1-\frac{y_1}{k}\right)=4y_1(k-y_1)$$

or $$x_1^2+4y_1(y_1-k)=0.$$

Generalising, the required locus is $x^2+4y(y-k)=0$.

Ex. 1 (b). Prove that if particles be projected from the same point in the same plane so as to describe equal parabolas, the vertices of their paths lie on a parabola. Find the vertex and latus rectum of this parabola. *(Kanpur 83)*

Solution. Let u be the velocity of projection and α the angle of projection.

Since the parabolas described for different values of u and α are equal therefore their latera recta will be equal, so we have

$(2u^2 \cos^2 \alpha)/g=\text{constant}$ or $u \cos \alpha=\text{constant}=k$ (say) (i)

Let (x_1, y_1) be the coordinates of the vertex of one of the trajectories, then

$$x_1=\frac{u^2 \sin \alpha \cos \alpha}{g}=\frac{ku \sin \alpha}{g} \quad \text{or} \quad \frac{gx_1}{k}=u \sin \alpha \qquad \text{(ii)}$$

Also $y_1=(u^2 \sin^2 \alpha)/2g$ (iii)

$\therefore$ From (ii) and (iii) we get $y_1=\left(\frac{gx_1}{k}\right)^2 \cdot \frac{1}{2g}=\frac{gx_1^2}{2k^2}$

$\therefore$ Locus of (x_1, y_1) *i.e.* of vertex is

$y=\frac{gx^2}{2k^2}$ or $x^2=\frac{2k^2}{g} y$, which is a parabola, whose vertex is $(0, 0)$ and latus rectum is $2k^2/g$. Hence the result.

***Ex. 2 Particles are projected simultaneously in the same vertical plane from the same point. Show that the locus of the foci of all the trajectories is a parabola when for each trajectory there is the same (a) horizontal velocity, (b) initial vertical velocity and (c) time of flight.**

Solution. Let u be the velocity of projection and α the angle of projection of one of the trajectories. Then if (x_1, y_1) be the coordinates of the focus, we have

$$x_1=\frac{u^2 \sin 2\alpha}{2g} \text{ and } y_1=-\frac{u^2 \cos 2\alpha}{2g} \qquad \text{...(i)}$$

(a) If the horizontal velocity is the same for each trajectory then

$$u \cos \alpha=k \text{ (say)} \qquad \text{(ii)}$$

Now $x_1=\frac{u^2 \sin \alpha \cos \alpha}{g}=\frac{ku \sin \alpha}{g}$, from (ii)

or $x_1^2=\frac{k^2u^2 \sin^2 \alpha}{g^2}=\frac{k^2}{g^2}(u^2-u^2 \cos^2 \alpha)=\frac{k^2}{g^2}(u^2-k^2)$

or $g^2x_1^2=k^2(u^2-k^2)$. (iv)

Also $y_1=-\frac{u^2}{2g} \cos 2\alpha=-\frac{u^2}{2g}(2 \cos^2 \alpha-1)=-\frac{1}{2g}(2k^2-u^2)$, from (ii)

or $2gy_1=u^2-2k^2$ .. (v)

Eliminating u^2 from (iv) and (v) we get

$$\frac{g^2 x_1^2}{k^2} - 2gy_1 = k^2 \quad \text{or} \quad x_1^2 = \frac{2gk^2}{g^2}\left(y_1 + \frac{k^2}{2g}\right)$$

∴ The locus of the focus (x_1, y_1) is

$$x^2 = \frac{2k^2}{g}\left(y + \frac{k^2}{2g}\right), \text{ which is a parabola.}$$

(b) If the initial velocity is constant, then ... (vi)

$$u \sin \alpha = k$$

∴ From (i), $x_1^2 = \dfrac{(u^2 \sin^2 \alpha)(u^2 \cos^2 \alpha)}{g^2} = \dfrac{k^2 (u^2 - k^2)}{g^2}$, from (iv)

or $$(g^2 x_1^2)/k^2 = u^2 - k^2 \qquad \text{... (vii)}$$

Also $y_1 = \dfrac{u^2 \cos 2\alpha}{2g} = -\dfrac{u^2}{2g}(1 - 2\sin^2 \alpha)$

$= -(1/2g)(u^2 - 2k^2)$, from (vi) ...(viii)

or $2gy_1 = -u^2 + 2k^2$.

Adding (vii) and (viii) we get $\dfrac{g^2 x_1^2}{k^2} + 2gy_1 = k^2$

or $$x_1^2 = \frac{k^2}{g^2}(k^2 - 2gy_1) = -\frac{2k^2}{g}\left(y_1 - \frac{k^2}{2g}\right)$$

∴ The locus of the vertex (x_1, y_1) is $x^2 = -\dfrac{2k^2}{g}\left(y - \dfrac{k^2}{2g}\right)$,

which is a parabola.

(c) If the time of flight is constant, then

$(2u \sin \alpha)/g = \text{constant}$ or $u \sin \alpha = \text{constant} = k$ (say)

Proceed further as in case (b) above.

Ex. 3. Particles are projected from the same point in a vertical plane with velocities which vary as $1/\sqrt{(\sin \theta)}$, θ being the angle of projection, find the locus of the vertices of the parabolas described *(Lucknow 84; Meerut 83)*

Solution. Here the velocity of projection $u = k/\sqrt{(\sin \theta)}$, ... (i) where k is some constant and θ is the angle of projection.

Let (x_1, y_1) be the coordinates of vertex of one of the trajectories, then $x_1 = \dfrac{\text{"}u^2 \sin \theta \cos \theta\text{"}}{g} = \dfrac{(k^2/\sin \theta) \sin \theta \cos \theta}{g}$, from (i) ...(ii)

or $$x_1 = (k^2 \cos \theta)/g$$

Also $y_1 = \dfrac{\text{"}u^2 \sin^2 \theta\text{"}}{2g} = \dfrac{(k^2 \sin \theta) \sin^2 \theta}{2g}$, from (i) ... (iii)

or $y_1 = (k^2 \sin \theta)/2g$ or $2y_1 = (k^2 \sin \theta)/g$

Eliminating θ between (ii) and (iii) we get

$$x_1^2 + (2y_1)^2 = (k^2/g)^2 \quad \text{or} \quad x_1^2 + 4y_1^2 = k^4/g^2$$

∴ The required locus of the vertex (x_1, y_1) is Ans.

$$x^2 + 4y^2 = k^4/g^2$$

Exercise on § 8

Ex. Show that the locus of the foci of all trajectories which pass through two given point

*§ 9. Two directions of

79)

From § 2 Page 1 we know that the equation of the path of a projectile is $y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$

If the projectile has to hit a given point (h, k), them

$$k = h \tan \alpha - \frac{gh^2}{2u^2 \cos^2 \alpha} = h \tan \alpha - \frac{gh^2}{2u^2} \sec^2 \alpha$$

or $$2u^2 k = 2u^2 h \tan \alpha - gh^2 (1 + \tan^2 \alpha)$$

or $$gh^2 \tan^2 \alpha - 2u^2 h \tan \alpha + (gh^2 + 2u^2 k) = 0, \quad \ldots(i)$$

which being a quadratic equation in $\tan \alpha$, shows that *in general there are two values of* $\tan \alpha$ *or* α, *i.e. two directions in which a particle can be projected with a given velocity so as to hit a given point.*

These two values of $\tan \alpha$ given by equation (i) will be real provided $(2u^2 h)^2 \geqslant 4gh^2 (gh^2 + 2u^2 k)$ or $u^4 \geqslant g (gh^2 + 2u^2 k)$

or $u^4 \geqslant g^2 h^2 + 2u^2 gk$ or $u^4 - 2u^2 gk + g^2 k^2 \geqslant g^2 h^2 + g^2 k^2$

or $(u^2 - gk)^2 \geqslant g^2 (h^2 + k^2)$ or $u^2 \geqslant g [k + \sqrt{h^2 + k^2}]$.

Hence the least value of u, the velocity of projection is given by

$$u^2 = g [k + \sqrt{(h^2 + k^2)}].$$

(*Kanpur 86; Lucknow 80; Meerut 85, 79*)

§ 10. Two times of flight to a given point.

If t be the time taken by the particle in reaching a point P (h, k) from the point of projection O (0, 0), then we get

$$h = (u \cos \alpha)\, t \text{ and } k = (u \sin \alpha)\, t - \tfrac{1}{2} g t^2.$$

Eliminating α between these equations, we get

$$h^2 + (k + \tfrac{1}{2} g t^2)^2 = u^2 t^2 \quad \text{or} \quad h^2 + k^2 + kgt^2 + \tfrac{1}{4} g^2 t^4 = u^2 t^2$$

or $$g^2 t^4 + 4(kg - u^2)\, t^2 + 4\,(h^2 + k^2) = 0$$

or $$t^4 + 4\left(\frac{k}{g} - \frac{u^2}{g^2}\right) t^2 + \frac{4}{g^2}\,(h^2 + k^2) = 0.$$

If t_1^2 and t_2^2 are the roots of this equation, then

$$t_1^2 + t_2^2 = -\left(\frac{k}{g} - \frac{u^2}{g^2}\right)$$

and $t_1^2 t_2^2 = (4/g^2)\,(h^2 + k^2)$ or $t_1 t_2 = (2/g) \sqrt{(h^2 + k^2)} = (2/g)\, OP$.

Solved Examples on § 9–§ 10.

**Ex. 1 Find the least velocity of projection of a particle from a given point A so that it may pass through another given point B.

(*Bhopal 81; Lucknow 83; Meerut 85*)

Solution. Same as § 9 above.

Ex. 2 (a). A stone is projected with velocity u from a height h to hit a point in the level at a horizontal distance R from the point

of projection. Show that the angle of projection is given by

$$R^2 \tan^2 \alpha - \frac{2u^2}{g} R \tan \alpha + R^2 - \frac{2hu^2}{g} = 0.$$

Hence deduce that the maximum range on this level for this velocity is $\sqrt{\left(\frac{u^2}{g^2} + \frac{2hu^2}{g}\right)}$ and that if R is this max. range and α the angle of projection to give the max. range, than tan $\alpha = u^2 g/R$ and tan $2\alpha = R'/h$.

Solution. Let O, the point of projection be taken as origin, the horizontal and vertical lines through O (as shown in the figure) be taken as coordinate axes.

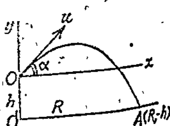


Fig. 22.

Then the equation of path of the stone is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2\alpha} \quad ...(i)$$

Let $A(R, -h)$ be the point which is to be hit by the stone. Then as A lies on (i), so we have,

$$-h = R \tan \alpha - \frac{gR^2}{2u^2 \cos^2 \alpha}$$

or $$-2u^2 h = 2u^2 R \tan \alpha - gR^2 (1 + \tan^2 \alpha)$$

or $$R^2 \tan^2 \alpha - \frac{2u^2}{g} R \tan \alpha + \left[R^2 - \frac{2u^2 h}{g}\right] = 0. \quad ..(ii)$$

Since the path traced out by the stone is real, therefore the roots of equation (ii) are also real.

Hence "$B^2 - 4AC \geqslant 0$",

or $$\frac{4u^4}{g^2} R^2 - 4R^2\left(R^2 - \frac{2u^2 h}{g}\right) \geqslant 0 \text{ or } \frac{u^4}{g^2} - \left(R^2 - \frac{2u^2 h}{g}\right) \geqslant 0$$

or $$\frac{u^4}{g^2} + \frac{2u^2 h}{g} \leqslant R^2 \text{ or } R \leqslant \left(\frac{u^4}{g^2} + \frac{2u^2 h}{g}\right)^{1/2} \quad (A)$$

$\therefore$ Maximum value of $R = R' = \left(\frac{u^4}{g^2} + \frac{2u^2 h}{g}\right)^{1/2}$. (iii)

Also if R' be the max. range and α the corresponding angle of projection, then from (ii)

$$R' \tan^2 \alpha - \frac{2u^2}{g} R' \tan \alpha + R'^2 - \frac{2u^2 h}{g} = 0$$

or $$R'^2 \tan^2 \alpha - \frac{2u^2}{g} R' \tan \alpha + \left(\frac{u^4}{g^2} + \frac{2u^2 h}{g}\right) - \frac{2u^2 h}{g} = 0, \text{ from (iii)}$$

or $\quad g^2R'^2\tan^2\alpha-2u^2.gR'\tan\alpha+u^4=0$

or $\quad (u^2-gR'\tan\alpha)^2=0 \quad$ or $\quad \tan\alpha=\dfrac{u^2}{gR'}$ ...(iv)

Also $\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}=\dfrac{2(u^2/gR')}{1-(u^4/g^2R'^2)}=\dfrac{2u^2gR'}{g^2R'^2-u^4}$

or $\quad \tan 2\alpha=\dfrac{2u^2gR}{g^2\left(\dfrac{u^4}{g^2}+\dfrac{2u^2h}{g}\right)-u^4}$, from (iii)

$$=\frac{2u^2gR'}{2u^2gh}=\frac{R'}{h}.$$

Hence proved.

*Ex. 2. (b) Show that the velocity required to project a particle from a height h to fall ... ... the point of projection is at least eq...

Solution. Refer F... ... of R we are given a and we are to find the least value of u.

As in Ex. 2 (a) we can find [See Result. (A) Page 34] that

$\dfrac{u^4}{g^2}+\dfrac{2u^2h}{g}\geqslant a^2 \quad$ or $\quad \dfrac{u^4}{g^2}+\dfrac{2u^2h}{g}+h^2\geqslant a^2+h^2$

or $\quad \left(\dfrac{u^2}{g}+h\right)^2\geqslant a^2+h^2 \quad$ or $\quad \dfrac{u^2}{g}+h\geqslant\sqrt{(a^2+h^2)}$

or $\quad u^2\geqslant g\,[\sqrt{(a^2+h^2)}-h]$

or $\quad u\geqslant\sqrt{[g\,\{\sqrt{(a^2+h^2)}-h\}]}$.

$\therefore$ The least value of u is $\sqrt{[g\,\{\sqrt{(a^2+h^2)}-h\}]}$.

Ex. 2. (c). A shot is fired from a gun at the top of a cliff of height h with velocity u ft./sec. Prove that if the range measured from the foot of the cliff is as great as possible, the elevation α is given by $\cos 2\alpha=gh/(u^2+gh)$.

Solution. As in example 2 (a) above, prove that

max. value of $R=\left(\dfrac{u^4}{g^2}+\dfrac{2u^2h}{g}\right)^{1/2}=R'$

or $\quad g^2R'^2=u^4+2u^2hg$ ...(i)

And $\tan\alpha=u^2/(gR')$ ...(ii)

$\therefore \cos 2\alpha=\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}=\dfrac{1-(u^2/gR')^2}{1+(u^2/gR')^2}$, from (ii)

$=\dfrac{g^2R'^2-u^4}{g^2R'^2+u^4}=\dfrac{(u^4+2u^2hg)-u^4}{(u^4+2u^2hg)+u^4}$, from (i)

$=\dfrac{2u^2hg}{2u^4+2u^2hg}=\dfrac{gh}{u^2+gh}$

Hence proved.

Ex. 2. (d) Prove that the minimum velocity required to project a particle from a height h to fall at a distance d from the point of projection is $\sqrt{[g\,(d-h)]}$ *(Avadh 83)*

Solution. Refer Fig. 22 Page 34 of this Chapter.

Proceeding as in Ex. 2 (a) Pages 33—34 we get

$\frac{u^4}{g^2}+\frac{2u^2h}{g}\geqslant R^2$ ...See result (A) of Ex. 2 (a) Page 34 (Note)

Here $R^2+h^2=d^2$ (given), since the distance of A from O is given as d.

or $R^2=d^2-h^2$

Then from above we get $\frac{u^4}{g^2}+\frac{2u^2h}{g}\geqslant d^2-h^2$

or $\frac{u^4}{g^2}+\frac{2u^2h}{g}+h^2\geqslant d^2$ or $\left(\frac{u^2}{g}+h\right)^2\geqslant d^2$

or $(u^2/g)+h\geqslant d$ or $u^2\geqslant g(d-h)$ or $u\geqslant\sqrt{[g(d-h)]}$

$\therefore$ The minimum value of $u=\sqrt{[g(d-h)]}$ Hence proved.

Ex. 3 A shot is fired with velocity u from the top of a cliff of height h and strikes the sea at a distance d from the foot of the cliff. Show that the possible times of flight are the roots of the equation $\frac{1}{4}g^2t^4-(gh+u^2)t^2+d^2+h^2=0$.

Solution. If α be the angle of projection, the point of projection be origin and the horizonal and upward drawn vertical lines through the point of projection (lying in the plane of flight) be taken as coordinates axes, then the coordinates of the point, where the shot strikes the sea, are $(d, -h)$. (See figure of last example)

Also $d=u\cos\alpha.t$ .. (i) and $-h=u\sin\alpha.t-\frac{1}{2}gt^2$, ...(ii)

where t is the time of flight of the particle

Eliminating α between (i) and (ii), we get

$$d^2+(-h+\tfrac{1}{2}gt^2)^2=u^2t^2 \text{ or } d^2+h^2+\tfrac{1}{4}g^2t^4-ght^2=u^2t^2$$

or $\frac{1}{4}g^2t^4-(gh+u^2)t^2+d^2+h^2=0$. Hence proved.

****Ex. 4. Show that the product of two times of flight from P to Q with a given velocity of projection is (2/g) PQ.**

Solution. Let P be the point of projection and the coordinates of Q be (h, k) referred to horizontal and upward drawn vertical lines through P (lying in the plane of flight) as axes. Let u and α be the velocity and angle of projection respectively.

Then $h=u\cos\beta.t$ and $k=u\sin\alpha.t-\frac{1}{2}gt^2$,

where t is the time of flight from P to Q.

Eliminating α between these two equations, we get

$$h^2+(k+\tfrac{1}{2}gt^2)^2=u^2t^2 \text{ or } h^2+k^2+kgt^2+\tfrac{1}{4}g^2t^4=u^2t^2$$

or $g^2t^4+4(kg-u^2)t^2+4(h^2+k^2)=0$ ..(i)

$\therefore$ If t_1 and t_2 be two times of flight from P to Q, then from (i) we get, product of the roots $=t_1^2t_2^2=4(h^2+k^2)/g^2$

or $t_1t_2=\frac{2\sqrt{(h^2+k^2)}}{g}=\frac{2PQ}{g}$, $\because PQ=\sqrt{(h^2+k^2)}$

****Ex. 5 Two points P and Q are at a distance a apart, their heights above the ground being h_1 and h_2. Prove that the least velocity**

with which a particle can be thrown from the ground level so as to pass through both the points is $\sqrt{[g(a+h_1+h_2)]}$.

Solution. Let O be the point of projection of the particle on the ground and u be its velocity of projection.

Also we are given that the distance between P and $Q=a$ and the height of Q above $P=(h_2-h_1)$.

In § 9 Page 33, we have proved that the least value of the velocity of projection $=\sqrt{g}\,[k+\sqrt{(h^2+k^2)}]^{1/2}=\sqrt{g}\,[k+a]^{1/2}$, where k is the height of P above O and a is the distance $\sqrt{(h^2+k^2)}$ *i.e.*, the distance OP.

$\therefore$ In this problem. the least value of v, the velocity at P to strike $Q=\sqrt{g}\,[(h_2-h_1)+a]^{1/2}$. ...(i)

Also for the motion from O to P,

$$v^2=u^2-2gh_1 \quad \text{or} \quad u^2=v^2+2gh_1. \qquad ...(i)$$

$\therefore$ From (ii) u is least when v is least hence least value of u, the velocity of projection, is given by,

$$u^2=g\,[h_2-h_1+a]+2gh_1 \text{ from (ii) and (i)}$$

or
$$u^2=g\,(h_1+h_2+a) \quad \text{or} \quad u=\sqrt{[g\,(h_1+h_2+a)]}$$

Ex. 6 If t_1 and t_2 be the times of flight from A to B and α the inclination of AB to the horizontal; prove that $t_1^2+2t_1t_2\sin\alpha+t_2^2$ is independent of α. (*Garhwal 81; Meerut 79*)

Solution. Let A be taken as origin and the coordinates of B be (h, k), referred to horizonal and upward drawn vertical lines (lying in the plane of flight) through A as axes.

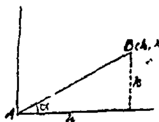


Fig. 23.

Then as in § 10 Page 33, the times of flight from A to B are given by $t^4+4\,(k/g-u^2/g^2)\,t^2+(4/g^2)\cdot(h^2+k^2)=0$ whence $t_1^2+t_2^2=$sum of the roots

and $\qquad =-4\,(k/g-u^2/g^2)$ ...(i)

$t_1^2\,t_2^2=$Product of the roots$=(4/g^2)\,(h^2+k^2)$. ...(ii)

Also as A is the origin and AB is inclined to the horizontal at an angle α, so $\tan\alpha=k/h$.

$$\therefore \quad \sin\alpha=\frac{k}{\sqrt{(h^2+k^2)}}. \qquad ...(iii)$$

$$\therefore \quad t_1^2+2t_1t_2\sin\alpha+t_2^2=(t_1^2+t_2^2)+2(t_1t_2)\sin\alpha$$

$$=-4\left(\frac{k}{g}-\frac{u^2}{g^2}\right)+2\left[\frac{2}{g}\sqrt{(h^2+k^2)}\right]\frac{k}{\sqrt{(h^2+k)^2}},$$

from (i), (ii) and (iii)

$=4u^2/g^2$, which is independent of h or k, hence α.

**Ex. 7 A particle is projected under gravity from A so as to pass through B; show that for a given velocity of projection there

are two paths. Show that if B has horizontal and vertical coordinates x, y referred to A, and the velocity of projection is $\sqrt{(2gh)}$, the angle between the two paths at B is a right angle if B lies on the ellipse $x^2+2y^2=2hy$

Solution. Take A as origin and α the angle of projection then the equation of the path of the particle is

$$y=x \tan \alpha-\frac{gx^2}{2u^2 \cos^2 \alpha}, \text{ where } u=\sqrt{(2gh)} \text{ (given)}$$

or $$y=x \tan \alpha-\frac{gx^2}{2.2gh \cos^2 \alpha}$$

$\because$ $B (x, y)$ lies on it, $\therefore$ $y=x \tan \alpha-(x^2/4h) \sec^2 \alpha$

or $$x^2 \sec^2 \alpha-4hx \tan \alpha+4hy=0. \quad ...(i)$$

Also differentiating (i) with respect to x, we get

$$2x \sec^2 \alpha-4h \tan \alpha+4h\,(dy/dx)=0.$$

or $$x \sec^2 \alpha-2h \tan \alpha+2hm=0, \quad ...(ii)$$

where $$m=dy/dx.$$

Solving (i) and (ii) for $\sec^2 \alpha$ and $\tan \alpha$, we have

$$\frac{\sec^2 \alpha}{-8h^2mx+8h^2y}=\frac{\tan \alpha}{4hxy-2hmx^2}=\frac{1}{-2hx^2+4hx^2}$$

or $$\sec^2 \alpha=\frac{4(y-mx)}{x} \text{ and } \tan \alpha=\frac{(2y-mx)}{x}.$$

Eliminating α between these two results, we get

$$1+\left[\frac{2y-mx}{x}\right]^2=\frac{4h(y-mx)}{x^2}, \quad \because 1+\tan^2 \alpha=\sec^2 \alpha$$

or $$x^2+4y^2+m^2x^2-4mxy=4hy-4mhx$$

or $$m^2x^2-4mx(y-h)+(x^2+4y^2-4hy)=0.$$

This being a quadratic equation in m, gives two values of m, i.e. dx/dy or the slopes of tangents of the two paths which pass through B. If m_1 and m_2 are the roots of this equation, then

m_1m_2=products of the roots$=(x^2+4y^2-4yh)/x^2$

If the angle between the two parts at B is a right angle, then $m_1m_2=-1$ or $(x^2+4y^2-4hy)/x^2=-1$

or $2x^2+4y^2-4hy=0$ or $x^2+2y^2=2hy$ which is the locus of B.

Exercises on § 9 – § 10

Ex. 1. Show that there are in general two directions in which a particle projected under gravity, with a given velocity from a given point O may pass through another given point P. Show that the product of the two times of flight from O to P in the different paths is independent of the initial velocity.

Ex. 2. A particle is projected from a point P, with given

velocity so as to pass through another point Q. Show that there are two times of transit t_1, t_2 such that $t_1t_2=(2/g)\,PQ$.

[Hint : See Ex. 4 Page 36.]

§ 11. Range and time of flight up an inclined plane. (*Gorakhpur 83*)

A particle is projected with a velocity u making an angle α with the horizon from a point O on an inclined plane inclined at an angle β to the horizon, in the vertical plane through the line of greatest slope. To determine the range on the inclined plane and time of flight, if the particle strikes the inclined plane.

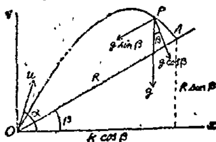


Fig. 24.

Let O be the point of projection, and A be the point where the particle strikes the inclined plane OA. Let $OA=R$. Then the co-ordinates of A, referred to horizontal and vertical lines through O (as shown in the figure) as axes, are $(R\cos\beta, R\sin\beta)$.

The equation of the path of the projectile is

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}$$

$\therefore$ $A\,(R\cos\beta,\ R\sin\beta)$ lies on it

$$\therefore R\sin\beta=R\cos\beta\tan\alpha-\frac{gR\cos^2\beta}{2u^2\cos^2\alpha}$$

or
$$-\sin\beta+\cos\beta\tan\alpha=\frac{gR\cos^2\beta}{2u^2\cos^2\alpha}$$

or
$$\frac{\sin(\alpha-\beta)}{\cos\alpha}=\frac{g\,R\cos^2\beta}{2u^2\cos^2\alpha}\ \text{or}\ R=\frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}\quad ...(i)$$

which gives the range on an inclined plane (*up the plane*).

****Maximum range up the inclined plane.**

We have proved above

$$R=\frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}=\frac{u^2}{g\cos^2\beta}\,[2\sin(\alpha-\beta)\cos\alpha]$$

$$=\frac{u^2}{g\cos^2\beta}\,[\sin(2\alpha-\beta)-\sin\beta].$$

$\therefore$ If u and β are constant, R is maximum when $\sin(2\alpha-\beta)$ is maximum *i.e*, when $2\alpha-\beta=\frac{1}{2}\pi$ ...(ii)

or
$$\alpha=\tfrac{1}{4}\pi+\tfrac{1}{2}\beta,\quad ...(iii)$$

which give the angle of projection for maximum range.

∴ Maximum range

$$=\frac{u^2}{g\cos^2\beta}[1-\sin\beta]=\frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)}=\frac{u^2}{g(1+\sin\beta)}.$$

∴ **Max. range up the inclined plane**$=\dfrac{u^2}{g(1+\sin\beta)}$. (iv)

(*Agra 86*)
(Note)

Also from (ii), we have $\alpha-\beta=\frac{1}{2}\pi-\alpha$.

i.e. the direction of projection bisects the angle between the vertical line at O and the line of greatest slope.

Also we observe that,

(a) the vertical line at O is perpendicular to the directrix of the parabolic path and

(b) the direction of projection is the tangent to the parabolic path at O.

Also in the case of parabola, the tangent at any point bisects the angle between the focal distance of the point and the perpendicular from the point on the directrix. Hence the line of greatest slope through O passes through the focus *i.e. when the range on an inclined plane is maximum, the focus of the parabolic path lies on the inclined plane.*

Time of flight :

The component of the velocity of projection at right angles to the inclined plane *i.e. OA* is $u\sin(\alpha-\beta)$. Also as shown in the figure 24 the components of acceleration of the particle along and perpendicular to the inclined plane are $g\sin\beta$ and $g\cos\beta$ respectively.

If T be the required time of flight from O to A, considering the motion perpendicular to the inclined plane we have

$0=u\sin(\alpha-\beta)$. $T-\frac{1}{2}g\cos\beta$. T^2; from "$s=ut+\frac{1}{2}ft^2$"

or
$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta},\quad ..(v)$$

which gives the required time of flight.

***§ 12. Range and time of flight *down* an inclined plane.**

Let O be the point of projection, u the velocity of projection and α the angle which u makes with the horizontal. Let the inclined plane make an angle β with the horizon.

∴ The components of initial velocity along and perpendicular to the inclined plane are $u\cos(\alpha+\beta)$ and $u\sin(\alpha+\beta)$ respectively. Also the acceleration of the particle along and perpendicular to the inclined plane are $g\sin\beta$ and $g\cos\beta$ respectively. (The negative sign of acceleration perpendicular to the inclined

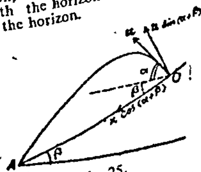


Fig. 25.

plane is due to the fact that its direction is opposite to $u \sin(\alpha+\beta)$, the component of initial velocity perpendicular to the inclined plane).

Let T be the time of flight from O to A, the point where the particle strikes the inclined plane. In this time the distance travelled by the particle perpendicular to the inclined plane is zero.

$\therefore$ Considering motion perpendicular to the inclined plane, from "$s=ut+\frac{1}{2}ft^2$", we have

$$0=u \sin(\alpha+\beta).T-\tfrac{1}{2}g \cos\beta.T^2$$

or $$T=\frac{2u \sin(\alpha+\beta)}{g \cos\beta}.$$

$\therefore$ **Time of flight down the inclined plane**

$$=\frac{\mathbf{2u \sin(\alpha+\beta)}}{\mathbf{g \cos\beta}} \quad ..(i)$$

Again considering the motion along and down the inclined plane, we have $OA=u\cos(\alpha+\beta)\,T+\frac{1}{2}g\sin\beta\,T^2$.

or **Range down the inclined plane=OA**

$$=u\cos(\alpha+\beta)\,\frac{2u\sin(\alpha+\beta)}{g\cos\beta}+\frac{g\sin\beta}{2}\cdot\frac{4u^2\sin^2(\alpha+\beta)}{g^2\cos^2\beta}$$

$$=\frac{2u^2\sin(\alpha+\beta)}{g\cos^2\beta}\left[\cos(\alpha+\beta)\cos\beta+\sin\beta\sin(\alpha+\beta)\right]$$

$$=\frac{\mathbf{2u^2\sin(\alpha+\beta)\cos\alpha}}{\mathbf{g\cos^2\beta}} \quad ..(ii)$$

[Note : W [illegible] $-\beta$ for β [illegible] hen the pa [illegible]

Maximum range down the inclined plane *(Agra 80)*

$$=\frac{\mathbf{u^2}}{\mathbf{g(1-\sin\beta)}}$$ (putting $-\beta$ for β in result (iv) of § 11 Page 40)

Solved Examples on § 11—§ 12

Ex. 1 (a). Show that for a given velocity of projection the maximum range down a plane of inclination α is greater than up the plane in the ratio $(1+\sin\alpha)/(1-\sin\alpha)$.

Solution. $\dfrac{\text{max. range down the plane}}{\text{max. range up the plane}}$

$$=\frac{u^2/[g(1-\sin\alpha)]}{u^2/[g(1+\sin\alpha)]}$$ ..See § 11 and 12 Pages 39-41

$$=(1+\sin\alpha)/(1-\sin\alpha).$$ Hence proved.

Ex. 1 (b). For a given velocity of projection the maximum range down an inclined plane is three times the range up the inclined plane, show the inclination of the plane to the horizontal is 30°.

Solution. Let u be the velocity of projection and β the inclination of the plane to the horizontal. Given that

max. range down the plane $=3\times$ max. range up the plane

$$\therefore \frac{u^2}{g(1-\sin\beta)}=3\times\frac{u^2}{g(1+\sin\beta)}$$

or $1+\sin\beta=3(1-\sin\beta)$ or $4\sin\beta=2$ or $\sin\beta=\frac{1}{2}$ or $\beta=30°$

Ex. 2. **A particle is projected with velocity u from a point on a plane inclined at an angle α to the horizontal. If r and r' be the maximum ranges up and down the inclined plane, prove that $(1/r)+(1/r')$ is independent of inclination of the plane.** *(Agra 81)*

Solution. $r=$ max. range up the plane $=\dfrac{u^2}{g(1+\sin\alpha)}$

and $r'=$ max. range down the plane $=\dfrac{u^2}{g(1-\sin\alpha)}$

$$\therefore \frac{1}{r}+\frac{1}{r'}=\frac{g(1+\sin\alpha)}{u^2}+\frac{g(1-\sin\alpha)}{u^2}=\frac{2g}{u^2},$$ which is independent of α. Hence proved.

***Ex. 3 (a).** **Show that if a gun be situated on an inclined plane, the maximum range in a direction at right angles to the line of greatest slope is a harmonic mean between the maximum ranges up and down the plane respectively.**

Solution Let u be the velocity of projection and β the inclination of the plane to the horizontal.

$$\therefore \text{Max. range up the plane}=\frac{u^2}{g(1+\sin\beta)}=R_1 \text{ (say)}$$

and max. range down the plane $=\dfrac{u^2}{g(1-\sin\beta)}=R_2$ (say)

$\therefore$ Harmonic mean of R_1 and R_2

$$=\frac{2R_1R_2}{R_1+R_2}=\frac{2u^4/g^2(1-\sin^2\beta)}{(u^2/g)\,[\{1/(1+\sin\beta)\}+\{1/(1-\sin\beta)\}]}$$

$=u^2/g=$ max. horizontal range.

$=$ max. range in a direction at right angles to the line of greatest slope.

Ex. 3 (b). **Prove that during the flight of particle projected at an elevation α on a line through the point of projection of inclination β, the direction of motion turns through an angle whose cotangent is $\frac{1}{2}\cos\beta\sec\alpha\operatorname{cosec}(\alpha-\beta)-\tan\alpha$.**

Solution. Let the particle be projected from the point O with velocity u at an angle α. After time t let it be moving at P with velocity v. Also let θ be the angle through which the direction of motion is turned, then considering the motion parallel and perpendicular to the original direction from "$v=u+ft$" we have

and $$v \cos \theta = u - g \sin \alpha . t \quad \ldots(i)$$
$$-v \sin \theta = 0 - g \cos \alpha . t \quad \ldots(ii)$$

Dividing (i) by (ii), $-\cot \theta = \frac{u - g \sin \alpha . t}{-g \cos \alpha . t} = -\frac{u}{gt \cos \alpha} + \tan \alpha$

$$\therefore \cot \theta = \frac{u}{g \cos \alpha} . \frac{1}{t} - \tan \alpha \quad \ldots(iii)$$

But the time of flight $t = \frac{2u \sin (\alpha - \beta)}{g \cos \beta}$ ...(iv)

$\therefore$ Putting the value of t from (iv) in (iii), we have

$$\cot \theta = \frac{u}{g \cos \alpha} \times \frac{g \cos \beta}{2u \sin (\alpha - \beta)} - \tan \alpha$$

$= \frac{1}{2} \cos \beta \sec \alpha \operatorname{cosec} (\alpha - \beta) - \tan \alpha$. Hence proved.

Ex. 4. Show that the greatest range up an inclined plane through the point of projection is equal to the distance through which a particle could fall freely during corresponding time of flight.

Solution. Let u be the velocity of projection. Let the plane be inclined at an angle β to the horizon.

We know (See § 11 Result (iii) Page 39) for max. range up the plane, the angle of projection $\alpha = \frac{1}{4}\pi + \frac{1}{2}\beta$.

Also time of flight up the inclined plane

$$= \frac{2u \sin (\alpha - \beta)}{g \cos \beta} = \frac{2u \sin (\frac{1}{4}\pi + \frac{1}{2}\beta - \beta)}{g \cos \beta} = \frac{2u \sin (\frac{1}{4}\pi - \frac{1}{2}\beta)}{g \cos \beta}$$

$$= \frac{2u [\sin \frac{1}{4}\pi \cos \frac{1}{2}\beta - \cos \frac{1}{4}\pi \sin \frac{1}{2}\beta]}{g \cos \beta} = \frac{2u [\cos \frac{1}{2}\beta - \sin \frac{1}{2}\beta]}{\sqrt{2} g \cos \beta}$$

$$= \frac{u\sqrt{2}}{g (\cos \frac{1}{2}\beta + \sin \frac{1}{2}\beta)} = T \text{ (say)} \quad \ldots(ii)$$

$\therefore$ The distance fallen by a particle in this time T

$$= \frac{1}{2}gT^2 = \frac{1}{2}g \left[\frac{2u^2}{g^2 (\cos \frac{1}{2}\beta + \sin \frac{1}{2}\beta)^2}\right] = \frac{u^2}{g (\cos \frac{1}{2}\beta + \sin \frac{1}{2}\beta)^2}$$

$= u^2/[g (1 + \sin \beta)] =$ max. range up the plane.

***Ex. 5. Show that the time of a particle describing an arc of its path cut off by a focal chord is equal to the time of falling vertically from rest through a height equal to the chord.**

Solution. In § 11 Page 39, we have proved that when the range on an inclined plane is maximum, the focus of the parabolic path lies on the inclined plane or max. range is the length of the focal chord through the point of projection. Let the focal chord under consideration make an angle β with the horizontal.

Proceed further as in Ex. 4 above.

***Ex. 6. If u be the velocity of projection and v_1 the velocity of striking the plane when projected so that range up the plane is maximum, and v_2 the velocity of striking the plane when projected so that the range down the plane is maximum. Prove that $u^2 = v_1 v_2$.**

Solution. Let the plane be inclined to the horizontal at an angle β.

$\therefore$ Max. range up the inclined plane$=h^2/[g(1+\sin\beta)]$

$\therefore$ Height h_1 of the point of striking above the point of projection $=[u^2/\{g(1+\sin\beta)\}].\sin\beta$. (Note)

If v_1 be the striking velocity at height h, then we have

$$v_1^2=u^2-2gh_1=u^2-2g\left[\frac{u^2\sin\beta}{g(1+\sin\beta)}\right]=u^2\left(\frac{1-\sin\beta}{1+\sin\beta}\right) \quad ...(i)$$

Similarly max. range down the inclined plane

$$=u^2/\{g(1-\sin\beta)\}$$

$\therefore$ Depth h_2 of the point of striking below the point of projection $=[u^2/g\{(1-\sin\beta)\}]\sin\beta$. (Note)

If v_2 be the striking velocity at depth h_2, then we have

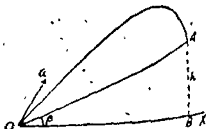
$$v_2^2=u^2+2gh_2=u^2+2g\left[\frac{u^2\sin\beta}{g(1-\sin\beta)}\right]=u^2\left(\frac{1+\sin\beta}{1-\sin\beta}\right) \quad ...(ii)$$

Multiplying (i) and (ii), we get $v_1^2v_2^2=u^4$ or $v_1v_2=u^2$.

****Ex. 7 (a). The angular elevation of an enemy's position on a hill h feet high is β. Show that in order to shell it the initial velocity of the projectile must not be less than $\sqrt{[hg(1+\text{cosec}\,\beta)]}$.**

(Meerut 83)

Solution. O is the point of projection of the shell and A is the position of enemy at a height h above the level of O. If u be the minimum initial velocity of the projectile to shell the enemy, then OA is the max. range up the inclined plane of angle β for this velocity.



(Fig. 26)

$$\therefore \quad OA=\frac{u^2}{g(1+\sin\beta)} \quad ...(i)$$

Also from $\triangle OAB$

$$OA=h\,\text{cosec}\,\beta \quad ...(ii)$$

$\therefore$ from (i) and (ii) we get $u^2/[g(1+\sin\beta)]=h\,\text{cosec}\,\beta$

or $\quad u^2=gh\,\text{cosec}\,\beta\,(1+\sin\beta)=gh\,(\text{cosec}\,\beta+1)$

or $\quad u=\sqrt{[gh(\text{cosec}\,\beta+1)]}$ Hence proved.

Ex. 7 (b). Determine the least velocity with which a ball can be thrown to reach to top of a cliff 40 m. high and $40\sqrt{3}$ m. away from the point of projection.

Solution. Refer Fig. 26 above.

Here A is the top of the cliff and the ball is thrown from the point O.

$\therefore \quad h=AB=40$ m. and $OB=40\sqrt{3}$m.

$\therefore \quad \tan\beta = \frac{AB}{OB} = \frac{40}{40\sqrt{3}} = \frac{1}{\sqrt{3}} \quad$ or $\beta = 30°$

$\therefore \quad OA = AB \operatorname{cosec} 30° = 40\ (2) = 80$ m.

If u is the least velocity with which ball can be thrown to reach A, then OA is the maximum range up the inclined plane of angle $\beta = 30°$ for this velocity.

$\therefore \quad OA = u^2/[g\ (1 + \sin\beta)]$, where $\beta = 30°$, $g = 9{\cdot}8$ m/sec.

or $\quad 80 = u^2/[(9{\cdot}8)\ (1 + \frac{1}{2})]$, $\quad \because \quad OA = 80$ m.

or $\quad u^2 = 80\ [(9{\cdot}8)\ (\frac{3}{2})] = 120\ (9{\cdot}8) = 1176$

or $\quad u = \sqrt{(1176)} = 14\sqrt{6}$ m./sec. **Ans.**

Ex. 8. If from a point on the side of a hill two bodies are projected in the vertical plane through the line of greatest slope with the same velocity but in directions at right angles to each other, show that difference of their ranges is independent of their angles of projection.

Solution. As the bodies are projected in directions at right angles to each other, so let their directions of projections make angles α and $90° + \alpha$ with the horizontal. Let the corresponding ranges be R and R' respectively. Let the side of the hill be inclined at an angle β to the horizontal.

Then $\quad R = \dfrac{2u^2 \sin(\alpha - \beta) \cos\alpha}{g \cos^2\beta}$

and $\quad R' = \dfrac{2u^2 \sin\{(90° + \alpha) - \beta\} \cos(90° + \alpha)}{g \cos^2\beta}$ **(Note)**

$\quad = \dfrac{2u^2 \cos(\alpha - \beta) \sin\alpha}{g \cos^2\beta}$ numerically, the negative sign indicates that this range is downwards.

$$\therefore \quad R' - R = \frac{2u^2 \cos(\alpha - \beta) \sin\alpha}{g \cos^2\beta} - \frac{2u^2 \sin(\alpha - \beta) \cos\alpha}{g \cos^2\beta}$$

$$= \frac{2u^2}{g \cos^2\beta} \left[\cos(\alpha - \beta) \sin\alpha - \sin(\alpha - \beta) \cos\alpha\right]$$

$$= \frac{2u^2}{g \cos^2\beta} \left[\sin\{\alpha - (\alpha - \beta)\}\right] = \frac{2u^2 \sin\beta}{g \cos^2\beta}.$$

****Ex. 9. A particle is projected at an angle α with the horizontal from the foot of the plane, whose inclination to the horizontal is β. Show that it will strike the plane at right angles if $\cot\beta = 2\tan(\alpha - \beta)$.** *(Lucknow 79; Meerut 82, 79)*

Solution. Let u be the velocity of projection, then the component of u along and perpendicular to the inclined plane are $u \cos(\alpha - \beta)$ and $u \sin(\alpha - \beta)$ respectively.

Also the acceleration along the plane is $g \sin \beta$ downwards.

The time of flight of the particle on the inclined plane

$$=\frac{2u \sin (\alpha-\beta)}{g \cos \beta} \quad \ldots(i)$$

Also as the particle strikes the inclined plane at A at right angles, so its velocity along the plane vanishes when it reaches A.

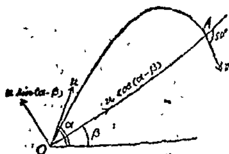


Fig. 27

$\therefore$ Considering the motion along the inclined plane from O to A from "$v=u+ft$", we have

$0=u \cos (\alpha-\beta)-g \sin \beta\, t$,

where t is the time taken from O to A.

or $$t=\frac{u \cos (\alpha-\beta)}{g \sin \beta} \quad \ldots(ii)$$

As the times given by (i) and (ii) are equal, so

$$\frac{u \cos (\alpha-\beta)}{g \sin \beta}=\frac{2u \sin (\alpha-\beta)}{g \cos \beta} \text{ or } \cot \beta=2 \tan (\alpha-\beta).$$

****Ex. 10. A particle projected with a velocity u strikes at right angles a plane through the point of projection inclined at an angle β to the horizon. Show that time of flight is $\dfrac{2u}{g\sqrt{(1+3 \sin^2 \beta)}}$, range on the plane is $\dfrac{2u^2}{g} \cdot \dfrac{\sin \beta}{(1+3 \sin^2 \beta)}$ and the vertical height of the point struck is $\dfrac{2u^2}{g} \cdot \dfrac{\sin^2 \beta}{(1+3 \sin^2 \beta)}$ above the point of projection.**

(Agra 82, 80; Avadh 83; Garhwal 79; Indore 79; Meerut 85, 81)

Solution. Let the angle of projection be α.

When the particle strikes the inclined plane at right angle, the velocity parallel to the plane vanishes and considering motion parallel to the inclined plane from "$v=u+ft$" we get

$0=u \cos (\alpha-\beta)-g \sin \beta . t$, where t is the time of flight

or $$t=\frac{u \cos (\alpha-\beta)}{g \sin \beta} \quad \text{(Also see last example)} \quad \ldots(i)$$

Also we know that the time of flight $=\dfrac{2u \sin (\alpha-\beta)}{g \cos \beta}$. $\ldots$(ii)

Equating (i) and (ii), we get $\dfrac{u \cos (\alpha-\beta)}{g \sin \beta}=\dfrac{2u \sin (\alpha-\beta)}{g \cos \beta}$

or $$\frac{\cos \beta}{\sin \beta}=\frac{2 \sin (\alpha-\beta)}{\cos (\alpha-\beta)}$$

or $\frac{\sin(\alpha-\beta)}{\cos\beta}=\frac{\cos(\alpha-\beta)}{2\sin\beta}=\frac{\sqrt{\{\sin^2(\alpha-\beta)+\cos^2(\alpha-\beta)\}}}{\sqrt{(\cos^2\beta+4\sin^2\beta)}}$ (Note)

$$=\frac{1}{\sqrt{(1+3\sin^2\beta)}},$$

where $\sin(\alpha-\beta)=\frac{\cos\beta}{\sqrt{(1+3\sin^2\beta)}}$; $\cos(\alpha-\beta)=\frac{2\sin\beta}{\sqrt{(1+3\sin^2\beta)}}$ (iii)

$\therefore$ Time of flight $=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$, from (ii)

$$=\frac{2u}{g\sqrt{(1+3\sin^2\beta)}}, \text{ from (iii)}$$

Also range on the inclined plane $=\frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$

$=\frac{2u^2\sin(\alpha-\beta)\cos(\alpha-\beta+\beta)}{g\cos^2\beta}$ (Note)

$=\frac{2u^2\sin(\alpha-\beta)}{g\cos^2\beta}\left[\cos(\alpha-\beta)\cos\beta-\sin(\alpha-\beta)\sin\beta\right]$

$=\frac{2u^2\cos\beta}{g\cos^2\beta\sqrt{(1+3\sin^2\beta)}}\left[\frac{2\sin\beta\cos\beta}{\sqrt{(1+3\sin^2\beta)}}-\frac{\cos\beta\sin\beta}{\sqrt{(1+3\sin^2\beta)}}\right]$, from (iii)

$=\frac{2u^2\sin\beta}{g(1+3\sin^2\beta)}=R$ (say)

or (*Agra 80*)

or $2u^2\sin\beta=Rg(1+3\sin^2\beta)$

$u^2=gR(1+3\sin^2\beta)/(2\sin\beta)$. (*Agra 86*)

And the vertical height of the point struck $=R\sin\beta$ $=(2u^2\sin^2\beta)/[g(1+3\sin^2\beta)]$. Hence proved.

Ex. 10 (a). The line joining C to D is inclined at an angle α to the horizon, show that the velocity required to shoot from C to D is $\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)$ times the least velocity required to shoot from D to C.

Solution. Let the least velocities required to shoot from C to D and D to C be u and v respectively.

For the projection up the plane (from C to D) the maximum range

$$CD=u^2/[g(1+\sin\alpha)]. \quad \text{(i)}$$

and for the projection down the plane (from D to C) the maximum range

$$DC=v^2/[g(1-\sin\alpha)]. \quad \text{...(ii)}$$

$\therefore$ from (i) and (ii) equating values of CD, we get

$$\frac{u^2}{v^2}=\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{(\sin\frac{1}{2}\alpha+\cos\frac{1}{2}\alpha)^2}{(\cos\frac{1}{2}\alpha-\sin\frac{1}{2}\alpha)^2} \quad \text{(Note)}$$

$$\therefore \frac{u}{v}=\frac{\sin\frac{1}{2}\alpha+\cos\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha-\sin\frac{1}{2}\alpha}=\frac{1+\tan\frac{1}{2}\alpha}{1-\tan\frac{1}{2}\alpha}=\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha).$$

****Ex. 11. Prove that if a particle is projected from O at an**

or $\sin\alpha\cos\beta=2\sin(\alpha-\beta)=2[\sin\alpha\cos\beta-\cos\alpha\sin\beta]$

or $2\cos\alpha\sin\beta=\sin\alpha\cos\beta$ or $2\tan\beta=\tan\alpha$

(ii) If the shot strikes the hill normally *i.e.* at right angles, then we can prove as in Ex. 9 Page 45 that

$$\cot\beta=2\tan(\alpha-\beta)=2\left[\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right]$$

or $\cot\beta(1+\tan\alpha\tan\beta)=2\tan\alpha-2\tan\beta$

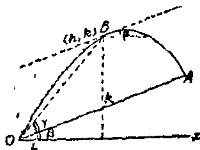
or $\cot\beta+\tan\alpha=2\tan\alpha-2\tan\beta$ or $\tan\alpha=\cot\beta+2\tan\beta$.

****Ex. 13. A stone is thrown at an angle α with the horizon from a point in a plane whose inclination to the horizon is β, the trajectory lying in the vertical plane containing the line of greatest slope. Show that if γ be the elevation of that point of the path which is most distant from inclined plane, then**

$$2\tan\gamma=\tan\alpha+\tan\beta. \qquad \textit{(Allahabad 79; Meerut 81 S)}$$

Solution. Let B be that point of the path which is most distant from the inclined plane, then the tangent at B to the trajectory must be parallel to the inclined plane.

Let the coordinates of B be (h, k) referred to the horizontal and vertical lines ... of pr... plane



$\therefore$ The equation of the trajectory is

(Fig, 29)

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha} \qquad ...(i)$$

$\because$ $B(h, k)$ lies on it, so $k=h\tan\alpha-\dfrac{gh^2}{2u^2\cos^2\alpha}$ ...(ii)

Also from (i) we get $\dfrac{dy}{dx}=\tan\alpha-\dfrac{gx}{u^2\cos^2\alpha}$, ...(iii)

which gives the inclination of the tangent to horizontal at any point of (i). At B, tangent to (i) is inclined at an angle β to the horizontal.

$\therefore$ From (iii) at B,

$\tan\beta=\tan\alpha-\dfrac{gh}{u^2\cos^2\alpha}$ or $\dfrac{gh}{u^2\cos^2\alpha}=\tan\alpha-\tan\beta$. ...(iv)

Also γ is the elevation of $B(h, k)$, so $\tan\gamma=k/h$ or $k=h\tan\gamma$

$\therefore$ from (ii), $h\tan\gamma=h\tan\alpha-[gh^2/(2u^2\cos^2\alpha)]$

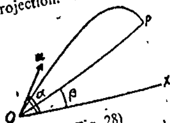
or $\tan\gamma=\tan\alpha-\frac{1}{2}[\tan\alpha-\tan\beta]$, from (iv)

or $2\tan\gamma=\tan\alpha+\tan\beta$. Hence proved.

***Ex. 14. Prove that the greatest range of a particle, projected with a given velocity, on an inclined plane, is four times the greatest vertical altitude above the inclined plane.**

elevation α and after time t the particle is at P, then $2 \tan \beta = \tan \alpha + \tan \theta$ where β and θ are the inclinations to the horizontal of OP and the direction of motion of the particle when at P.

Solution. Let u be the velocity of projection. After time t, considering the motion in the horizontal and vertical directions, we have



(Fig. 28)

Horizontal component of velocity at $P = u \cos \alpha$.

And vertical component of velocity at $P = u \sin \alpha - gt$.

$\therefore$ If θ be the angle which the direction of motion of the particle at P makes with the horizontal,

then
$$\tan \theta = \frac{\text{vertical component of velocity at } P}{\text{horizontal component of velocity at } P} \quad \text{...(i)}$$
$$= \frac{u \sin \alpha - gt}{u \cos \alpha} = \tan \alpha - \frac{gt}{u \cos \alpha}.$$

Also t is the time taken by the particle in moving from O to P. Treating this motion as if it is taking place on the inclined plane OP inclined at an angle β to the horizon, we have time of flight t
$$= \frac{2u \sin (\alpha - \beta)}{g \cos \beta}.$$

Substituting this value in (i), we have
$$\tan \theta = \tan \alpha - \frac{g}{u \cos \alpha} \cdot \frac{2u \sin (\alpha - \beta)}{g \cos \beta}$$
$$= \tan \alpha - \frac{2}{\cos \alpha \cos \beta} [\sin \alpha \cos \beta - \cos \alpha \sin \beta].$$

or $\tan \theta = \tan \alpha - 2 \tan \alpha + 2 \tan \beta.$

or $\tan \alpha + \tan \theta = 2 \tan \beta.$ Hence proved.

***Ex. 12.** A shot is fired at an angle α to the horizontal up a hill of inclination β to the horizontal. Show that it strikes the hill (*Agra 86*)

(i) horizontally if $\tan \alpha = 2 \tan \beta$.

(ii) normally if $\tan \alpha = 2 \tan \beta + \cot \beta$.

Solution. Let u be the veloctiy of projection and T be the time of flight.

Then $$T = \frac{2u \sin (\alpha - \beta)}{g \cos \beta}. \quad \text{...(i)}$$

Also as the shot strikes the hill horizontally after time T from its start, so in time T the vertical component of velocity has vanished. $\therefore$ from "$v = u + ft$", we get

$0 = u \sin \alpha - g . T$ or $T = (u \sin \alpha)/g$ ...(ii)

Equating (i) and (ii), we have $\dfrac{u \sin \alpha}{g} = \dfrac{2u \sin (\alpha - \beta)}{g \cos \beta}$

or $\sin\alpha\cos\beta=2\sin(\alpha-\beta)=2[\sin\alpha\cos\beta-\cos\alpha\sin\beta]$

or $2\cos\alpha\sin\beta=\sin\alpha\cos\beta$ or $2\tan\beta=\tan\alpha$

(ii) If the shot strikes the hill normally *i.e.* at right angles, then we can prove as in Ex. 9 Page 45 that

$$\cot\beta=2\tan(\alpha-\beta)=2\left[\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right]$$

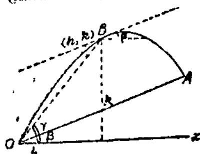
or $\cot\beta(1+\tan\alpha\tan\beta)=2\tan\alpha-2\tan\beta$

or $\cot\beta+\tan\alpha=2\tan\alpha-2\tan\beta$ or $\tan\alpha=\cot\beta+2\tan\beta$.

****Ex. 13. A stone is thrown at an angle α with the horizon from a point in a plane whose inclination to the horizon is β, the trajectory lying in the vertical plane containing the line of greatest slope. Show that if γ be the elevation of that point of the path which is most distant from inclined plane, then**

$$2\tan\gamma=\tan\alpha+\tan\beta.$$ *(Allahabad 79; Meerut 81 S)*

Solution. Let B be that point of the path which is most distant from the inclined plane, then the tangent at B to the trajectory must be parallel to the inclined plane.



(Fig, 29)

Let the coordinates of B be (h, k) referred to the horizontal and vertical lines through the point of projection O and lying in the plane of flight as coordinate axes.

∴ The equation of the trajectory is $y=x\tan\alpha-\dfrac{gx^2}{2u^2\cos^2\alpha}$ ...(i)

∵ $B(h, k)$ lies on it, so $k=h\tan\alpha-\dfrac{gh^2}{2u^2\cos^2\alpha}$ ...(ii)

Also from (i) we get $\dfrac{dy}{dx}=\tan\alpha-\dfrac{gx}{u^2\cos^2\alpha}$, ...(iii)

which gives the inclination of the tangent to horizontal at any point of (i). At B, tangent to (i) is inclined at an angle β to the horizontal.

∴ From (iii) at B,

$\tan\beta=\tan\alpha-\dfrac{gh}{u^2\cos^2\alpha}$ or $\dfrac{gh}{u^2\cos^2\alpha}=\tan\alpha-\tan\beta$. ...(iv)

Also γ is the elevation of $B(h, k)$, so $\tan\gamma=k/h$ or $k=h\tan\gamma$

∴ from (ii), $h\tan\gamma=h\tan\alpha-[gh^2/(2u^2\cos^2\alpha)]$

or $\tan\gamma=\tan\alpha-\frac{1}{2}[\tan\alpha-\tan\beta]$, from (iv)

or $2\tan\gamma=\tan\alpha+\tan\beta$. Hence proved.

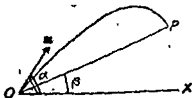
***Ex. 14. Prove that the greatest range of a particle, projected with a given velocity, on an inclined plane, is four times the greatest vertical altitude above the inclined plane.**

elevation α and after time t the particle is at P, then $2 \tan \beta = \tan \alpha + \tan \theta$ where β and θ are the inclinations to the horizontal of OP and the direction of motion of the particle when at P.

Solution. Let u be the velocity of projection. After time t, considering the motion in the horizontal and vertical directions, we have

Horizontal component of velocity at $P = u \cos \alpha$.

And vertical component of velocity at $P = u \sin \alpha - gt$.



(Fig. 28)

$\therefore$ If θ be the angle which the direction of motion of the particle at P makes with the horizontal,

then $$\tan \theta = \frac{\text{vertical component of velocity at } P}{\text{horizontal component of velocity at } P}$$

$$= \frac{u \sin \alpha - gt}{u \cos \alpha} = \tan \alpha - \frac{gt}{u \cos \alpha}. \quad ...(i)$$

Also t is the time taken by the particle in moving from O to P. Treating this motion as if it is taking place on the inclined plane OP inclined at an angle β to the horizon, we have time of flight t

$$= \frac{2u \sin (\alpha - \beta)}{g \cos \beta}.$$

Substituting this value in (i), we have

$$\tan \theta = \tan \alpha - \frac{g}{u \cos \alpha} \cdot \frac{2u \sin (\alpha - \beta)}{g \cos \beta}$$

$$= \tan \alpha - \frac{2}{\cos \alpha \cos \beta} [\sin \alpha \cos \beta - \cos \alpha \sin \beta].$$

or $\tan \theta = \tan \alpha - 2 \tan \alpha + 2 \tan \beta$.

or $\tan \alpha + \tan \theta = 2 \tan \beta$. Hence proved.

***Ex. 12.** **A shot is fired at an angle α to the horizontal up a hill of inclination β to the horizontal. Show that it strikes the hill**

(i) **horizontally if $\tan \alpha = 2 \tan \beta$.**

(ii) **normally if $\tan \alpha = 2 \tan \beta + \cot \beta$.** *(Agra 86)*

Solution. Let u be the velocity of projection and T be the time of flight.

Then $$T = \frac{2u \sin (\alpha - \beta)}{g \cos \beta}. \quad ...(i)$$

Also as the shot strikes the hill horizontally after time T from its start, so in time T the vertical component of velocity has vanished. $\therefore$ from "$v = u + ft$", we get

$$0 = u \sin \alpha - g . T \quad \text{or} \quad T = (u \sin \alpha)/g \quad ...(ii)$$

Equating (i) and (ii), we have $\dfrac{u \sin \alpha}{g} = \dfrac{2u \sin (\alpha - \beta)}{g \cos \beta}$

or $\sin\alpha\cos\beta=2\sin(\alpha-\beta)=2[\sin\alpha\cos\beta-\cos\alpha\sin\beta]$

or $2\cos\alpha\sin\beta=\sin\alpha\cos\beta$ or $2\tan\beta=\tan\alpha$

(ii) If the shot strikes the hill normally *i.e.* at right angles, then we can prove as in Ex. 9 Page 45 that

$$\cot\beta=2\tan(\alpha-\beta)=2\left[\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right]$$

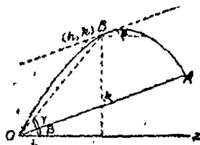
or $\cot\beta(1+\tan\alpha\tan\beta)=2\tan\alpha-2\tan\beta$

or $\cot\beta+\tan\alpha=2\tan\alpha-2\tan\beta$ or $\tan\alpha=\cot\beta+2\tan\beta$.

****Ex. 13. A stone is thrown at an angle α with the horizon from a point in a plane whose inclination to the horizon is β, the trajectory lying in the vertical plane containing the line of greatest slope. Show that if γ be the elevation of that point of the path which is most distant from inclined plane, then**

$$2\tan\gamma=\tan\alpha+\tan\beta.$$ *(Allahabad 79; Meerut 81 S)*

Solution. Let B be that point of the path which is most distant from the inclined plane, then the tangent at B to the trajectory must be parallel to the inclined plane.



(Fig, 29)

Let the coordinates of B be (h, k) referred to the horizontal and vertical lines through the point of projection O and lying in the plane of flight as coordinate axes.

$\therefore$ The equation of the trajectory is $y=x\tan\alpha-\dfrac{gx^2}{2u^2\cos^2\alpha}$ ...(i)

$\because$ $B(h, k)$ lies on it, so $k=h\tan\alpha-\dfrac{gh^2}{2u^2\cos^2\alpha}$ ...(ii)

Also from (i) we get $\dfrac{dy}{dx}=\tan\alpha-\dfrac{gx}{u^2\cos^2\alpha}$, ...(iii)

which gives the inclination of the tangent to horizontal at any point of (i). At B, tangent to (i) is inclined at an angle β to the horizontal.

$\therefore$ From (iii) at B,

$\tan\beta=\tan\alpha-\dfrac{gh}{u^2\cos^2\alpha}$ or $\dfrac{gh}{u^2\cos^2\alpha}=\tan\alpha-\tan\beta$. ...(iv)

Also γ is the elevation of $B(h, k)$, so $\tan\gamma=k/h$ or $k=h\tan\gamma$

$\therefore$ from (ii), $h\tan\gamma=h\tan\alpha-[gh^2/(2u^2\cos^2\alpha)]$

or $\tan\gamma=\tan\alpha-\frac{1}{2}[\tan\alpha-\tan\beta]$, from (iv)

or $2\tan\gamma=\tan\alpha+\tan\beta$. Hence proved.

... particle, projected ... times the greatest

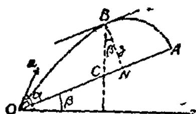


Fig. 30

Solution. Let B be the point on the trajectory, whose vertical altitude BC (BC is the vertical line through B meeting the inclined plane in C) is the greatest. From B draw BN perpendicular to the inclined plane OA. Let $BN=z$. Let the plane OA be inclined at an angle β to the horizontal. Then from the figure it is evident that $\angle CBN=\beta$.

$\therefore$ From $\triangle BNC$, we have $BC=z \sec B$.

$\therefore$ If BC is greatest, then z is greatest *i.e.* B is the point of the path of the particle which is most distant from the inclined plane; and if B is most distant from the inclined plane OA, then the tangent at B to the trajectory is parallel to OA *i.e.* the direction to velocity of the particle at B is parallel to OA and as such its component of velocity perpendicular to the inclined plane OA vanishes at B. (Note)

If u and α be the velocity and angle of projection respectively of the particle, then component of its initial velocity perpendicular to the inclined plane$=u \sin(\alpha-\beta)$ and acceleration perpendicular to the inclined plane$=g\cos\beta$ (towards the plane).

$\therefore$ From "$v^2=u^2+2fs$" for the motion perp. to the inclined plane from O to B, we have

$$0=u^2\sin^2(\alpha-\beta)-2g\cos\beta\, z, \text{ where } z=BN$$

or
$$z=\frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta} \quad \ldots(i)$$

If the range OA on the inclined plane is maximum, then angle of projection $\alpha=\frac{1}{4}\pi+\frac{1}{2}\beta$ or $\alpha-\beta=\frac{1}{4}\pi-\frac{1}{2}\beta$.

Substituting in (i) we get $z=\dfrac{u^2}{2g\cos\beta}\sin^2(\frac{1}{4}\pi-\frac{1}{2}\beta)$

$$=\frac{u^2}{4g\cos\beta}[1-\cos 2(\tfrac{1}{4}\pi-\tfrac{1}{2}\beta)], \because 2\sin^2\tfrac{1}{2}\theta=1-\cos\theta$$

$$=\frac{u^2}{4g\cos\beta}[1-\cos(\tfrac{1}{2}\pi-\beta)]=\frac{u^2}{4g\cos\beta}(1-\sin\beta)$$

or
$$z\sec\beta=\frac{u^2(1-\sin\beta)}{4g\cos^2\beta}=\frac{u^2(1-\sin\beta)}{4g(1-\sin^2\beta)}=\frac{u^2}{4g(1+\sin\beta)}$$

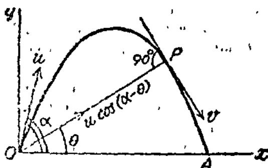
$=\frac{1}{2}$ [max. range on the inclined plane].

***Ex. 5.** **A particle is projected from O at an elevation α. Prove that there are two positions P on its path at which the direction of velocity is perpendicular to OP. Prove that for real positions to exist, α is not less than $\cos^{-1}(\frac{1}{3})$; also if in the two positions, OP makes angles θ_1, θ_2 with the horizontal, prove that**

$$\theta_1+\theta_2=\alpha.$$

Solution. Let u be the velocity of projection. Let the inclination of OP to the horizontal be θ. Then time taken by the particle in moving from O to P

$=2\,[u \sin(\alpha-\theta)]/(g \cos\theta)$ ...(i)



(Fig. 31)

Considering the motion parallel to OP we find that velocity of the particle parallel to OP vanishes at P, since the direction of motion at P is perpendicular to OP.

$\therefore$ from "$v=u+ft$", we find that

$0=u\cos(\alpha-\theta)-g\sin\theta.t$, where t is the time taken from O to P.

or $\qquad t=[u\cos(\alpha-\theta)]/(g\sin\theta)$ ...(ii)

$\because$ (i) and (ii) give the time from O to P.

$$\therefore \quad \frac{2u\sin(\alpha-\theta)}{g\cos\theta}=\frac{u\cos(\alpha-\theta)}{g\sin\theta}$$

or $2\sin\theta\sin(\alpha-\theta)=\cos\theta\cos(\alpha-\theta)$

or $2\sin\theta\,[\sin\alpha\cos\theta-\cos\alpha\sin\theta]=\cos\theta\,[\cos\alpha\cos\theta+\sin\alpha\sin\theta]$

or $2\sin\alpha\sin\theta\cos\theta-2\cos\alpha\sin^2\theta-\cos^2\theta\cos\alpha-\sin\alpha\sin\theta\cos\theta=0$

or $2\sin^2\theta\cos\alpha-\sin\alpha\sin\theta\cos\theta+\cos^2\theta\cos\alpha=0$

or $2\tan^2\theta\cos\alpha-\sin\alpha\tan\theta+\cos\alpha=0$. ...(iii)

This is a quadratic equation in $\tan\theta$, hence gives two values of $\tan\theta$ and corresponding to each value of $\tan\theta$ *i.e.* θ there will be one position of P.

Let the two roots of (ii) be $\tan\theta_1$ and $\tan\theta_2$.

Then $\tan\theta_1+\tan\theta_2=$ sum of the roots$=\dfrac{\sin\alpha}{2\cos\alpha}=\frac{1}{2}\tan\alpha$

and $\tan\theta_1\tan\theta_2=$ product of the roots$=\dfrac{\cos\alpha}{2\cos\alpha}=\frac{1}{2}$

$$\therefore \tan(\theta_1+\theta_2)=\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}=\frac{\frac{1}{2}\tan\alpha}{1-\frac{1}{2}}=\tan\alpha$$

or $\qquad \theta_1+\theta_2=\alpha$. Hence proved.

Also from (iii), if $\tan\theta$ is real, then "B^2-4AC" $\geqslant 0$

or $\quad \sin^2\alpha-4\,(2\cos\alpha)(\cos\alpha)\geqslant 0$ or $\sin^2\alpha-8\cos^2\alpha\geqslant 0$

or $\quad 1-\cos^2\alpha-8\cos^2\alpha\geqslant 0$ or $1-9\cos^2\alpha\geqslant 0$

or $\quad 9\cos^2\alpha\leqslant 1$ or $\cos\alpha\leqslant\frac{1}{3}$ or $\alpha\geqslant\cos^{-1}(\frac{1}{3})$ (Note)

Hence α can not be less than $\cos^{-1}(\frac{1}{3})$.

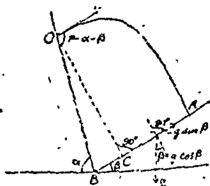
**Ex. 16. Two inclined planes intersect in a horizontal line, their inclinations to the horizon being α and β; if a particle is projected at right angles to the former from a point in it so as to strike the other at right angles, the velocity of projection is

$$\sin\beta\left\{\frac{2ag}{\sin\alpha-\sin\beta\cos(\alpha+\beta)}\right\}^{1/2},$$

a being the distance of the point of projection from the intersection of the planes.

Solution. Let O be the point of projection and u be the velocity of projection. Let the particle strike the other plane at A at right angles. From O draw OC perpendicular to the inclined plane BA. $\angle OBC$ = angle between the planes $=\pi-\beta-\alpha$.

$\therefore$ direction of u is at right angles to OB and OC is at right angles to BA, therefore angle between directions of u and OC is also $\pi-\alpha-\beta$.



(Fig. 32)

$\therefore$ The components of u along and perpendicular to OC are $u\cos(\pi-\alpha-\beta)$ and $u\sin(\pi-\alpha-\beta)$. Also the components of acceleration along and perpendicular to OC are $g\cos\beta$ and $-g\sin\beta$ respectively (as shown in the figure). Also $OB=a$ (given).

$\therefore$ From $\Delta\, OBC$,

$OC=a\sin(\pi-\alpha-\beta)=a\sin(\alpha+\beta)$

Considering the motion parallel to OC from O to A from "$s=ut+\frac{1}{2}ft^2$", we have

$$OC=u\cos(\pi-\alpha-\beta)\,t+\tfrac{1}{2}g\cos\beta.t^2,$$

where t is the time taken in moving from O to A

or $\quad a\sin(\alpha+\beta)=-u\cos(\alpha+\beta).t+\frac{1}{2}g\cos\beta\, t^2.$ ...(i)

Also as the particle strikes the plane BA at right angles at A so the component of velocity perpendicular to OC vanishes at A.

Hence from "$v=u+ft$", we have for motion perp. to OC

$$0=u\sin(\pi-\alpha-\beta)-g\sin\beta.t$$

or $$t=\frac{u\sin(\alpha+\beta)}{g\sin\beta}$$

Substituting this value in (i), we get

$$a\sin(\alpha+\beta)=-\frac{u^2\cos(\alpha+\beta)\sin(\alpha+\beta)}{g\sin\beta}+\frac{g\cos\beta\,u^2\sin^2(\alpha+\beta)}{2g^2\sin^2\beta}$$

or $\quad 2ag\sin^2\beta=-2u^2\sin\beta\cos(\alpha+\beta)+u^2\cos\beta\sin(\alpha+\beta).$

$=u^2[\{\cos\beta\sin(\alpha+\beta)-\sin\beta\cos(\alpha+\beta)\}-\sin\beta\cos(\alpha+\beta)]$

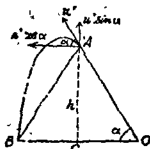
(Note)

$=u^2[\{\sin(\alpha+\beta-\beta)\}-\sin\beta\cos(\alpha+\beta)]$

or $u=\left[\dfrac{2ag\sin^2\beta}{\sin\alpha-\sin\beta\cos(\alpha+\beta)}\right]^{1/2}=\sin\beta\left[\dfrac{2ag}{\sin\alpha-\sin\beta\cos(\alpha+\beta)}\right]^{1/2}$

***Ex. 17. Two inclined planes of equal altitudes h, and inclined at the same angle α to the horizon, are placed back to back upon a horizontal plane. A ball is projected from the foot of one plane along its surface and in a direction making an angle β with its line of intersection with the horizontal plane. After flying over the ridge it falls at the foot of other plane, show that the velocity of projection is $\frac{1}{2}\sqrt{(gh)}\operatorname{cosec}\beta\sqrt{(8+\operatorname{cosec}^2\alpha)}$.**

Solution. OAB is the normal cross-section of the given planes by a plane perpendicular to their common ridge. Let O be the point of projection and u be the velocity of projection.



(Fig. 33)

Since the ball is projected from O along the surface of the inclined plane containing O and in a direction making an angle β with its line of intersection with the horizontal plane, therefore the components of u along OA and horizontally are $u\sin\beta$ and $u\cos\beta$ respectively. The component $u\cos\beta$ (horizontal) remains constant throughout the motion and it would cause the particle to fall at some point D instead of B, which is exactly opposite to O, such that $BD=u\cos B.T$, where T is the time of flight. Here D lies on the line of intersection of the second inclined plane with the horizontal plane.

Due to velocity component $u\sin\beta$ the ball will move first along the line OA then leave the plane at A and finally describe the parabolic path AB.

Hence we have two displacements of the ball, one due to velocity component $u\cos\beta$ and the other due to velocity component $u\sin\beta$. We shall consider them separately and at the end their results will be combined.

From $\triangle AOC$, we get $OA=h\operatorname{cosec}\alpha$ and $OC=h\cot\alpha=CB$. Let u' be the velocity of the ball when it reaches A. Then from "$v^2=u^2+2fs$" for the velocity component $u\sin\beta$, we have

$u'^2=u^2\sin^2\beta-2g\sin\alpha.OA=u^2\sin^2\beta-2g\sin\alpha.h\operatorname{cosec}\alpha$

or $u'^2=u^2\sin^2\beta-2gh.$

$\therefore$ At A, the horizontal and vertical components of velocity u' are $u'\cos\alpha$ and $u'\sin\alpha$ respectively. Let t be the time taken in moving from A to B, then considering the motion in horizontal and

vertical directions we get $(u' \cos \alpha)\, t = BC = h \cot \alpha$ ...(ii)

and $-h = (u' \sin \alpha)\, t - \frac{1}{2} g t^2$ (Note) ...(iii)

From (ii) we get $t = h/(u' \sin \alpha)$

$\therefore$ from (iii) we get

$$-h = h - \tfrac{1}{2} g \left(\frac{h^2}{u'^2 \sin^2 \alpha} \right) \text{ or } u'^2 = \frac{gh \operatorname{cosec}^2 \alpha}{4} \quad \text{...(iv)}$$

Substituting this value of u'^2 in (i) we get

$\frac{1}{4} gh \operatorname{cosec}^2 \alpha = u^2 \sin^2 \beta - 2gh$ or $u^2 \sin^2 \beta = \frac{1}{4} gh\, (8 + \operatorname{cosec}^2 \alpha)$

or $u = \frac{1}{2} \sqrt{(gh)} \operatorname{cosec} \beta \sqrt{(8 + \operatorname{cosec}^2 \alpha)}$. Hence proved.

As for the component of velocity $u \cos \beta$, it being horizontal will have no effect on the vertical motion of the ball.

***§ 13. Envelope of the trajectories with the same velocity and point of projection.** *(Gorakhpur 84, 82)*

Let u be the velocity of projection and α the angle of projection.

Then the equation of the trajectory is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad \text{...(i)}$$

Differentiating both sides of (i) with respect to the parameter 'α', we have $0 = x \sec^2 \alpha - \frac{gx^2}{2u^2}\, 2 \sec^2 \alpha \tan \alpha$ (Note)

or $\sec^2 \alpha\, (u^2 - gx \tan \alpha) = 0$

or $\tan \alpha = u^2/gx$, since $\sec^2 \alpha \neq 0$

Substituting this value of $\tan \alpha$ in (i), we get the equation of the envelope as $y = x \left(\frac{u^2}{gx} \right) - \frac{gx^2}{2u^2} \left(1 + \frac{u^4}{g^2 x^2} \right) = \frac{u^2}{g} - \frac{gx^2}{2u^2} - \frac{u^2}{2g}$

or $y - \frac{u^2}{2g} = -\frac{gx^2}{2u^2}$ or $x^2 = -\frac{2u^2}{g} \left(y - \frac{u^2}{2g} \right)$ .. (ii)

Also if h be the height of the directrix above the point of projection, then $u^2 = 2gh$.

$\therefore$ from (ii) we have $x^2 = -4h\,(y - h)$ ...(iii)

as the equation of the envelope.

This equation represents a parabola, whose axis is vertical and vertex is the point $(0, h)$ *i.e.* the vertex lies on the common directrix of the trajectories and whose focus is the point of projection.

Note. This enveloping parabola touches all the trajectories externally and hence no projectile can go beyond this parabola *i.e.* [illegible] the same velo-

[illegible] ll the parabolic

[illegible] bola. To find

[illegible] we should find

Solved Examples on § 13.

***Ex. 1.** In vacuum particles are projected in all directions from a point O with a velocity u, g being assumed constant. Show that all their paths lie within an enveloping paraboloid. If their be a ceiling at a height k above O, show that its portion within reach of such particles is a circle of area $\pi \frac{u^2}{g}\left(\frac{u^2}{g}-2k\right)$.*

Solution. We know (proved in § 13 Page 54) that for different angles of projection all the trajectories in a plane will lie within the enveloping parabola given by $x^2=-\frac{2u^2}{g}\left(y-\frac{u^2}{2g}\right)$ ...(i)

[See result (ii) of § 13 Page 54]

If the velocity of projection remains u, then the envelope in every other plane through the point of projection will be an equal parabola. Hence all the paths lie within an enveloping paraboloid which envelopes all the enveloping parabolas in different planes through O, the point of projection and is generated by revolving (i) about y-axis.

The ceiling is given by $y=k$. ...(ii)

Let (ii) meet (i) at $x=x_1$, then we get

$$x_1^2=-\frac{2u^2}{g}\left(k-\frac{u^2}{2g}\right)=\frac{u^2}{g}\left(\frac{u^2}{g}-2k\right)$$

∴ The portion of the ceiling within reach of the particles is a circle of radius x_1 and hence its area $=\pi x_1^2=\pi\,(u^2/g)\,(u^2/g-2k)$.

***Ex. 2.** A rocket fired vertically upwards bursts at its highest point h feet above the ground. If each fragment starts with the same velocity u, prove that all fragments on reaching the ground lie within a circle of radius $u\,(u^2+2gh)^{1/2}/g$.*

Solution. Taking the point where the rocket bursts as origin, the equation of the enveloping parabola is

$$x^2=-\frac{2u^2}{g}\left(y-\frac{u^2}{2g}\right)$$ ..(i)

T[illegible] [illegible]ow this point, so it is $y=-h$...(ii)

[illegible] $x^2=-(2u/g)\,(-h-u^2/2g)$

or [illegible] $2gh)$

∴ Max. range on the ground $=x=(u/g)\,(u^2+2gh)^{1/2}$

...See Note at the end of § 13 Page 54

∴ All fragments on reaching the ground lie within a circle radius $(u/g)\,(u^2+2gh)^{1/2}$.

****Ex. 3.** A shot is fired wi[illegible] from the top of mountain which is in the form [illegible] that the farthest points which ca[illegible] tance (measured in a straight line) $a-\sqrt{(a^2-4ah)}$ [illegible] in order that a particle may clear a [illegible] of projection from the highest point*

Solution. Take O, the top of the mountain as origin and axis as shown in the figure.

Then the equation of the enveloping parabola is

$$x^2 = -\frac{2u^2}{g}\left(y - \frac{u^2}{2g}\right), \text{ where } u \text{ is the velocity of projection.}$$

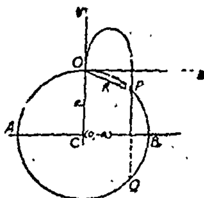
Here $u^2 = 2gh$ (given).

$\therefore$ The equation of the enveloping parabola is

$$x^2 = -4h\,(y - h) \quad \text{.. (i)}$$

Let P be the farthest point reached by the shot on the hemispherical mountain. Circle AOP is the cross-section of the mountain by the vertical plane through O and P. Then if C be the centre of this hemisphere the coordinates of C are $(0, -a)$ and consequently the equation of the circle with centre C is $\quad x^2+(y+a)^2=a^2$

or $\quad x^2+y^2+2ay=0 \quad \text{...(ii)}$



(Fig. 34)

Let P be (x_1, y_1), then from (i) and (ii) we have

$$x_1^2 = -4h\,(y_1 - h) \quad \text{...(iii)}$$

and

$$x_1^2 + y_1^2 + 2ay_1 = 0 \quad \text{...(iv)}$$

Solving (iii) and (iv) we get $y_1^2 + (2a - 4h)\,y_1 + 4h^2 = 0$,

whence $y_1 = \dfrac{(4h-2a) \pm \sqrt{[(2a-4h)^2 - 16h^2]}}{2}$

$= (2h - a) \pm \sqrt{[(a^2 - 4ah)]}$

Let $OP = R$, then

$R^2 = x_1^2 + y_1^2 = -$ [illegible]

$= -2a\,[(2h$ [illegible] $-4ah)$

$= a^2 + (a^2 - 4$ [illegible]

or $\quad R = a \pm \sqrt{(a^2 - 4ah)}. \quad \text{...(v)}$

This gives two values of R *i.e.* OP. These two values correspond to two points where (i) meets (ii) viz. P and Q in the figure one above and one below the diameter AB.

$\therefore \quad OP = R = a - \sqrt{(a^2 - 4ah)}$ (Note)

If R is real, we have from (v), $a^2 - 4ah \geqslant 0$ or $4ah \leqslant a^2$ or $h \leqslant \frac{1}{4}a$.

$\therefore$ Maximum value of $h = \frac{1}{4}a$ and in this case R has only one value and the velocity of projection $= \sqrt{(2gh)} = \sqrt{(2g \cdot \frac{1}{4}a)} = \sqrt{(\frac{1}{2}ga)}$.

Hence in order that a particle may clear a sphere of radius a the velocity of projection from the highest point should not be less than $\sqrt{(\frac{1}{2}ga)}$.

Exercises on § 13

Ex. 1. If any number of particles are projected from the same point with equal velocities in different directions, prove that the particles will lie on a circle, at a fixed time. (*Gorakhpur 84, 82*)

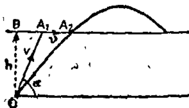
Ex 2: If a particle is projected with a velocity u from a given point S so that its path lies in a given vertical plane, prove that all possible paths lie within a parabola of latus rectum $2u^2/g$ and focus S.

(Hint. Find the envelope of the paths).

§ 14. Solved Examples on Projection to hit a moving object.

***Ex. 1. An aeroplane is flying with constant velocity v and at a constant height h. Show that if a gun is fired point blank at the aeroplane after it has passed directly over the gun and when the angle of elevation as seen from the gun is α, the shell will hit the aeroplane provided $2(V\cos\alpha - v)\, v \tan^2\alpha = gh$, where V is the initial velocity of the shell, the path being assumed to be parabolic.**

Solution. O is the point of projection of the shell. A_1 is the position of the aeroplane when the shot was fired at. Fired point blank means the intial velocity V of the shell has its direction along OA_1 or the angle of projection is α. The aeroplane is flying at a height h along the line BA_1 with velocity v in the direction as shown in the figure. A_2 is the point where the shell can hit the aeroplane if they reach A_2 at the same time, the shell starting from O and the aeroplane flying from A. Let the shell hit the aeroplane after time t.



(Fig. 35)

The distance moved by aeroplane in time $t = A_1A_2 = v.t.$ ...(i)

The horizontal distance moved by the shell in time t

$= BA_2 = (V\cos\alpha)\, t.$ ...(ii)

Also from $\triangle OA_1B$, we have $BA_1 = h\cot\alpha$ ...(iii)

Now $BA_2 = BA_1 - A_1A_2$

or $(V\cos\alpha)t = h\cot\alpha + v.t$, from (i), (ii) and (iii)

or $t(V\cos\alpha - v) = h\cot\alpha$ or $t = h\cot\alpha/(V\cos\alpha - v)$. (iv)

Considering the vertical motion of the shell from O to A_2, we get $h = (V\sin\alpha)\, t - \frac{1}{2}gt^2 = (V\sin\alpha - \frac{1}{2}gt)\, t$

$$= \left[V\sin\alpha - \tfrac{1}{2}g\left(\frac{h\cot\alpha}{V\cos\alpha - v}\right)\right]\left(\frac{h\cot\alpha}{V\cos\alpha - v}\right), \text{ from (iv)}$$

or $2h(V\cos\alpha - v)^2 = [2V\sin\alpha(V\cos\alpha - v) - gh\cot\alpha]\, h\cot\alpha$

or $2(V\cos\alpha - v)^2 = 2V\cos\alpha(V\cos\alpha - v) - gh\cot^2\alpha$

or $2(V\cos\alpha - v)^2 - 2V\cos\alpha(V\cos\alpha - v) = -gh\cot^2\alpha$

or $2(V\cos\alpha - v)[(V\cos\alpha - v) - V\cos\alpha] = -gh\cot^2\alpha$

or $2v(V\cos\alpha - v) = gh\cot^2\alpha$ or $2v(V\cos\alpha - v)\tan^2\alpha = gh.$

or $u \cos\alpha \quad v = nv$ or $v = (u \cos\alpha)/(n+1)$.

$\therefore$ from (i), $\tan\theta = \dfrac{u \sin\alpha}{u\cos\alpha - [u\cos\alpha)/(n+1)]} = \dfrac{(n+1)\sin\alpha}{n\cos\alpha}$

$= [1+(1/n)]\tan\alpha$. Hence proved.

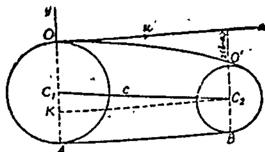
**Ex. 3. The radii of the front and hind wheels of a carriage are a and b and c is the distance between the axle trees; a particle of mud driven from the highest point of the hind wheel is observed to alight on the highest point of the front wheel, show that the velocity of carriage is $\sqrt{[g(c+b-a)(c+a-b)/\{4(b-a)\}]}$. *(Meerut 86)*

Solution. Let u be the velocity of carriage, then the velocity of the highest point of the hind wheel is $2u$ horizontally. (Note)

$\therefore$ The velocity of the mud particle is $2u$ horizontally.

But the carriage is moving horizontally with a velocity u.

The velocity of the mud-particle relative to the carriage $= 2u - u = u$, acting horizontally.



(Fig. 37)

Let O be the highest point of the hind wheel. Let C_1 and C_2 be the centres of hind and front wheels respectively. From C_2 draw C_2K perpendicular to the diameter OA.

C_1C_2 = distance between the centres $= c$ (given) ; $C_1K = b-a$ and $C_2K^2 = C_1C_2^2 - C_1K^2 = c^2 - (b-a)^2 = x_1^2$ (say).

Take O as origin and choose the axes as shown in the figure. Then the coordinates of O' the highest point of the front wheel are $(x_1, -y_1)$, where x_1 is given by (i) ...(ii)

and $y_1 = OA - O'B = 2b - 2a = 2(b-a)$.

Also the equation of the path traced out by the mud particle is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \quad ...(iii)$$

But here $\alpha = 0$, as the mud-particle leaves O horizontally

$\therefore$ The equation of the path traced out by the mud-particle is $y = -gx^2/(2u^2)$, putting $\alpha = 0$ in (iii).

$\because$ $O'(x_1, -y_1)$ is a point on this path, so we have $-y_1 = -gx_1^2/2u^2$

or $$2(b-a) = \frac{g}{2u^2}[c^2 - (b-a)^2], \text{ from (i) and (ii)}$$

or $$u^2 = \frac{g[c^2-(b-a)^2]}{4(b-a)} = \frac{g(c+b-a)(c-b+a)}{4(b-a)}$$

or $u = \sqrt{[g(c+b-a)(c+a-b)/\{4(b-a)\}]}$. Hence proved.

"$s=ut+\frac{1}{2}ft^2$" we have

$$-a\cos\theta=-(v\sin\theta)\,t-\tfrac{1}{2}gt^2 \quad \text{(Note)}$$

or $$a\cos\theta=v\sin\theta.\left[\frac{a(1-\sin\theta)}{v\cos\theta}\right]+\tfrac{1}{2}g\left[\frac{a(1-\sin\theta)}{v\cos\theta}\right]^2 \quad \text{...from (i)}$$

or $$a\cos\theta=\frac{a\sin\theta(1-\sin\theta)}{\cos\theta}=\frac{ga^2(1-\sin\theta)^2}{2v^2\cos^2\theta}$$

or $$a[\cos^2\theta-\sin\theta+\sin^2\theta]=\frac{ga^2(1-\sin\theta)^2}{2v^2\cos\theta}$$

or $$(1-\sin\theta)=\frac{ga(1-\sin\theta)^2}{2v^2\cos\theta} \quad \text{or} \quad 2v^2\cos\theta=ga(1-\sin\theta).$$

Ex. 5. A shot fired with velocity V at an elevation α strikes a point P on a horizontal plane through the point of projection. If the point P is receding from the gun with velocity v, show that the elevation must be changed to θ, where

$$\sin 2\theta=\sin 2\alpha+(2v/V)\sin\theta$$

Solution We know that the range of a particle on an inclined plane $=(2/g)$ (Horizontal comp. of vel.) (initial vert. comp. of vel.) ... (i)

Let O be the point of projection and α the angle of projection, then originally range $OP=(2/g)\,(V\cos\alpha)\,(V\sin\alpha)$. ... (ii)

Now the angle of projection is θ, when the point P is receding from the gun with a velocity v. Let us assume that point P is stationary, then initial horizontal and vertical components of velocity of the shot are $(V\cos\theta-v)$ and $V\sin\theta$ respectively.

$\therefore$ from (i), range $OP=(2/g)\,(V\cos\theta-v)\,(V\sin\theta)$ ... (iii)

$\therefore$ from (ii) and (iii) we have

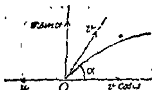
$$(2/g)\,(V\cos\alpha)(V\sin\alpha)=(2/g)\,(V\cos\theta-v)\,(V\sin\theta)$$

or $V\sin 2\alpha=V\sin 2\theta-2v\sin\theta$ or $V\sin 2\theta=V\sin 2\alpha+2v\sin\theta$

or $\sin 2\theta=\sin 2\alpha+(2v/V)\sin\theta$ Hence proved.

****Ex. 6. A battle ship is streaming ahead with velocity u. A gun is mounted on the ship so as to point straight backwards and is set at an angle of elevation α. If v be the velocity of projection relative to the gun show that the range is $(2v/g)\sin\alpha\,(v\cos\alpha-u)$ and the angle for maximum range is $\cos^{-1}\{(u+\sqrt{(u^2+8v^2)})\}$.**

Solution. As the ship is moving horizontally with a velocity u in a direction opposite to that of the projection from the gun and the horizontal and vertical components of velocity of projection relative to the gun are $v\cos\alpha$ and $v\sin\alpha$ respectively, so initially the actual horizontal components of velocity $=v\cos\alpha-u$ and the vertical component of velocity $=v\sin\alpha$.



(Fig. 39)

Also we know that horizontal range of a particle $=(2/g)$ (horizontal comp. of velocity) $\times$ (initial vertical comp. of velocity).

∴ If R be the required range, then

$$R=(2/g)\ (v\cos\alpha-u)\ (v\sin\alpha) \quad ...(i)$$

Now if R is maximum, then $dR/d\alpha=0$ and $d^2R/d\alpha^2=-ve$.
From (i), $dR/d\alpha=(2/g)\ [(v\cos\alpha-u)\ (v\cos\alpha)$

$$+(v\sin\alpha)\ (-v\sin\alpha)]$$

$$=(2/g)\ [v^2(\cos^2\alpha-\sin^2\alpha)-uv\cos\alpha]$$

If $dR/d\alpha=0$, then $v^2(\cos^2\alpha-\sin^2\alpha)-uv\cos\alpha=0$

or $v(2\cos^2\alpha-1)-u\cos\alpha=0$, ∵ $\cos^2\alpha-\sin^2\alpha=2\cos^2\alpha-1$

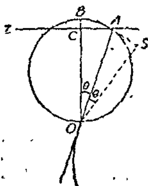
or $2v\cos^2\alpha-u\cos\alpha-v=0$ or $\cos\alpha=\{u\pm\sqrt{(u^2+8v^2)}\}/4v$.

The negative sign before the radical renders the value of $\cos\alpha$ negative *i.e.* α is obtuse which is against the problem. (Note)

Hence $\cos\alpha=\{u+\sqrt{(u^2+8v^2)}\}/4v$ Hence proved

§ 16. Particles suffered to describe parabolic paths.

****Ex. 1. Particles slide down the chord of a vertical circle terminating in the lowest point. Show that the locus of the foci of the path subsequently described is a cardioid.** *(Gorakhpur 85, 79)*



(Fig. 40)

Solution. BO is the vertical diameter through O, the lowest point of the circle. OA is a chord of the circle terminating at O. Let the polar coordinates of A be (r, θ) referred to O as pole and OB as initial line.

A particle slides from A and reaches O along the chord OA and let v be its velocity when it reaches O.

Then $v=\sqrt{\{2g\sin(90°-\theta)\ OA\}}$, from "$v^2=u^2+2fs$"

$=\sqrt{(2g.OA\cos\theta)}=\sqrt{(2g.OC)}$, ... (i)

where AC is the horizontal line through A.

After reaching O the particle leaves the circle and moves freely under gravity and thus traces out parabolic path beyond O with velocity of projection as v.

Also velocity at any point of the parabolic path is that due to fall from the directrix and from (i) velocity v at $O=\sqrt{(2g\ CO)}$. Hence the directrix of the subsequent path is the horizontal line AC.

Now OA is a tangent to the parabola at O and OC is the perpendicular from O to the directrix AC. Also we know that tangent at any point of the parabola bisects the angle between its focal distance and the perpendicular from the point on the directrix. Therefore if we draw a line OS making $\angle AOS=\angle COA=\theta$ (say) and take $OS=OC$, then S is the focus of the parabolic path.

Let (r_1, α) be the polar coordinates of S.

Then $\alpha=\angle COS=2\theta$ ...(ii)

and $r_1=OS=OC=OA\cos\theta=(2a\cos\theta)\cos\theta$,

∵ $OA=OB\cos\theta=2a\cos\theta$ and a is the radius of the circle

$\therefore \quad r_1 = 2a\cos^2\theta = a\,(1+\cos 2\theta) = a\,(1+\cos\alpha)$, from (ii)

$\therefore$. The locus of $S\,(r_1, \alpha)$ is $r = a\,(1+\cos\theta)$, which is the standard equation of a cardioid.

Ex. 2. **Particles slide down the diameters of a verical circle, prove that the locus of the foci of their subsequenent paths is a circle.**

Solution. OCP is a diameter of the circle with centre C. A particle slides from O along the diameter OP and leaves the circle at P, subsequent motion being free the particle traces out a parabolic path beyond P.

OP is the tangent to the parabolic path at P and PK is the perpdndicular from P on the directrix OZ, which is a horizontal line through O,

(See last example)

$\because \quad \angle OKP = 90°$

$\therefore \quad K$ is a point on the circle (angle in a seme-circle). Make $\angle POS = \angle KOP$ and let $OS = OK$. Then S is the focus of the parabolic path.

In $\triangle$' OKP and OSP, $OK = OS$, (by construction)

OP is common and $\angle KOP = \angle POS$ (by construction)

$\therefore \quad \triangle$' OKP and OSP are congruent.

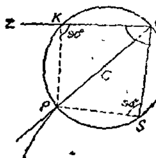


Fig. 41)

In particular $\angle OSP$ is a right angle and as such S will lie on the circle. Hence the locus of the focus S is the given circle.

*Ex. 3. **A number of bodies slide from rest down the chords of a vertical circle starting from its highest point and afterwards move freely. Prove that the locus of the foci of their subsequent paths is a circle whose radius is half that of given circle.**

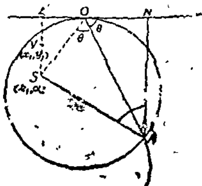
Find the locus of the vertices of the subsequent parabolic paths.

Solution. Let a be the radius of the circle. Let O, the highest point of the circle be taken as pole and the horizontal line ON be taken as the initial line.

OA is any chord of the circle through O, such that

$\angle NOA = \theta$ (say)

A particle slides from O along OA and leaves the circle at A and subsequent motion being free traces out a parabolic path beyond A. The horizontal line ON through O is the directrix of this parabola, since the velocity at A is that due to fall from the level of directrix and is also due to the fall from the horizontal



(Fig. 42)

line ON. From A draw AN perpendicular to the directrix. Make $\angle NAO = \angle OAS$ and cut off $AS = AN$. Then S is the focus of the parabolic path. Join OS

Since $\triangle^s$ OAN and OAS are congruent,

$\therefore$ $\angle SOA = \angle NOA = \theta$:

$\therefore$ If S be (r_1, α), we have $\alpha = \angle SON = 2\theta$ (see figure) . (i)

and $r_1 = OS = OA \cos\theta = (2a \sin\theta)\cos\theta$.

since $OA = 2a \sin\theta$, where a is the radius of the circle.

or $r_1 = a \sin 2\theta = a \sin\alpha$, from (i)

Hence the locus of focus S (r, α) is $r = a \sin\theta$, which is a circle of radius $\frac{1}{2}a$. (This fact can be seen very easily by changing $r = a \sin\theta$ to cartesian form).

Let (x_1, y_1) be the cartesian coordinates of the vertex V of the parabolic path, where V is the middle point of the perpendicular SZ drawn from the focus S to the directrix OS.

Now $x_1 = OZ = OS \cos(\pi - 2\theta) = -a \sin 2\theta \cos 2\theta$,

$\because$ $OS = a \sin 2\theta$ (proved)

or $x_1 = -a \sin 2\theta \cos 2\theta$ ...(ii)

And $y_1 = VZ = \frac{1}{2} SZ = \frac{1}{2}. OS \sin(\pi - 2\theta) = \frac{1}{2} a \sin 2\theta \sin 2\theta$

or $y_1 = \frac{1}{2} a \sin^2 2\theta$. ...(iii)

Eliminating θ between (ii) and (iii) we have

$$x_1^2 = a^2 \sin^2 2\theta \cos^2 2\theta = a^2\left(\frac{2y_1}{a}\right)\left(1 - \frac{2y_1}{a}\right), \text{ from (iii)}$$

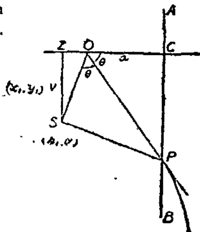
or $x_1^2 = 2y_1(a - 2y_1)$ or $x_1^2 + 4y_1^2 = 2ay_1$.

$\therefore$ The locus of $V(x_1, y_1)$ is $x^2 + 4y^2 = 2ay$, which is the equation of an ellipse.

Ex. 4. Particles are allowed to fall from rest under gravity from a fixed point to a fixed vertical straight line along various rectilinear chords. If they be suffered to describe parabolic trajectories after leaving these chords show that the locus of their foci is a circle and that of the verticies is an ellipse.

Solution. O is the fixed point and AB is the fixed vertical straight line. OP is any chord through O meeting the line AB in P. A particle slides from O along OP and leaves the chord OP at P and the subsequent motion being free, the particle traces out a parabolic path beyond P. From O draw a horizontal line meeting AB in C. Let $AC = a$ (constant since O and AB are fixed) and $\angle COP = \theta$.

Now OP is a tangent to the parabolic path at P and PC is perpendicular to the directrix OC



(Fig. 43)

$\therefore$ The locus of $S(r_1, \theta)$ is $r=a$, which represents a circle.

From focus S draw SZ, perpendicular to the directrix OC.

Then V, the middle point of SZ, is the vertex of the parabolic path. Let the cartesian coordinates of V be (x_1, y_1).

Then $x_1=OZ=OS\cos(\pi-2\theta)=-a\cos 2\theta$, since $OS=a$

or $x_1=-a\cos 2\theta$ ...(i)

And $y_1=VZ=\frac{1}{2}SZ=\frac{1}{2}OS.\sin(\pi-2\theta)=\frac{1}{2}a\sin 2\theta$

or $2y_1=a\sin 2\theta$ .. (ii)

Squaring and adding (i) and (ii) we have $x_1^2+4y_1^2=a^2$.

$\therefore$ The locus of $V(x_1, y_1)$ is $x^2+4y^2=a^2$, which is the equation of an ellipse.

MISCELLANEOUS SOLVED EXAMPLES

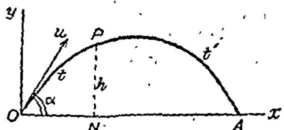
Ex. 1. (a). If t be the time in which a projectile reaches a point P in its path and t' the time from P till it reaches the horizontal plane through the point of projection, show that height of P above the horizontal is $\frac{1}{2}gtt'$.

Also prove that the greatest height of the projectile is $\frac{1}{8}g(t+t')^2$.

Solution. Let O be the point of projection, A is the point where the particle strikes the horizontal plane through O, t and t' are the times taken by the particle in moving from O to P and P to A respectively.

Consider the vertical motion from O to P, from "$s=ut+\frac{1}{2}ft^2$", we have, $h=u\sin\alpha.t-\frac{1}{2}gt^2$, ...(i)

where u and α are the velocity and angle of projection and h is the required height PN of P above OA.



(Fig. 44)

Also time of flight $=(2u\sin\alpha)/g$

or $t+t'=(2u\sin\alpha)/g$

or $u\sin\alpha=\frac{1}{2}g(t+t')$ ..(ii)

Substituting this value in (i) we get

$$h=\frac{1}{2}g(t+t')t-\frac{1}{2}gt^2=\frac{1}{2}gtt'$$

Also the greatest height of the projectile

$$=\frac{u^2\sin^2\alpha}{2g}=\frac{g^2(t+t')^2}{4\times 2g}, \text{ from (ii)}$$

$$=\frac{1}{8}g(t+t')^2.$$ Hence proved.

Ex. 1. (b). A particle is projected with velocity u from a point on an inclined plane. If v_1 be is velocity on striking the plane when

the range up the plane is maximum and v_2 the velocity on striking the plane when the range down the plane is maximum, prove that

$$u^2 = v_1 v_2.$$

Solution. If β be the inclination of the given plane to the horizontal, then the maximum range R of the particle up the inclined plane is given by

$$R = u^2/[g(1+\sin\beta)] \quad ..(i)$$

Also we know that the velocity v of the particle at a height h above the point of projection is given by

$$v^2 = u^2 - 2gh. \quad \text{(Note)} \quad ...(ii)$$

$\therefore$ If v_1 be its velocity on striking the inclined plane when the range up the plane is maximum *i.e.* R, then

'h' $= R\sin\beta$ (See Fig. 26 Page 44) (Note)

and from (ii) we have $v_1^2 = u^2 - 2gR\sin\beta$, where R is given by (i)

or $$v_1^2 = u^2 - \frac{2g\,u^2 \sin\beta}{g(1+\sin\beta)} = \frac{u^2(1-\sin\beta)}{(1+\sin\beta)} \quad ...(iii)$$

Changing β by $-\beta$ in (iii) *i.e.* for the range down the plane we have $v_2^2 = u^2[1+\sin\beta]/(1-\sin\beta)$ .. (iv)

$\therefore$ From (iii) and (iv) we get $v_1^2 v_2^2 = u^4$ or $v_1 v_2 = u^2$.

Hence proved.

***Ex. 2. At any instant a projectile is moving with velocity u in a direction making an angle α to the horizon. After an interval of time t, the direction of its path makes an angle β with the horizontal. Prove that $u\cos\alpha = gt/(\tan\alpha - \tan\beta)$.**

Solution. After time t let v be the velocity of the projectile making an angle β (given) with the horizontal. The horizontal and vertical components of this velocity are $v\cos\beta$ and $v\sin\beta$ respectively.

Also the horizontal and vertical components of velocity when the particle is moving with a velocity u making an angle α with the horizontal, are $u\cos\alpha$ and $u\sin\alpha$ respectively.

Also as the horizontal component of velocity remains constant throughout the motion, so $v\cos\beta = u\cos\alpha$ ...(i)

Considering the vertical motion from "$v = u + ft$" we have

$$v\sin\beta = u\sin\alpha - gt \quad (ii)$$

From (ii), we get $gt = u\sin\alpha - v\sin\beta$ ...(iii)

$= u\sin\alpha - [(u\cos\alpha)/\cos\beta]\sin\beta$, from (i)

$= u\cos\alpha\,[\tan\alpha - \tan\beta]$

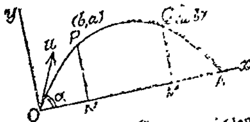
or $\therefore u\cos\alpha = gt/(\tan\alpha - \tan\beta)$. Hence proved.

Ex. 3. A ball is projected so as just to clear two walls, the first of height a at a distance b from the point of projection and the second of height b and at a distance a from the point of projection. Show that the range on the horizontal plane is $(a^2+ab+b^2)/(a+b)$ and that the angle of projection exceeds $\tan^{-1} 3$.

289-66

Solution. Let O be the point of projection of the ball.

Let u and α be the velocity and angle of projection of the ball so that it just clears the walls PN and QM i.e. trajectory traced out by the ball passes through P and Q, the top of the walls.



(Fig. 45)

Choose O as origin and the axes as shown in the figure. Then according to the problem the coordinates of P and Q are (b, a) and (a, b) respectively.

Also the equation of the trajectory is

$$y=x \tan \alpha-\frac{gx^2}{2u^2 \cos^2 \alpha} \quad \ldots(i)$$

$\because$ $P(b, a)$ and $Q(a, b)$ lie on (i), so we have

$$b=a \tan \alpha-\frac{ga^2}{2u^2 \cos^2 \alpha} \ldots(ii) \text{ and } a=b \tan \alpha-\frac{gb^2}{2u^2 \cos^2 \alpha} \quad \ldots(iii)$$

Multiplying (ii) by a and (iii) by b and subtracting we get

$$0=(a^2-b^2) \tan \alpha-\frac{g(a^3-b^3)}{2u^2 \cos^2 \alpha}$$

or $$2u^2(a^2-b^2) \tan \alpha \cos^2 \alpha=g(a^3-b^3)$$

or $$\frac{2u^2 \sin \alpha \cos \alpha}{g}=\frac{a^3-b^3}{a^2-b^2}=\frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)}$$

or Range on the horizontal plane $=(a^2+ab+b^2)/(a+b)$.

Again from (ii), $b-a \tan \alpha=-ga^2/2u^2 \cos^2 \alpha$ .. (iv)

and from (iii), $a-b \tan \alpha=-gb^2/2u^2 \cos^2 \alpha$ .. (v)

Dividing (iv) by (v) we get $\dfrac{b-a \tan \alpha}{a-b \tan \alpha}=\dfrac{a^2}{b^2}$

or $b^2(b-a \tan \alpha)=a^2(a-b \tan \alpha)$ or $[a^2b-ab^2] \tan \alpha=a^3-b^3$ (Note)

or $$\tan \alpha=\frac{a^3-b^3}{ab(a-b)}=\frac{(a-b)^3+3ab(a-b)}{ab(a-b)}$$

$$=3+\frac{(a-b)^3}{ab(a-b)}=3+\frac{(a-b)^2}{ab}$$

$=3+$some positive quantity.

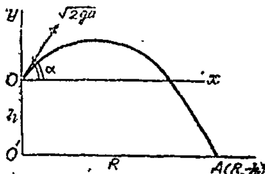
or $\tan \alpha>3$ or $\alpha>\tan^{-1} 3$. Hence proved.

****Ex. 4. A particle is projected under gravity with velocity $\sqrt{(2ag)}$ from a point at a height h above the level plane. Show that the angle of projection θ for the maximum range on the plane is given by $\tan^2 \theta=a/(a+h)$ and the maximum range is $2\sqrt{\{a(a+h)\}}$.** (*Allahabad 83*)

Solution. Let the particle be projected from O at an angle α with the horizontal. Then the equation of the path of the projectile is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad \ldots(i)$$

referred to O as origin and the horizontal and vertical lines through O (as shown in the figure) as coordinate axes and u as the velocity of projection.



(Fig. 46)

If the particle strikes the level at $(R, \ h)$ then $(R, -h)$ is a point on (i).

$$\therefore \quad -h = R \tan \alpha - \frac{gR^2}{2u^2 \cos^2 \alpha}$$

or $\quad -2u^2h = 2u^2 R \tan \alpha - g R^2 \sec^2 \alpha$

or $\quad -2u^2h = 2u^2 R \tan \alpha - g R^2 (1 + \tan^2 \alpha)$

or $\quad gR^2 \tan^2 \alpha - 2u^2 R \tan \alpha + (gR^2 - 2u^2h) = 0, \quad \ldots(ii)$

which is a quadratic equation in $\tan \alpha$.

Since the path of the projectile is real, therefore the values of $\tan \alpha$ given by (ii) are real and condition for the same is

"$b^2 - 4ac \geqslant 0$"

or $\quad 4u^4 R^2 - 4 (gR^2)(gR^2 - 2u^2h) \geqslant 0$ or $u^4 - g^2R^2 + 2u^2gh \geqslant 0$

or $\quad (4g^2a^2) - g^2R^2 + 2 (2ga) gh \geqslant 0$, since $u^2 = 2ga$

or $\quad 4a^2 - R^2 + 4ah \geqslant 0$ or $4a^2 + 4ah \geqslant R^2$

or $\quad R^2 \leqslant 4a (a+h)$ or $R \leqslant 2\sqrt{\{a (a+h)\}}$.

$\therefore$ Maximum range on the level plane

$= $ maximum value of $R = 2\sqrt{\{a (a+h)\}}$

Also if $\alpha = \theta$ when R is maximum *i.e.* $2\sqrt{\{a (a+h)\}}$ then from (ii) we have

$$4ga (a+h) \tan^2 \theta - 2 (2ga) . 2\sqrt{\{a (a+h)\}} \tan \theta + 4ga (a+h) - 2 (2ga) h = 0$$

(remembering that $u^2 = 2ga$)

or $\quad (a+h) \tan^2 \theta - 2\sqrt{\{a (a+h)\}} \tan \theta + a = 0$

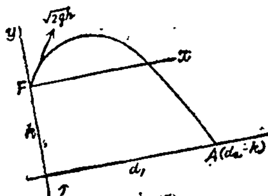
or $\quad [\sqrt{(a+h)} \tan \theta - \sqrt{a}]^2 = 0$

or $\quad \tan \theta = \sqrt{a}/\sqrt{(a+h)}$ or $\tan^2 \theta = a/(a+h)$. Hence proved.

***Ex. 5.** **A fort and ship are both armed with guns which give** ... **guns in the fort are**

Solution. Let F be the fort situated at a height k above the sea level. The initial velocity of the projectile of the gun of the fort $=\sqrt{(2gh)}=u$ (say).

Then referred to F as origin and the coordinate axes as shown in the figure, the equation of the path of the projectile of the gun of the fort is



(Fig. 47)

$$y=x\tan\alpha-\frac{gx^2}{2u^2\cos^2\alpha}. \qquad \ldots(i)$$

$\because$ the point $A(d_1, -k)$ lies on it

$\therefore \quad -k=d_1\tan\alpha-\dfrac{gd_1^2}{2u^2\cos^2\alpha}$

But d_1 is maximum (given), $\therefore \dfrac{d}{d\alpha}(d_1)=0$,

Hence from (i) differentiating both sides with respect to α

we get $0=\left(\dfrac{dd_1}{d\alpha}\tan\alpha+d_1\sec^2\alpha\right)-\dfrac{g}{2u^2}\dfrac{d}{d\alpha}(d_1^2\sec^2\alpha)$

or $\quad 0=\dfrac{dd_1}{d\alpha}\tan\alpha+d_1\sec^2\alpha$

$$-\frac{g}{2u^2}\left[d_1^2\,2\sec^2\alpha\tan\alpha+\sec^2\alpha\,2d_1\frac{dd_1}{d\alpha}\right]$$

or $\quad \dfrac{dd_1}{d\alpha}\left[\tan\alpha-\dfrac{gd_1\sec^2\alpha}{u^2}\right]=\left[\dfrac{gd_1^2\tan\alpha}{u^2}-d_1\right]\sec^2\alpha$

or $\quad \dfrac{dd_1}{d\alpha}\left[\tan\alpha-\dfrac{gd_1\sec^2\alpha}{2gh}\right]=\left[\dfrac{gd_1^2\tan\alpha}{2gh}-d_1\right]\sec^2\alpha$, since $u^2=2gh$ (given)

$\therefore \dfrac{dd_1}{d\alpha}=0$ gives $\dfrac{d_1^2\tan\alpha}{2h}-d_1=0$ or $\tan\alpha=\dfrac{2h}{d_1}$

$\therefore$ From (i), $-k=d_1\left(\dfrac{2h}{d_1}\right)-\dfrac{gd_1^2}{2u^2}\left(1+\dfrac{4h^2}{d_1^2}\right)$

$$=2h-\frac{gd_1^2}{2\times 2gh}\left(\frac{d_1^2+4h^2}{d_1^2}\right), \quad \because u^2=2gh$$

$$=2h-\frac{d_1^2}{4h}-h=h-\frac{d_1^2}{4h}$$

or $\quad d_1^2=4h(h+k)$ or $d_1=2\sqrt{[h(h+k)]}$.

For the gun in the ship, putting $-k$ for k, we get
$d_2=2\sqrt{\{h(h-k)\}}$

$$\therefore \frac{d_1}{d_2}=\frac{2\sqrt{\{h(h+k)\}}}{2\sqrt{\{h(h-k)\}}}=\sqrt{\left(\frac{h+k}{h-k}\right)}.$$ Hence proved.

***Ex. 6 (a).** **A ship is under fire from the guns of a fort k feet above sea level. Assuming that two guns of the ship and the fort fire with a velocity $\sqrt{(2gh)}$, prove that the width of the zone under the fire from the fort which the ship has to cross before being able to reply is $2\sqrt{h}\,[\sqrt{(h+k)}-\sqrt{(h-k)}]$.**

Solution. As in the last example we can prove that if d_1 and d_2 are the greatest horizontal ranges at which the fort and ship respectively, can engage, then $d_1=2\sqrt{\{h(h+k)\}}$ and $d_2=2\sqrt{\{h(h-k)\}}$.

$\therefore$ Required width of the zone $=d_1-d_2$ (Note)

$=2\sqrt{\{h(h+k)\}}-2\sqrt{\{h(h-k)\}}$

$=2\sqrt{h}\,[\sqrt{(h+k)}-\sqrt{(h-k)}]$. Hence proved.

Ex. 6 (b). **A fort is at the top of a hill of height h above the sea level. Prove that the greatest horizontal distance at which gun in the ship can hit the fort is $2\sqrt{[k(k-h)]}$, where $\sqrt{(2gk)}$ is the muzzle velocity of the shot.** *(Gorakhpur 86)*

Hint : Do as Ex. 5 Page 69.

****Ex. 7.** **A fort is on the edge of a cliff of height k. Show that there is an annular region of area $8\pi hk$ in which the fort is out of range of the ship but the ship is not out of range of the fort, where $\sqrt{(2gh)}$ is the velocity of the shells used by both.**

Solution. As in Ex. 5 Page 69 we can prove that if d_1 and d_2 are the greatest horizontal ranges at which the fort and ship, respectively, can engage, then $d_1=2\sqrt{\{h(h+k)\}}$ and $d_2=2\sqrt{\{h(h-k)\}}$.

Now required *area of an annular region in which the fort is out of range of the ship but the ship is not out of range of the fort*

$=$area between two concentric circles of radii d_1 and d_2. (Note)

$=\pi d_1^2-\pi d_2^2=\pi(d_1^2-d_2^2)=\pi[4h(h+k)-4h(h-k)]=8\pi hk.$

***Ex. 8.** **Two particles are projected simultaneously with the same speed V in the same vertical plane with angles of elevation θ and 2θ, where $\theta<45°$. Show that their velocities are parallel after a time $(V/g)\cos\frac{1}{2}\theta\operatorname{cosec}\frac{3}{2}\theta$.**

Solution. We know (from § 2 Page 1) that if u and α be the velocity and angle of projection, then x and y components of velocity after time t are $u\cos\alpha$ and $(u\sin\alpha-gt)$.

$\therefore$ If ϕ be the angle which the direction of velocity after time t makes with the x-axis, we have

$$\tan\phi=(u\sin\alpha-gt)/(u\cos\alpha). \quad ...(i)$$

In this question let the velocities of the given particles be parallel after time t, then from (i) we have

$$\frac{V\sin\theta-gt}{V\cos\theta}=\frac{V\sin 2\theta-gt}{V\cos 2\theta}$$ (Note)

or $(V \sin \theta - gt) \cos 2\theta = (V \sin 2\theta - gt) \cos \theta$

or $gt [\cos \theta - \cos 2\theta] = V (\sin 2\theta \cos \theta - \cos 2\theta \sin \theta)$

or $gt [2 \sin \frac{3}{2}\theta \sin \frac{1}{2}\theta] = V \sin (2\theta - \theta) = V \sin \theta$

or $t = (V \sin \theta)/[g \{2 \sin \frac{3}{2}\theta \sin \frac{1}{2}\theta\}]$

$= (2V \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta)/[g \{2 \sin \frac{3}{2}\theta \sin \frac{1}{2}\theta\}]$

$= (V/g) \cos \frac{1}{2}\theta \operatorname{cosec} \frac{3}{2} \theta.$ *Hence proved.*

Ex. 9. A gun fires a shell with a muzzle velocity u, show that the farthest horizontal distance at which an aeroplane at a height h can be hit is $(u/g)\sqrt{(u^2 - 2gh)}$ and the gun's elevation then is $\tan^{-1} \{u/\sqrt{(u^2 - 2gh)}\}$. *(Kanpur 86; Lucknow 82, 79)*

Solution Let α be the angle of projection of the shell.

Let the point of projection of the shell be taken as origin and the horizontal line (lying in the plane of flight) and vertical line through the point of projection be taken as x and y axes respectively.

Then the equation of the trajectory of the shell is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad \text{...(i)}$$

If R be the horizontal distance of the aeroplane from the point of projection, the coordinates of the position of the aeroplane when it is hit by the shell, are (R, h).

$\because$ (R, h) lies on (i), so $h = R \tan \alpha - \dfrac{gR^2 \sec^2 \alpha}{2u^2}$ ...(ii)

If R be maximum, then $\dfrac{dR}{d\alpha} = 0$ and $\dfrac{d^2R}{d\alpha^2} =$ negative

Differentiating both sides of (ii) with respect to α, we have

$$0 = \left(R \sec^2 \alpha + \frac{dR}{d\alpha} \tan \alpha\right) - \frac{g}{2u^2}\left[2R \sec^2 \alpha \frac{dR}{d\alpha} + 2R^2 \sec^2 \alpha \tan \alpha\right]$$

or $$\frac{dR}{d\alpha}\left[\tan \alpha - \frac{gR^2 \sec^2 \alpha}{u^2}\right] = \frac{gR^2 \sec^2 \alpha \tan \alpha}{u^2} - R \sec^2 \alpha$$

If $\dfrac{dR}{d\alpha} = 0$, then $\dfrac{g R^2 \sec^2 \alpha \tan \alpha}{u^2} - R \sec^2 \alpha = 0$

or $\tan \alpha = u^2/gR$, since $\sec^2 \alpha \neq 0$ ...(iii)

Substituting this value of $\tan \alpha$ in (ii), we get

$$h = R\left(\frac{u^2}{gR}\right) - \frac{gR^2}{2u^2}\left(1 + \frac{u^4}{g^2R^2}\right) = \frac{u^2}{g} - \frac{gR^2}{2u^2} - \frac{u^2}{2g} = \frac{u^2}{2g} - \frac{gR^2}{2u^2}.$$

or $\dfrac{gR^2}{2u^2} = \dfrac{u^2}{2g} - h = \dfrac{u^2 - 2gh}{2g}$ or $R^2 = \dfrac{u^2}{g^2}(u^2 - 2gh)$

or $R = (u/g) \sqrt{(u^2 - 2gh)}$

Also from (ii) we get

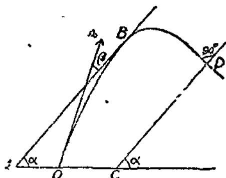
$$\tan \alpha = \frac{u^2}{gR} = \frac{u^2}{u\sqrt{(u^2 - 2gh)}} \text{ or } \alpha = \tan^{-1}\left[\frac{u}{\sqrt{(u^2 - 2gh)}}\right].$$

**Ex. 10. Two parallel straight lines in the same vertical plane are each inclined to the horizon at an angle α. A particle is projected from a point mid-way between them so as to graze one of the lines and then to strike the other at right angles, show that the direction of projection makes with either of the lines an angle

$$\tan^{-1}[(\sqrt{2}-1)\cot\alpha].$$

Solution. O is the point mid-way between the given parallel straight lines AB and CD inclined at an angle α to the horizontal.

The particle is projected from O and touches (grazes) the line AB at A and strikes the line CD at right angles.



(Fig. 48)

Let the velocity of projection be u making an angle β with either of the lines AB and CD. The resolved parts of this velocity u parallel and perpendicular to AB are $u\cos\beta$ and $u\sin\beta$ respectively and acceleration in these directions are $-g\sin\alpha$ and $-g\cos\alpha$ respectively. (Note)

Let t_1 and t_2 be the times taken by the particle in moving from O to B and O to D respectively.

Since O is the point mid-way between AB and CD therefore B and D are two points at equal distances on opposite sides of O. Hence if we consider the motion of the particle perpendicular to AB, then the distance travelled by the particle from O to B at right angles to $AB = -$(the distance travelled by the particle from O to D at right angles to AB).

i.e. $\quad (u\sin\beta . t_1 - \frac{1}{2}g\cos\alpha\, t_1^2) = -(u\sin\beta\, t_2 - \frac{1}{2}g\cos\alpha\, t_2^2)$

or $\quad (u\sin\beta)(t_1+t_2) = \frac{1}{2}g\cos\alpha\,(t_1^2+t_2^2)$. ...(i)

Also as the particle touches AB at B, so the velocity of the particles at right angles to AB vanishes at E. So from "$v=u+ft$", we have $0=u\sin\beta - g\cos\alpha\, t_1$ or $t_1=(u\sin\beta)/(g\cos\alpha)$ ...(ii)

Again the particle strikes CD at right angles at D, so the velocity of the particle parallel to the line AB vanishes at D, so from "$v=u+ft$", we have $0=u\cos\beta - g\sin\alpha . t_2$

or $\quad t_2=(u\cos\beta)/(g\sin\alpha)$. ...(iii)

Substituting the values of t_1, t_2 from (ii) and (iii) in (i) we get

$$(u\sin\beta)\left[\frac{u\sin\beta}{g\cos\alpha}+\frac{u}{g}\cdot\frac{\cos\beta}{\sin\alpha}\right]=\frac{1}{2}g\cos\alpha\left[\frac{u^2\sin^2\beta}{g^2\cos^2\alpha}+\frac{u^2\cos^2\beta}{g^2\sin^2\alpha}\right]$$

or $$\frac{\sin^2\beta}{\cos\alpha}+\frac{\sin\beta\cos\beta}{\sin\alpha}=\frac{\sin^2\beta}{2\cos\alpha}+\frac{\cos^2\beta\cos\alpha}{2\sin^2\alpha}$$

or $$\frac{\sin^2\beta}{2\cos\alpha}+\frac{\sin\beta\cos\beta}{\sin\alpha}-\frac{\cos^2\beta\cos\alpha}{2\sin^2\alpha}=0$$

or $\sin^2\beta\sin^2\alpha+2\sin\beta\cos\beta\sin\alpha\cos\alpha-\cos^2\beta\cos^2\alpha=0$

or $\tan^2\beta+2\tan\beta\cot\alpha-\cot^2\alpha=0$,

dividing each term by $\cos^2\beta\sin^2\alpha$

or $\tan\beta=\frac{1}{2}[-2\cot\alpha\pm\sqrt{(4\cot^2\alpha+4\cot^2\alpha)}]$

$=-\cot\alpha\pm\sqrt{2}\cot\alpha$

Since β is an acute angle, hence $\tan\beta$ is +ve.

$\therefore$ $\tan\beta=\sqrt{2}\cot\alpha-\cot\alpha=(\sqrt{2}-1)\cot\alpha$,

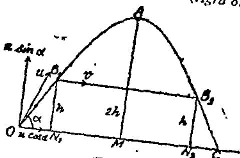
or $\beta=\tan^{-1}[(\sqrt{2}-1)\cot\alpha]$.

Hence proved.

***Ex. 11. A stone is thrown in such a manner that it would just hit a bird at the top of a tree and afterwards reach a height double that of the tree. If at the moment of throwing the stone the bird flies away horizontally, show that not withstanding this, the stone will hit the bird if its horizontal velocity to that of the bird be as $(\sqrt{2}+1):2$.** (*Agra 83*)

Solution. Let O be the point of projection of the stone and u be its velocity of projection making an angle α with the horizon.

The horizontal and vertical components of the velocity of projection of the stone are $u\cos\alpha$ and $u\sin\alpha$ respectively.



(Fig. 49)

B_1N_1 is tree of height h (say). Then the maximum height upto which the stone rises during its motion$=2h$ (given).

At the moment of throwing the stone the bird begins to fly away horizontally along B_1B_2 with velocity v (say) from the top B_1 of the tree. Let the stone strike the bird at B_2.

Let the stone be at a height h after time t. Then considering the vertical motion of the stone, we get

$h=u\sin\alpha\, t-\frac{1}{2}gt^2$ or $gt^2-2u\sin\alpha . t+2h=0$ ...(i)

or $t=\dfrac{2u\sin\alpha\pm\sqrt{(4u^2\sin^2\alpha-8gh)}}{2g}=\dfrac{\sqrt{(4hg)}\pm\sqrt{(4hg-2gh)}}{g}$,

$\because$ $u^2\sin^2\alpha=4gh$

or $t=(2\pm\sqrt{2})\sqrt{(h/g)}$.

If t_1 and t_2 be the two roots of the equation (ii), then

$t_1=(2-\sqrt{2})\sqrt{(h/g)}$ and $t_2=(2+\sqrt{2})\sqrt{(h/g)}$. ...(iii)

Here t_1 is the time taken by the stone in moving from O to B_1 and t_2 is the time from O to B_2. Also $t_2>t_1$.

The stone will hit the bird at B_2 if stone and bird reach B_2 at the same time i.e. if the bird takes time t_2 in flying from B_1 to B_2.

Then $B_1B_2=vt_2$ ...(iv)

Also for the horizontal motion of the stone, we get

$B_1B_2=(u\cos\alpha)(t_2-t_1)$. . (v)

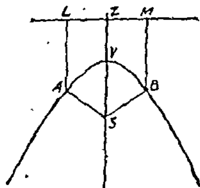
$\therefore$ From (iv) and (v) we get $vt_2=(u\cos\alpha)(t_2-t_1)$

or $\dfrac{u\cos\alpha}{v}=\dfrac{t_2}{t_2-t_1}=\dfrac{(2+\sqrt{2})\sqrt{(h/g)}}{(2+\sqrt{2})\sqrt{(h/g)}-(2-\sqrt{2})\sqrt{(h/g)}}$

$=\dfrac{2+\sqrt{2}}{2\sqrt{2}}=\dfrac{\sqrt{2}+1}{2}$. Hence proved.

***Ex. 12. Prove that the locus of the foci of all trajectories passing through two given points is a hyperbola.**

Solution. A and B are two given points. The trajectory passes through A and B. Let S be the focus and LM be the directrix of this parabolic path (trajectory). Join SA and SB. Also from A and B draw AL and BM perpendiculars to the directrix LM.



(Fig. 50)

Then $SA=AL$ and $SB=BM$ ...(properties of parabola)

$\therefore$ $SB-SA=BM-AL$

=vertical distance between A and B

=constant, since A and B are given points.

$\therefore$ S is a point, the differences of whose distances from two points A and B is constant. Also in the case of hyperbola we know that the difference of focal distances of any point on it is constant (remember that their are two foci of a hyperbola). Hence the focus of the parabola lies on a hyperbola whose foci are A and B.

***Ex. 13. Three particles are projected simultaneously and in the same vertical plane from a point with velocity u, v, w in directions making angles α, β, γ with the horizontal. Show that the area of the triangle formed by the particles at any time t is proportional to the square of the time elapsed from the instant of projection, and that the three particles will always lies in the same straight line if**

$$\frac{\sin(\beta-\gamma)}{u}+\frac{\sin(\gamma-\alpha)}{v}+\frac{\sin(\alpha-\beta)}{w}=0.$$

Solution. Take the point of projection as origin and the horizontal line (lying in the plane of flight) and the vertical line through the point of projection as coordinates axes.

After time t, let the coordinates of the positions of the particles be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

Then $x_1=(u\cos\alpha)\,t$ and $y_1=(u\sin\alpha)\,t-\frac{1}{2}gt^2$,

$x_2=(v\cos\beta)\,t$ and $y_2=(v\sin\beta)\,t-\frac{1}{2}gt^2$,

and $x_3=(w\cos\gamma)\,t$ and $y_3=(w\sin\gamma)\,t-\frac{1}{2}gt^2$.

$\therefore$ Area of the triangle formed by these points

$=\frac{1}{2}[(x_1y_2+x_2y_3+x_3y_1)-(x_2y_1+x_3y_2+x_1y_3)]$

$=\frac{1}{2}\Sigma\,(x_1y_2-x_2y_1)$

$=\frac{1}{2}\Sigma\,[(u\cos\alpha t)(v\sin\beta t-\frac{1}{2}gt^2)-(v\cos\beta t)(u\sin\alpha t-\frac{1}{2}gt^2)]$

$=\frac{1}{2}\Sigma\,[uv(\cos\alpha\sin\beta-\sin\alpha\cos\beta)\,t^2-\frac{1}{2}gt^3(u\cos\alpha-v\cos\beta)]$

$=\frac{1}{2}t^2\,\Sigma\,[uv\sin(\beta-\alpha)]-\frac{1}{4}gt^3\,\Sigma\,(u\cos\alpha-v\cos\beta)$

$=\frac{1}{2}t^2\,\Sigma\,[uv\sin(\beta-\alpha)],\ \because\ \Sigma\,(u\cos\alpha-v\cos\beta)=0$ (Note)

which is proportional to t^2.

These three particles will lie in a straight line if this area of the triangle formed by these three particles is zero;

i.e. $\Sigma\,[uv\sin(\beta-\alpha)]=0$

or $uv\sin(\alpha-\beta)+vw\sin(\beta-\gamma)+wu\sin(\gamma-\alpha)=0$

or $\dfrac{\sin(\alpha-\beta)}{w}+\dfrac{\sin(\beta-\gamma)}{u}+\dfrac{\sin(\gamma-\alpha)}{v}=0,$

dividing each term by uvw. Hence proved.

Ex. 15. *Three particles are projected from the same point in the same vertical plane with velocities u, v, w at elevations α, β, γ respectively. Prove that the foci of their path will lie on a straight line if* $\dfrac{\sin 2(\beta-\gamma)}{u^2}+\dfrac{\sin 2(\gamma-\alpha)}{v^2}+\dfrac{\sin 2(\alpha-\beta)}{w^2}=0.$

Solution. Coordinates of foci of the parabolic paths of three particles are $\left(\dfrac{u^2\sin 2\alpha}{2g},-\dfrac{u^2\cos 2\alpha}{2g}\right)$, $\left(\dfrac{v^2\sin 2\beta}{2g},-\dfrac{v^2\cos 2\beta}{2g}\right)$, and $\left(\dfrac{w^2\sin 2\gamma}{2g},-\dfrac{w^2\cos 2\gamma}{2g}\right)$ respectively. (See § 3 Page 3–5)

The foci will lie in the same line, if

$$\begin{vmatrix} (u^2\sin 2\alpha)/2g & -(u^2\cos 2\alpha)/2g & 1 \\ (v^2\sin 2\beta)/2g & -(v^2\cos 2\beta)/2g & 1 \\ (w^2\sin 2\gamma)/2g & -(w^2\cos 2\gamma)/2g & 1 \end{vmatrix}=0$$

or $$\begin{vmatrix} u^2\sin 2\alpha & u^2\cos 2\alpha & 1 \\ v^2\sin 2\beta & v^2\cos 2\beta & 1 \\ w^2\sin 2\gamma & w^2\cos 2\gamma & 1 \end{vmatrix}=0$$

or $u^2\sin 2\alpha\,(v^2\cos 2\beta-w^2\cos 2\gamma)-u^2\cos 2\alpha\,(v^2\sin 2\beta-w^2\sin 2\gamma)$
$+(v^2w^2\sin 2\beta\cos 2\gamma-v^2w^2\cos 2\beta\sin 2\gamma)=0$

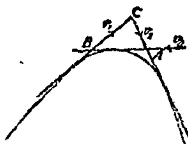
or $\Sigma\,u^2v^2\,[\sin 2\alpha\cos 2\beta-\cos 2\alpha\sin 2\beta]=0$

or $u^2v^2\sin 2(\alpha-\beta)+v^2w^2\sin 2(\beta-\gamma)+w^2u^2\sin 2(\gamma-\alpha)=0$

or $\dfrac{\sin 2(\alpha-\beta)}{w^2}+\dfrac{\sin 2(\beta-\gamma)}{u^2}+\dfrac{\sin 2(\gamma-\alpha)}{v^2}=0.$

dividing each term by $u^2 v^2 w^2$. Hence proved.

Ex. 15. The tangents to a projectile's path form a triangle ABC, the velocities are v_1 along BC, v_2 along CA and v_3 along AB, show that
$$\frac{BC}{v_1}+\frac{CA}{v_2}+\frac{AB}{v_3}=0.$$



Solution. Let the tangent BC, CA and AB which form the triangle ABC be inclined at angles θ_1, θ_2 and θ_3 respectively to the horizon.

Then in $\triangle ABC$, we get

Fig. 61.

$\angle A$=angle between CA and AB which are inclined at angles θ_2 and θ_3 to the horizontal, *i.e.* $\angle A=(\theta_2-\theta_3)$. **(Note)**

Similarly $\angle B=\theta_3-\theta_1$

$\therefore \quad \angle C=\pi-(\angle A+\angle B)=[\pi-(\theta_2-\theta_1)]$.

Now from $\triangle ABC$, we have $\dfrac{BC}{\sin A}=\dfrac{CA}{\sin B}=\dfrac{AB}{\sin C}$

or $$\frac{BC}{\sin(\theta_2-\theta_3)}=\frac{CA}{\sin(\theta_3-\theta_1)}=\frac{AB}{\sin(\theta_1-\theta_2)}=k \text{ (say)} \quad ..(i)$$

Also we know that the horizontal component of velocity remains constant throughout the motion, so we get
$$v_1\cos\theta_1=v_2\cos\theta_2=v_3\cos\theta_3$$

or $$\frac{v_1}{\cos\theta_2\cos\theta_3}=\frac{v_2}{\cos\theta_3\cos\theta_1}=\frac{v_3}{\cos\theta_1\cos\theta_2}=\lambda \text{ (say)} \quad ...(ii)$$

$\therefore$ From (i) and (ii) we get
$$\frac{BC}{v_1}=\frac{k\sin(\theta_2-\theta_3)}{\lambda\cos\theta_2\cos\theta_3}=\frac{k(\sin\theta_2\cos\theta_3-\cos\theta_2\sin\theta_3)}{\lambda\cos\theta_2\cos\theta_3}$$

or $\dfrac{BC}{v_1}=\dfrac{k}{\lambda}(\tan\theta_2-\tan\theta_3)$.

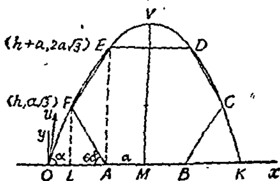
Similarly $\dfrac{CA}{v_2}=\dfrac{k}{\lambda}(\tan\theta_3-\tan\theta_1)$ and $\dfrac{AB}{v_3}=\dfrac{k}{\lambda}(\tan\theta_1-\tan\theta_2)$

Adding these we get $\dfrac{BC}{v_1}+\dfrac{CA}{v_2}+\dfrac{AB}{v_3}=0$. Hence proved.

***Ex. 16. A regular hexagon stands with one side on the ground and a particle is projected so as to graze its four upper vertices. Show that the velocity of the particle on reaching the ground is to its least velocity as $\sqrt{(31)}:\sqrt{3}$.**

Solution. Let the particle be projected from the point O with a velocity u making an angle α with the horizontal. Let the parti-

cle again strike the ground at K after grazing the four upper points F, E, D and C of the hexagon $ABCDEF$. The velocity with which the particle strikes at K is u and the least velocity will be $u \cos \alpha$ at the highest point V of the parabolic path.



Let $2a$ be the length of each side of the hexagon. From F draw FL perpendicular to OA Let $OL=h$.

$AL=2a \cos 60°$

$=a$ and

$FL=2a \sin 60°=a\sqrt{3}$.

$\therefore EA=2FL=2a\sqrt{3}$.

Let O be the origin. OK the x-axis

(Fig. 52)

and the vertical line through O be y-axis. Then the coordinates of B and E are $(h, a\sqrt{3})$ and $(h+a, 2a\sqrt{3})$ respectively.

Also the equation of parabolic path is

$$y=x \tan \alpha-\frac{gx^2}{2u^2 \cos^2 \alpha} \qquad ...(i)$$

$\because$ $E(h+a, 2a\sqrt{3})$ and $F(h, a\sqrt{3})$ lie on (i)

$\therefore$ $2a\sqrt{3}=(h+a) \tan \alpha-[g(h+a)^2/(2u^2 \cos^2 \alpha)]$ ...(ii)

and $a\sqrt{3}=h \tan \alpha-[gh^2/(2u^2 \cos^2 \alpha)]$. ...(iii)

Substracting (iii) from (ii) we have

$$a\sqrt{3}=a \tan \alpha-\frac{g(2ha+a^2)}{2u^2 \cos^2 \alpha} \quad \text{or} \quad \sqrt{3}=\tan \alpha-\frac{g(2h+a)}{2u^2 \cos^2 \alpha} \qquad ...(iv)$$

Also range on the horizontal plane $=OK=(2u^2 \sin \alpha \cos \alpha)/g$

or $2.OM=(2u^2 \sin \alpha \cos \alpha)/g$, M is the mid-point of OK

or $OL+LA+AM=(u^2 \sin \alpha \cos \alpha)/g$

or $h+a+a=(u^2 \sin \alpha \cos \alpha)/g$

or $u^2=g(h+2a)/\sin \alpha \cos \alpha$ ...(v)

Substituting this value in (iv), we get

$$\sqrt{3}=\tan \alpha-\frac{g(2h+a) \sin \alpha \cos \alpha}{2g(h+2a) \cos^2 \alpha}=\tan \alpha-\frac{(2h+a) \tan \alpha}{(2h+4a)}$$

$$\text{or} \quad \sqrt{3}=\frac{3a \tan \alpha}{(2h+4a)} \quad \text{or} \quad \tan \alpha=\frac{(2h+4a)}{a\sqrt{3}} \qquad ...(vi)$$

$\therefore$ from (v), $2(h+2a)=(2u^2 \sin\alpha \cos\alpha)/g$

or $\quad a\sqrt{3} \tan\alpha=(2u^2 \sin\alpha \cos\alpha)/g$, from (vi)

or $\quad u^2 \cos^2\alpha=\frac{1}{2}(a\sqrt{3}g)$. ...(vii)

Also from (iii), $a\sqrt{3}=h \tan\alpha-[ga^2/(2u^2 \cos^2\alpha)]$

$$=\frac{h(2h+4a)}{a\sqrt{3}}-\frac{gh^2}{a\sqrt{3}g}, \text{ from (vi) and (viii)}$$

or $\quad 3a^2=h^2+4ah \quad$ or $\quad h^2+4ah-3a^2=0$

or $\quad h=\frac{1}{2}[-4a\pm\sqrt{(16a^2+12a^2)}]=-2a+a\sqrt{7}$, $\because$ h is +ve

or $\quad h+2a=a\sqrt{7}$ ...(viii)

$\therefore$ From (vi) we get $\tan\alpha=\dfrac{2a\sqrt{7}}{a\sqrt{3}}=\dfrac{2\sqrt{7}}{\sqrt{3}}$.

Hence $\cos\alpha=\dfrac{\sqrt{3}}{\sqrt{\{(2\sqrt{7})^2+(\sqrt{3})^2\}}}=\dfrac{\sqrt{3}}{\sqrt{(31)}}$ or $\sec^2\alpha=\dfrac{31}{3}$

$\therefore$ From (vii), $u^2=\dfrac{a\sqrt{3}g}{2}\sec^2\alpha=\dfrac{a\sqrt{3}g}{2}\times\dfrac{31}{3}=\dfrac{31\,ag}{2\sqrt{3}}$

$\therefore$ $\dfrac{\text{velocity on reaching ground}}{\text{least velocity}}=\dfrac{u}{u\cos\alpha}=\sec\alpha$

$$=(\sec^2\alpha)^{1/2}=\sqrt{(31)}/\sqrt{3}.$$ Hence proved.

Ex. 17. A particle is projected so as to graze the four upper corners of a regular hexagon whose side is c and which is placed vertically with one side on the table. Show that the range on the table is $c\sqrt{7}$ and that the square of time of flight is $28c/g\sqrt{3}$.

Solution. Putting $2a=c$ or $a=\frac{1}{2}c$ in the last example we can prove $u^2=31cg/2\sqrt{3}=31\,ag/4\sqrt{3}$.

And $\cos\alpha=\dfrac{\sqrt{3}}{\sqrt{(31)}}$, $\sin\alpha=\dfrac{2\sqrt{7}}{\sqrt{(31)}}$.

$\therefore$ Range on the table

$$=\frac{2u^2\sin\alpha\cos\alpha}{g}=\frac{2\times31\,cg\times\sqrt{7}\times2\sqrt{3}}{4\sqrt{3}\times g\sqrt{(31)}\times\sqrt{(31)}}=c\sqrt{7}.$$

And square of the time of flight

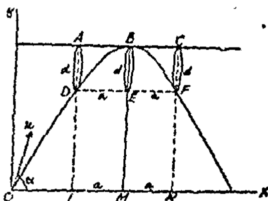
$$=\left(\frac{2u\sin\alpha}{g}\right)^2=\frac{4u^2\sin^2\alpha}{g^2}=\frac{4\times31\,cg\times28}{4\sqrt{3}g^2\times31}=\frac{28\,c}{g\sqrt{3}}$$

***Ex. 18. A particle is projected so as just to pass through three equal rings of diameter d, placed in parallel vertical planes at distances a apart, with their highest point in the same horizontal straight line at a height h above the point of projection. Prove that its angle of elevation must be $\tan^{-1}[\{2\sqrt{(hd)}\}/a]$.**

Solution. Let O be the point of projection and the horizontal and vertical lines through O (as shown in the figure) be the coordinate axes.

Then if u and α be the velocity and angle of projection respectively, the equation of the path of the particle is

$y=x \tan \alpha$
$-\{gx^2/(2u^2 \cos^2 \alpha)\}$ ... (i)



(Fig. 53)

AD, BE and CF are given rings each of diameter d.

The horizontal distance between the rings are given to be a (as shown in the figure).

The rings are so placed that the particle will graze the lowest points of the side rings AD and CF and the highest point of the middle ring BE.

Now BM = maximum height attained by the particle
$=(u^2 \sin^2 \alpha)/2g$

Let $BM=h$, then $h=(u^2 \sin^2 \alpha)/2g$ or $u^2=2gh/(\sin^2 \alpha)$ ...(ii)

Let the coordinates of D be (x_1, y_1).

Then $y_1=h-d$ and $x_1=OL=OM-a$
$=$(half the horizontal range)$-a$

or $x_1=[(u^2 \sin \alpha \cos \alpha)/g]-a=2h \cot \alpha-a$, from (ii)

Now as $D(x_1, y_1)$ lies on (i), $\therefore$ $y_1=x_1 \tan \alpha-\dfrac{gx_1^2}{2u^2 \cos^2 \alpha}$

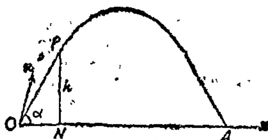
or $h-d=(2h \cot \alpha-a) \tan \alpha-\dfrac{g(2h \cot \alpha-a)^2}{4gh \cot^2 \alpha}$, from (ii)

or $h-d=2h-a \tan \alpha-\dfrac{(4h^2 \cot^2 \alpha-4ah \cot \alpha+a^2) \tan^2 \alpha}{4h}$

$=2h-a \tan \alpha-h+a \tan \alpha-[(a^2 \tan^2 \alpha)/4h]$

or $a^2 \tan^2 \alpha=4hd$ or $\tan \alpha=2\sqrt{(hd)}/a$ or $\alpha=\tan^{-1}[2\sqrt{(hd)}/a]$.

***Ex. 19. Particles are projected simultaneously from same point in different directions with such velocities so as just to clear a vertical wall of height h and to fall in a narrow ditch parallel to the wall; show that all the particles fall into the ditch at the same moment.**



(Fig. 54)

Solution. Let O be the point of projection. Let u and α be the velocity and angle of projection. PN is the wall of height h and A is the point in the ditch where one of the particles falls. Let $ON=a$ and $OA=R$ (say).

Let t_1 and t_2 be the times taken in moving from O to P and O to A respectively.

Then $a=u\cos\alpha . t_1$ ...(i); $h=u\sin\alpha . t_1-\frac{1}{2}gt_1^2$, ...(ii)

$t_2=(2u\sin\alpha)/g$ ...(iii) and $R=u\cos\alpha . t_2$ ...(iv)

From (i) and (iv) we get $a/R=t_1/t_2$.

From (ii), $h=(\frac{1}{2}gt_2)\,t_1-\frac{1}{2}gt_1^2$, $\because$ from (iii) $u\sin\alpha=\frac{1}{2}gt_2$

$$=\tfrac{1}{2}gt_1^2\left[\frac{t_2}{t_1}-1\right]=\tfrac{1}{2}gt_1^2\left[\frac{R}{a}-1\right], \text{ from (v)}$$

$$=\tfrac{1}{2}g\,(a^2t_2^2/R^2)\,[(R/a)-1], \quad \because \text{ from (v) } t_1=at_2/R$$

or $$t_2^2=2h/[g\,(a/R)^2\,\{(R/a)-1\}]$$

$\therefore$ t_2 depends only on the ratio $\dfrac{a}{R}$ *i.e.* $\dfrac{ON}{OA}$ which is constant.

Hence t_2 is the same for all trajectories.

Ex. 20. A trapezium with parallel sides of length $2a$ and $4a$ and non-parallel sides each of length $2a$ is placed on the ground with its plane vertical and with the longest side in contact with the ground. A ball is projected from the corner lying on the ground so as just to graze the other three corners of the trapezium. Show that the greatest height reached by the particle is $4a/\sqrt{3}$ and that the total time of flight is $\sqrt{[32a/(g\sqrt{3})]}$.

Solution. Let $ABCD$ be the given trapezium whose sides are as follows :—

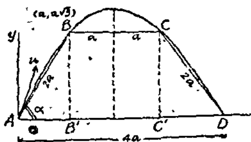
$AD=4a$;
$AB=BC=CD=2a$.

From B and C draw BB' and CC' perpendiculars to AD. Then

$B'C'=2a$

$\therefore$ $AB'=C'D=\frac{1}{2}(2a)=a$

$BB'=\sqrt{(AB^2-AB'^2)}$
$=\sqrt{(4a^2-a^2)}$
$=a\sqrt{3}$



(Fig. 55)

Let u and α be the velocity and angle of projection of the ball from the point A. Take

A as origin and the horizontal line AD and vertical line through A as coordinate axes. Then the coordinates of B are $(a, a\sqrt{3})$.

Also the equation of the trajectory of the ball is

$$y = x \tan\alpha - [gx^2/(2u^2\cos^2\alpha)]$$

$\because$ $B(a, a\sqrt{3})$ lies on it, so we have

$$a\sqrt{3} = a\tan\alpha - [(ga^2/(2u^2\cos^2\alpha)] \quad ...(i)$$

Also horizontal range $= AD$

or $(2u^2\sin\alpha\cos\alpha)/g = 4a$ or $(u^2\sin\alpha\cos\alpha)/g = 2a$ ...(ii)

Substituting the value of u^2 from (ii) in (i) we get

$$\sqrt{3} = \tan\alpha - \frac{ga\sin\alpha\cos\alpha}{4ag\cos^2\alpha} = \tan\alpha - \tfrac{1}{4}\tan\alpha$$

or $$\tan\alpha = \tfrac{4}{3}\sqrt{3} = 4/\sqrt{3} \quad ...(iii)$$

$\therefore$ From (i) we get

$$\sqrt{3} = \frac{4}{\sqrt{3}} - \frac{ga}{2u^2\cos^2\alpha} \text{ or } \frac{ga}{2u^2\cos^2\alpha} = \frac{4}{\sqrt{3}} - \sqrt{3} = \frac{1}{\sqrt{3}}$$

or $$u^2\cos^2\alpha = \tfrac{1}{2}ga\sqrt{3} \quad ...(iv)$$

From (ii) we have $(u^2\sin^2\alpha)(u^2\cos^2\alpha)/g^2 = 4a^2$,

squaring both sides of (ii)

or $$u^2\sin^2\alpha = \frac{4a^2g^2}{u^2\cos^2\alpha} = \frac{8a^2g^2}{ag\sqrt{3}} = \frac{8ag}{\sqrt{3}}.$$

$\therefore$ time of flight

$$= \text{"}\frac{2u\sin\alpha}{g}\text{"} = \sqrt{\left(\frac{4u^2\sin^2\alpha}{g^2}\right)} = \sqrt{\left(\frac{4\times 8ag}{g^2\sqrt{3}}\right)} = \sqrt{\left(\frac{32a}{g\sqrt{3}}\right)}$$

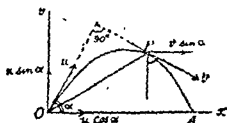
Maximum height attained by the particle

$$= \text{"}\frac{u^2\sin^2\alpha}{2g}\text{"} = \frac{8ga}{2g\sqrt{3}} = \frac{4a}{\sqrt{3}}.$$

Hence proved.

Ex. 21. *A particle projected with velocity $\sqrt{(2gh)}$ so that its range on horizontal plane through the point of projection is $4r$. When it is at a distance l from the point of projection it is moving at right angles to the original direction of motion, prove that*

$$l^2r^2 = h^3(l - h)$$



(Fig. 56)

Solution. Let the particle be projected from O with velocity $\sqrt{(2gh)} = u$ (say) making an angle α with the horizontal.

Let after time t the particle reach P with velocity v moving at right angles to the original direction.

Since the horizontal component of velocity remains constant throughout the motion, so we have

$$v \sin \alpha = u \cos \alpha \quad \text{or} \quad v = u \cot \alpha \qquad ...(i)$$

Considering the vertical motion from O to P from "$v=u+ft$" we have $\quad -v \cos \alpha = u \sin \alpha - gt$

or $\quad gt = u \sin \alpha + v \cos \alpha = u \sin \alpha + [u \cos^2 \alpha / \sin \alpha]$, from (i)

or $\quad gt = \dfrac{u (\sin^2 \alpha + \cos^2 \alpha)}{\sin \alpha} = \dfrac{u}{\sin \alpha} \quad$ or $\quad t = \dfrac{u}{g \sin \alpha} \qquad ...(ii)$

If the coordinates of P be (x_1, y_1), then we get

$$x_1 = u \cos \alpha . t = u \cos \alpha \, (u/g \sin \alpha), \text{ from (ii)}$$
$$= (u^2 \cot \alpha)/g = 2h \cot \alpha, \qquad \text{since } u^2 = 2gh$$

or $\quad x_1 = 2h \cot \alpha$

and $\quad y_1 = u \sin \alpha \, t - \frac{1}{2} g t^2$

$$= u \sin \alpha \, (u/g \sin \alpha) - \tfrac{1}{2} g \, (u/g \sin \alpha)^2, \text{ from (ii)}$$
$$= \frac{u^2}{g} - \frac{u^2}{2g \sin^2 \alpha} = \frac{u^2}{2g} (2 - \operatorname{cosec}^2 \alpha)$$

or $\quad y_1 = h_1 (2 - \operatorname{cosec}^2 \alpha) \qquad ...(iii) \qquad ...$since $u^2 = 2gh$

Now we are given $OP = l$

$\therefore \quad l^2 = OP^2 = x_1^2 + y_1^2 = (2h \cot \alpha)^2 + h^2 (2 - \operatorname{cosec}^2 \alpha)^2$

or $\quad l^2 = 4h^2 \cot^2 \alpha + 4h^2 - 4h^2 \operatorname{cosec}^2 \alpha + h^2 \operatorname{cosec}^4 \alpha$

or $\quad l^2 = h^2 \operatorname{cosec}^4 \alpha \quad$ or $\quad l = h \operatorname{cosec}^2 \alpha \qquad ...(iv)$

Also horizontal range $= r$ (given)

or $\quad (2u^2 \sin \alpha \cos \alpha)/g = 4r$

or $\quad r = (u^2 \sin \alpha \cos \alpha)/2g = (2gh \sin \alpha \cos \alpha)/2g, \quad \because \; u^2 = 2gh$

or $\quad r = h \sin \alpha \cos \alpha$

$\therefore$ From (iv) and (v) $l^2 r^2 = h^2 \operatorname{cosec}^4 \alpha \times h^2 \sin^2 \alpha \cos^2 \alpha$

or $\quad l^2 r^2 = h^4 \cot^2 \alpha = h^3 (h \cot^2 \alpha) = h^3 (h \operatorname{cosec}^2 \alpha - h)$

or $\quad l^2 r^2 = h^3 (l - h)$, from (iv). $\qquad$ Hence proved.

Ex. 22. Particles slide from rest down the chords of a vertical plane curve, having its highest point as the pole and a horizontal line through it as the initial line ; all these chords pass through the pole. If the particles be allowed to move freely after leaving the chords, prove that the locus of the foci of the subsequent paths is given by $r = f(\frac{1}{2}\theta) \cos \theta$, where $r = f(\theta)$ is the equation of the curve.

Solution. OP is the curve given by $r = f(\theta)$. O is the highest point of this curve and OP any chord. Let P be (r, θ) referred to O as pole and horizontal line OX through O as initial line. The velocity of particle when it reaches P, after sliding along the chord

OP_1 is due to a fall from Ox, hence Ox is the directrix of the parabolic path which the particle traces after leaving the chord at P. The line OP is a tangent to the parabolic path at P. Make $\angle OPS = \angle OPM$, where PM is the perpendicular from P on the directrix Ox. Make $PS = PM$, then S is the focus of the parabolic path. Join OS. Let S be (r', ϕ).

Then $\phi = \angle SOx = 2\theta$, as $\angle POx = \theta$ and $\triangle SOP \equiv \triangle POM$

or $\phi = 2\theta$ ...(i)

And $r' = OS = OP \cos\theta$

$\therefore$ PS is perpendicular to OS as PM is perpendicular to Ox

or $r' = r\cos\theta$, as $OP = r$

$= f(\theta)\cos\theta$, as $r = f(\theta)$

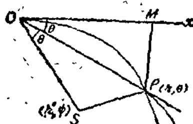
is the equation of the curve

or $r' = f(\frac{1}{2}\phi)\cos\frac{1}{2}\phi$,

since from (i) we have $\theta = \frac{1}{2}\phi$

$\therefore$ The locus of $S\ (r', \phi)$ is

$r = f(\frac{1}{2}\theta)\cos\frac{1}{2}\theta$.



(Fig. 57)

Hence proved.

****Ex. 23. If v_1, v_2, v_3 are the velocities at three points P, Q, R on the path of a projectile where the inclinations to the horizon of these velocities are $\alpha, \alpha-\beta, \alpha-2\beta$ and if t_1, t_2 be the times of describing PQ, QR respectively, prove that $v_3 t_1 = v_1 t_2$;**

$$(1/v_1) + (1/v_3) = (2\cos\beta)/v_2 \quad \text{. . (Garhwal 82)}$$

Solution. We know that horizontal component of velocity remains constant throughout the motion of a projectile.

$\therefore$ $v_1\cos\alpha = v_2\cos(\alpha-\beta) = v_3\cos(\alpha-2\beta)$ ...(i)

Also for the vertical motion of the particle from P to Q we have $v_2\sin(\alpha-\beta) = v_1\sin\alpha - gt_1$ ...(ii)

and from Q to R we have

$v_3\sin(\alpha-2\beta) = v_2\sin(\alpha-\beta) - gt_2$ ...(iii)

Now $\dfrac{1}{v_1} + \dfrac{1}{v_3} = \dfrac{\cos\alpha}{v_2\cos(\alpha-\beta)} + \dfrac{\cos(\alpha-2\beta)}{v_2\cos(\alpha-\beta)}$, from (i) (Note)

$= \dfrac{\cos\alpha + \cos(\alpha-2\beta)}{v_2\cos(\alpha-\beta)} = \dfrac{2\cos(\alpha-\beta)\cos\beta}{v_2\cos(\alpha-\beta)}$

$= (2\cos\beta)/v_2$ Hence proved.

Also from (ii), $gt_1 = v_1\sin\alpha - v_2\sin(\alpha-\beta)$ ...(iv)

From (iii), $gt_2 = v_2\sin(\alpha-\beta) - v_3\sin(\alpha-2\beta)$ ...(v)

Multiplying (iv) by v_3 and (v) by v_1 and subtracting, we get

$$g(v_3 t_1 - v_1 t_2) = v_1 v_3 \sin\alpha - v_2 v_3 \sin(\alpha-\beta) - v_2 v_1 \sin(\alpha-\beta) + v_3 v_1 \sin(\alpha-2\beta)$$

$$= v_1 v_3 \left[\sin\alpha - \left(\frac{v_2}{v_1}\right)\sin(\alpha-\beta) - \left(\frac{v_2}{v_3}\right)\sin(\alpha-\beta) + \sin(\alpha-2\beta)\right]$$

$$= v_1 v_3 \left[\sin\alpha - v_2\left(\frac{1}{v_1}+\frac{1}{v_3}\right)\sin(\alpha-\beta) + \sin(\alpha-2\beta)\right]$$

$= v_1 v_3 [\sin\alpha - 2\cos\beta \sin(\alpha-\beta) + \sin(\alpha-2\beta)]$,

since $(1/v_1)+(1/v_3)=(2\cos\beta)/v_2$

$= v_1 v_3 [\{\sin\alpha + \sin(\alpha-2\beta)\} - 2\cos\beta\sin(\alpha-\beta)]$

$= v_1 v_3 [\{2\sin(\alpha-\beta)\cos\beta\} - 2\cos\beta\sin(\alpha-\beta)]$

$= v_1 v_3 (0) = 0$

$\therefore\ v_3 t_1 - v_1 t_2 = 0,\ \because\ g \neq 0$

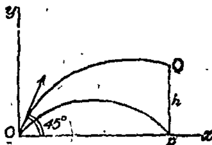
or $\quad v_3 t_1 = v_1 t_2$. Hence proved.

Ex. 24. A shot projected with velocity V at an elevation 45° reaches a point P on the horizontal plane through the point of projection. Show that in order to hit a mark h feet above P if the shot is projected at same elevation, the velocity of projection must be increased to $V^2/\sqrt{(V^2-gh)}$.

Solution. Let O be the point of projection of the shot. Take the horizontal line (lying in the plane of flight) and the upward drawn vertical line through O as x and y-axes respectively.

According to the problem, as the angle of projection is 45° and the velocity of projection is V, so we have $OP = V^2/g$

....See § 6 (b) P. 6 of this chapter



(Fig. 58)

Let Q be the point at a height h above P, then the coordinates of Q are $(V^2/g, h)$ referred to Ox and Oy as axes.

(See Fig. 58 above).

Let V_1 be the velocity of projection to hit Q. Then the equation of this path is

$y = x\tan\alpha - (gx^2/2u^2\cos^2\alpha)$, where $\alpha=45°$, $u=V_1$

or $\quad y = x - (gx^2/V_1^2)$, $\because\ \tan\alpha=1$, $\cos\alpha=1/\sqrt{2}$ when $\alpha=45°$

As $Q\ (V^2/g,\ h)$ lies on it, so we have

$$h=\frac{V^2}{g}-\frac{g}{V_1^2}\left(\frac{V^2}{g}\right)^2=\frac{V^2}{g}-\frac{V^4}{gV_1^2}$$

or $gh=V^2-(V^4/V_1^2)$ or $(V^4/V_1^2)=V^2-gh$

or $V_1^2=V^4/(V^2-gh)$ or $V_1=V^2/\sqrt{(V^2-gh)}$. Hence proved.

Ex. 25. Prove that when a shot is projected form a gun at any angle of elevation, the shot as seen from the point of projection will appear to descend past vertical target with uniform velocity.



(Fig. 59)

Solution. Let O, the point of projection of the shot, be taken as origin and horizontal and vertical lines (as shown in the figure) be taken as coordinate axes. Let u and α be the velocity and angle of projection respectively.

AB is the given vertical target. Let $OB=d$ (say).

Let the particle be at P after time t. PM is perpendicular from P to x-axis. Produce OP to meet the target in A

PM=vertical distance travelled by the shot in time t.

$$=u\sin\alpha . t-\tfrac{1}{2}gt^2 \qquad ...(i)$$

and OM=horizontal distance moved by the shot in time t.

$$=u\cos\alpha . t.$$

Also from similar Δ's OPM and OAB we get

$$\frac{AB}{OB}=\frac{PM}{OM} \text{ or } AB=\frac{PM}{OM}OB.$$

or $$AB=\left(\frac{u\sin\alpha . t-\frac{1}{2}gt^2}{u\cos\alpha . t}\right)\times d=\left(\tan\alpha-\frac{gt}{2u\cos\alpha}\right)d.$$

If (x, y) be the coordinates of A, then

$$x=OB=d \text{ and } y=AB=\left(\tan\alpha-\frac{gt}{u\cos\alpha}\right)d$$

$$\therefore \frac{dx}{dt}=0 \text{ and } \frac{dy}{dt}=-\frac{gd}{2u\cos\alpha}=\text{negative constant.}$$

$\therefore$ the shot as seen from O will appear to descend vertically downwards with uniform velocity.

Ex. 26. Obtain the equation of the path of projectile in the form $y=x \tan \alpha \left(1-\frac{x}{R}\right)$, where R is the horizontal range and α the angle of projection. *(Allahabad 79; Lucknow 80)*

Solution. We know that the equation of the path of the projectile is

$$y=x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad \ldots \text{(i)}$$

where u and α are the velocity and angle of projection of the projectile respectively.

Also if R be the horizontal range of the projectile, then we have

$$R=\frac{2u^2 \sin \alpha \cos \alpha}{g} \quad \ldots \text{(ii)}$$

...See § 6 (b) Pages 6—7

or $\frac{g}{u^2}=\frac{2 \sin \alpha \cos \alpha}{R}$ or $\frac{g}{u^2 \cos^2 \alpha}=\frac{2 \sin \alpha \cos \alpha}{R \cos^2 \alpha}$

or
$$\frac{g}{2u^2 \cos^2 \alpha}=\frac{\tan \alpha}{R} \quad \ldots \text{(iii)}$$

Substituting this value of $\frac{g}{2u^2 \cos^2 \alpha}$ from (iii) in (i) we get

$$y=x \tan \alpha - \frac{\tan \alpha}{R}\left(x^2\right) \quad \text{or} \quad y=x \tan \alpha \left(1-\frac{x}{R}\right),$$

which is the required form of the equation of the path of the projectile. This can also be written as

$$yR/(xR-x)=\tan \alpha. \qquad \textit{(Lucknow 80)}$$

MISCELLANEOUS EXERCISES ON PROJECTILES

Ex. 1. A particle is projected with velocity u at an elevation α. Prove that if $h<(u^2 \sin \alpha)/2g$, the particle will be again at a height h after a time $(2/g)\sqrt{(u^2 \sin^2 \alpha - 2gh)}$ after being at the same *height the first time.*

Ex. 2. A cricketer in a long field can judge a catch and secure it easily, if the height of the ball above the ground lies between k_1 and k_2. Prove that he must estimate his distance from the batsman within a length $R\left\{\left(1-\frac{k_2}{h}\right)^{1/2} \sim \left(1-\frac{k_1}{h}\right)^{1/2}\right\}$, where $2R$ is the horizontal range and h the greatest height of the ball. *(Lucknow 82)*

Ex. 3. State whether the following statement is true or false :—'The velocity of a projectile, at its maximum height in a parabolic path, is zero.'

Ans. False. (The vertical velocity is zero)

Ex. 4. A particle is projected so as just to clear a wall of height b, at a horizontal distance a, and to have a range c from the point of projection. *Show that the initial velocity is given by*

$$2u^2/g=[a^2(c-a)^2+b^2c^2]/ab(c-a).$$

Constrained Motion

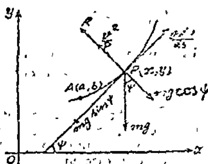
(On Smooth Plane Curves)

§ 1. Definition. If a particle is compelled to move along a given curve or surface (smooth or rough), then the motion of the particle is called the constrained motion. *(Meerut 93)*

***§ 2. Motion on a smooth curve in a vertical plane.** *(Kanpur 90; Rohilkhand III 92)*

A heavy particle is made to move on a smooth curve in a vertical lane, to discuss the motion.

Let P be the position of the particle moving on the plane curve at time t. Let A be a fixed point on the curve, from which arc is measured and particle is supposed to start. Let arc $AP=s$ and the tangent to the curve at P make an angle ψ with the x-axis, which is a horizontal line in the plane of the curve. Then y-axis is a vertical line in this plane, the axis being rectangular.



(Fig. 1)

∴ The forces acting on the particle at P are its weight mg acting vertically downwards and the normal reaction R between the particle and the plane curve acting in the direction of the inward drawn normal to the curve at P.

∴ The equations of motion in the tangential and inward drawn normal senses are

$$mv\frac{dv}{ds}=-mg\sin\psi \qquad \text{.....(i)}$$

and $$m(v^2/\rho)=R-mg\cos\psi, \qquad \text{...(ii)}$$

where v is the velocity of the particle at P and ρ is the radius of curvature of the curve at P.

Also we know from Differential Calculus, that if P be (x, y) then $\sin\psi=dy/ds$, so from (i) we get

$$v\frac{dv}{ds}=-g\frac{dy}{ds} \quad \text{or} \quad 2v\,dv=-2g\,dy \qquad \text{(Note)}$$

Integrating, $v^2=-2gy+C$, where C is constant of integration. If A be (a, b) and u be the velocity of particle at A, then

$$u^2=-2gb+C \quad \text{or} \quad C=u^2+2gb$$

$\therefore$ We have $\quad v^2=-2gy+u^2+2gb$

or
$$v^2=u^2-2g(y-b)=u^2-2gh, \qquad \text{...(iii)}$$

where $h=y-b=$ height of P above A.

The velocity v of the particle at P is given by (iii) and from (ii) we can find the value of the normal reaction at P.

When the particle leaves the curve, its contact with the curve is broken and consequently at that point the normal reaction R between the particle and the curve vanishes. (Remember)

MOTION IN A VERTICAL CIRCLE

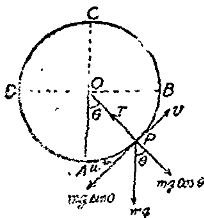
§ 3. . A heavy particle is tied to one end of a light inextensible string whose other end is attached to a fixed point. It is projected with a given velocity from its vertical position of equilibrium; to determine the subsequent motion. *(Avadh 87; Bundelkhand 87; Gorakhpur 90, 88; Lucknow 92, 89; Rohilkhand 85)*

Let O be the fixed point and OA the string in the vertical position of equilibrium. Let the particle be projected horizontally from A with velocity u.

Let the particle be at P after time t, such that $\angle POA=\theta$.

The forces acting on the particle in this position are :—
(i) weight of the particle mg (say) acting vertically downwards and
(ii) tension T in the string.

$\therefore$ The equations of motion in the tangential and normal (inward drawn) senses are



(Fig. 2)

$$mv\frac{dv}{ds}=-mg\sin\theta \qquad \text{...(i)}$$

or
$$m\frac{v^2}{\rho}=T-mg\cos\theta. \qquad \text{...(ii)}$$

Also if the arc $AP=s$, then $s=a\theta$, where a is the length of the string.

$\therefore$ From (ii), $\dfrac{d^2s}{dt^2}=-g\sin\theta, \quad \because \quad v\dfrac{dv}{ds}=\dfrac{d^2s}{dt^2}$

or $$a\frac{d^2\theta}{dt^2}=-g\sin\theta,\quad \because\ s=a\theta.$$

Multiplying both sides by $2\,(d\theta/dt)$, we get

$$2a\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}=-2g\sin\theta\frac{d\theta}{dt}$$

Integrating $a\,(d\theta/dt)^2=2g\cos\theta+C$, where C is constant of integration

or $[a\,(d\theta/dt)]^2=2ag\cos\theta+C'$, where $C'=aC$.

Initially $a\,(d\theta/dt)$=linear velocity=u and $\theta=0$.

$\therefore$ $u^2=2ga+C'$ or $C'=u^2-2ga$

$\therefore$ $[a\,(d\theta/dt)]^2=2ga\cos\theta+u^2-2ga$

or $$v^2=u^2+2ag\cos\theta-2ag, \quad \text{...(ii)}$$

since linear velocity $v=d\theta/dt$.

$\therefore$ from (ii), $(m/a)\,(u^2+2ga\cos\theta-2ga)=T-mg\cos\theta$, $\because\ \rho=a$

or $$T=(m/a)\,(u^2+3ga\cos\theta-2ga) \quad \text{...(iii)}$$

Let velocity vanish when $\theta=\theta_1$ (say), then from (iii)

$$0=u^2+2ga\cos\theta_1-2ga \text{ or } \cos\theta_1=\frac{2ga-u^2}{2ga} \quad \text{...(iv)}$$

Let the height of this point, where velocity vanishes, be h_1 above the lowest point A.

Then $h_1=a-a\cos\theta_1$ **(Note)**

$$=a-a\left(\frac{2ga-u^2}{2ga}\right)=\frac{u^2}{2g} \quad \text{...(vi)}$$

Let T vanish when $\theta=\theta_2$ (say).

Then from (iv), $0=(m/a)\,[u^2+3ga\cos\theta_2-2ga]$

or $$\cos\theta_2=(2ga-u^2)/3ag \quad \text{...(vii)}$$

Let the height of the point, where T vanishes be h_2 above the lowest point A.

Then $$h_2=a-a\cos\theta_2=a-a\left(\frac{2ga-u^2}{3ga}\right)=\frac{u^2+ag}{3g}$$

or $$h_2=(u^2+ag)/3g \quad \text{...(viii)}$$

Case I. If velocity v vanishes before tension T.

In this case $h_1<h_2$

or $(u^2/2g)<\{(u^2+ag)/3g\}$, from (vi) and (viii)

or $3u^2<2u^2+2ag$ or $u^2<2ag$ is the condition for velocity vanishing before tension. In the case the string remaining taut, so if $u^2<2ag$ the particle will trace its path back *i.e.*, will oscillate. *(Gorakhpur 88)*

Also $\cos\theta<1$ or θ will be acute and the particle will not rise upto the level of O.

Case II. If velocity v and tension T vanish together.

In this case $h_1=h_2$ or $u^2/2g=(u^2+ag)/3g$, from (vi) and (viii)

or $$u^2=2ag.$$

$\therefore$ from (v) and (vi) $\cos\theta_1=0=\cos\theta_2$

or $$\theta_1=\tfrac{1}{2}\pi=\theta_2.$$

i.e. the particle will rise upto the level of the fixed point O and oscillate in the semi-circle BAD if $u^2=2ag$. *(Gorakhpur 92)*

Case III. Condition for tracing complete circles.

(Bnadelkhand 87)

At the highest point C, $\theta=\pi$.

$\therefore$ from (iv) at C, $T=\dfrac{m}{a}\left[u^2-3ag-2ag\right]=\dfrac{m}{a}\left[u^2-5ag\right]$

and from (iii) at C, $v^2=u^2-2ga-2ga=u^2-4ag.$

$\therefore$ If $u^2>5ag$, tension as well as velocity will not vanish even at the highest point and particle will go on tracing circles. *(Gorakhpur 90)*

If however $u^2=5ag$, then velocity will not vanish at the highest point C whereas the tension will vanish at C. Hence in this case the string will be momertarily slack but the particle will go on tracing circles.

i. e. the least velocity with which the particle must start from the lowest point so at to describe complete circles is given by

$$u^2=5ag.$$ *(Kumaun 88)*

Case IV. Tension vanishing before velocity.

In this case $h_2<h_1$ or $(u^2+ag)/3g<u^2/2g$, from (vi) and (viii)

or $$2ag+2u^2<3u^2 \quad\text{or}\quad u^2>2ag.$$

If $u^2>2ag$ but $<5ag$. (*See* case *III* above), the tension vanishes *i.e.* the string becomes slack but the velocity being not zero, the particle will leave the circular path and trace out a parabolic path.

§ 4. *Discuss the motion of a particle projected from the lowest point with velocity u and moving along the inside of a smooth vertical circle.* *(Avadh 90, Purvanchal 90)*

Sol. The discussion is exactly the same as in § 3 above replacing tension T in the string by reaction R between the particle and the circle.

Note. *What difference will it make, if there was a thin smooth tube rather than the rim ?*

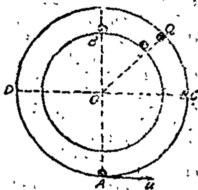
In this case the particle will start moving from the lowest point A on the outer side within the tube, with velocity u. (See figure 3 Page 5).

In the case $u^2<2ag$. the particle oscillate about A on the outer side within the tube.

In the case $u^2=2ag$, the particle will oscillate in the semicircle CAD on the outer side within the tube.

In the case $2ag < u^2 < 5ag$, the particle will leave contact with the tube at Q (say) but since it is moving within the tube so it cannot come out of it and hence will change side of the tube and will start moving on the inner side within the tube (see adjoining Fig. 3).

In the case $u^2 \geqslant 5ag$, as above the particle will move in circle partly on the outer side and partly on the inner side of the tube (See Fig. 3)



(Fig. 3)

Solved Examples on § 3 and § 4.

*Ex. 1. A heavy particle hangs from a point O, by string of length a. It is projected horizontally with a velocity v such that $v^2=(2+\sqrt{3})\ ag$; show that the string becomes slack when it has described an angle $\cos^{-1}(-1/\sqrt{3})$.

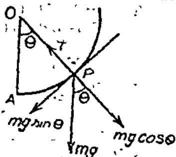
Sol. Let the particl
and $\angle AOP=\theta$, where A is

The forces acting on t ... (i) its weight mg acting vertically downwards and (ii) the tension T in the string acting towards O.

$\therefore$ The equation of motion in the tangential and inward drawn normal senses are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad \text{...(i)}$$

and $$m\frac{v^2}{\rho}=T-mg\cos\theta \quad \text{...(ii)}$$



Also $s=a\theta$, so that $\dfrac{d^2s}{dt^2}=a\dfrac{d^2\theta}{dt^2}$

$\therefore$ from (i), $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Hence $\{a\,(d\theta/dt)\}^2=2ag\cos\theta+\sqrt{3}ag$ ...(iii)

$\therefore$ from (ii) and (iii) we get

or

The st ...(iv)

$\therefore$ From (iv) we get $3g\cos\theta+\sqrt{3}g=0$ or $\cos\theta=-1/\sqrt{3}$

or $\theta=\cos^{-1}(-1/\sqrt{3})$. Hence proved.

*Ex. 2. A particle at the end of a string of length l, the upper end of which is fixed is projected horizontally with a velocity which is k times that due to a fall l. If the particle leaves the circular path show that it does so at the height $(l/3)(1+2k^2)$ above the lowest point. Also find the greatest height that the particle attains during the entire motion. *(Lucknow 91)*

Sol. OA is the equilibrium position of the string with end O fixed. Let the particle be projected horizontally from A with a velocity u, then we have

$u=k\sqrt{(2gl)}$ (given)

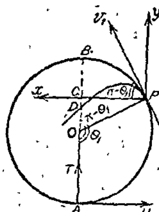
or $u^2=2k\,gl$ ...(i)

Proceeding as in § 3 Page 2 we can prove that velocity v and tension T at any point at an angular distance θ from A are given by

$v^2=u^2+2gl\cos\theta-2gl$ ...(ii)

and $T=(m/l)[u^2+3gl\cos\theta-2gl]$ ....(iii)

Let the particle leave the circular path at P, such that $\angle AOP=\theta_1$ and velocity at P of the particle $=v_1$ (say).



(Fig. 5)

Then at P, $T=0$, $\theta=\theta_1$

$\therefore$ from (i) and (iii) we have $0=(m/l)[2k^2gl+3gl\cos\theta_1-2gl]$

or $\cos\theta_1=\frac{2}{3}(1-k^2)$. ...(iv)

From P draw PC perpendicular to the vertical diameter AB.

Then the height of P above $A=AC=AO+OC$

$=l+OP\cos(\pi-\theta_1)=l-l\cos\theta_1$

$=l-\frac{2}{3}l(1-k^2)$, from (iv)

$=\frac{1}{3}l(1+2k^2)$.

Also from (ii) at P, we have

$v_1^2=2k^2gl+2gl\cos\theta_1-2gl$

$=2k^2gl+2gl.\frac{2}{3}(1-k^2)-2gl=\frac{2}{3}gl\,(k^2-1)$ ...(vi)

Beyond P, the particle leaves the circular path and traces out a parabolic path with v_1 and $(\pi-\theta_1)$ as velocity and angle of projection.

$\therefore$ The maximum height attained by the particle above P

$$=\frac{v_1^2\sin^2\theta_1}{2g}=\frac{v_1^2(1-\cos^2\theta_1)}{2g}$$

$$=\frac{2gl\,(k^2-1)}{3\times 2g}\left\{1-\frac{4(1-k^2)^2}{9}\right\}, \text{ from (iv) and (vi)}$$

$$=(1/27)\,l\,(k^2-1)\,[9-4(1-2k^2+k^4)]$$

$$=(1/27)\,l\,(k^2-1)\,(5+8k^2-4k^4)$$

$\therefore$ Required max. height attained by the particle above A

$=AK+$the max. height attained above P

$=\frac{1}{3}l\,(1+2k^2)+(1/27)\,l\,(k^2-1)\,(5+8k^2-4k^4)$

$=(1/27)\,\{4+15k^2-12k^4-4k^6\}=(1/27)\,l\,(1+2k^2)^2\,(4-k^2)$.

Ex. 3. A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts to it a velocity $2\sqrt{(gl)}$, prove that the cord becomes slack when the particle has risen to height $\frac{2}{3}$ above the fixed point, and find the height of the highest point of the parabola subsequently described.

Sol. OA is the equilibrium position of the string with end O fixed. Let the particle be projected horizontally from A with a velocity u, where $u=2\sqrt{(gl)}$ (given)

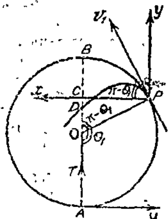
or $u^2=4gl$ ...(i)

Proceeding as in § 3 Page 2 we can prove that velocity v and tension T at any point at an angular distance θ from A are given by

$v^2=u^2+2gl\cos\theta-2gl$ ...(ii)

And $T=(m/l)\,[u^2+3gl\cos\theta-2gl]$ ...(iii)

Let the cord become slack at P, such that $\angle AOP=\theta_1$ and velocity at P of the particle $=v_1$ (say).



(Fig. 6)

$\because$ At P, $T=0$, $\theta=\theta_1$

$\therefore$ From (i) and (ii) we have

$0=(m/l)\,[4gl+3gl\cos\theta_1-2gl]$ or $\cos\theta_1=-\frac{2}{3}$ ...(iv)

From P draw PC perpendicular to the vertical diameter AB.

$\therefore$ The height of P above the fixed point $O=OC=l\cos(\pi-\theta_1)$
$=-l\cos\theta_1=\frac{2}{3}l$, from (iv)

Also from (ii) and (i) at P, we have

$v_1^2=4gl+2gl\cos\theta_1-2gl=4gl+2gl\,(-\frac{2}{3})-2gl=\frac{2}{3}gl$ ...(v)

Beyond P, the particle leaves the circular path and traces out a parabolic path with v_1 and $(\pi-\theta_1)$ as velocity and angle of projection respectively.

$\therefore$ The maximum height attained by the particle above P

$=\dfrac{v_1^2\sin^2\theta_1}{2g}=\dfrac{2gl}{3\times 2g}\,(1-\cos^2\theta_1)$, from (v)

$=\frac{1}{3}l\,[1-(4/9)]$, from (iv)

$=(5/27)\,l$.

$\therefore$ Required height of the highest point of the parabolic path above the fixed point $O=OC+$the max. height attained above P

$=\dfrac{2l}{3}+\dfrac{5l}{27}=\dfrac{23l}{27}$ **Ans.**

Ex. 4. A particle projected along the inner side of a smooth vertical circle of rad[illegible]
Show that if $2ga < u^2$ [illegible]
arriving at the highest [illegible]
rectum is $2(u^2-2ga)^3/(27g^3a^2)$.

Sol. In this case there will be reaction R between the particle and the circle instead of tension T in § 3 Page 2.

$\therefore$ Here we can prove as in § 3 Page 2 that velocity v and reaction R at any point of the given circle are given by

$$v^2=u^2+2ga\cos\theta-2ga \qquad \text{...(i)}$$

and
$$R=(m/a)[u^2+3ga\cos\theta-2ga]. \qquad \text{...(ii)}$$

The particle will leave the circle when $R=0$. Let $\theta=\theta_1$ at the point where the particle leaves the circle.

Then from (ii), $0=(m/a)[u^2+3ga\cos\theta_1-2ga]$

or
$$\cos\theta_1=-[(u^2-2ga)/3ga] \qquad \text{...(iii)}$$

Since $2ga < u^2 < 5ga$ so $\cos\theta_1=-$ve and greater than -1 *i.e.* θ_1 is obtuse and less than π.

i.e. the particle leaves the circle before arriving at the highest point (where $\theta=\pi$) but above the level of the centre of the circle (where $\theta=\frac{1}{2}\pi$).

Let $v=v_1$ at the point where $R=0$ *i.e.*, at $\theta=\theta_1$.

Then from (i), $v_1^2=u^2+2ga\cos\theta_1-2ga$

$$=u^2+[2ga(2ga-u^2)/3ga]-2ga, \text{ from (iii)}$$

or
$$v_1^2=\tfrac{1}{3}(u^2-ga) \qquad \text{...(iv)}$$

Also the direction of v_1 makes an angle $(\pi-\theta_1)$ with the horizontal (as in last example).

Beyond this point (where $R=0$) the particle traces out a parabolic path with v_1 and $(\pi-\theta_1)$ as velocity and angle of projection.

$\therefore$ Latus rectum of the parabola

$$=\frac{2u^2\cos^2\alpha}{g}=\frac{2v_1^2\cos^2(\pi-\theta_1)}{g}$$

$$=\frac{2v_1^2\cos^2\theta_1}{g}=\frac{2(u^2-2ga)(u^2-2ga)^2}{g\times 3\times 9g^2a^2}, \text{ from (iii) and (iv)}$$

$$=2(u^2-2ga)^3/(27g^3a^2).$$

Hence proved.

****Ex. 5.** A heavy particle hangs by an inextensible string of length a from a fixed point and is then projected horizontally with a velocity $\sqrt{(2gh)}$. If $(5/2)\,a > h > a$, prove that the circular motion ceases when the greatest height ever reached by the particle above the point of projection is $(4a-h)(a+2h)^2/(27a^2)$. *(Meerut II' 91)*

Sol. We can prove as in § 3 Page 2 that velocity v and tension T at any point P, whose angular distance from the lowest point A is θ, are given by
$$v^2=u^2+2ag\cos\theta-2ga \qquad \text{...(i)}$$

and
$$T=(m/a)[u^2+3ga\cos\theta-2ga]$$

When the circular motion ceases, the string becomes slack therefore $T=0$.

Hence from (ii) if $\theta=\theta_1$ when $T=0$, we get

$u^2+3ag\cos\theta_1-2ag=0$

or $\cos\theta_1=\dfrac{2ag-u^2}{3ag}=\dfrac{2ag-2gh}{3ag}$,

$\because\ u^2=2gh$ (given)

or $\cos\theta_1=(2a-2h)/3a$ ...(iii)

And if v_1 be the velocity of the particle when circular motion ceases

$v_1^2=2gh+2ag\cos\theta_1-2ag$,

$\because\ u^2=2gh$ (given)

or $v_1^2=2gh+2ag\,[(2a-2h)/(3a)]-2ag$, from (iii)

$=\frac{2}{3}g\,(h-a)$.

Also height of B above the point of projection A

$=AO+ON$

$=a+a\cos(\pi-\theta_1)$ ...(see figure)

$=a-a\cos\theta_1=a+[2a\,(h-a)/3a]$, from (iii)

$=\frac{1}{3}(a+2h)$.

(Fig. 7)

(Meerut III' 91)

When the circular motion ceases at B, the particle begins to describe parabolic path with a velocity v_1 making an angle $(\pi-\theta_1)$ with the horizontal line through B_1

$\therefore$ The maximum height reached by the particle above B

$$=\frac{\text{"}u^2\sin^2\alpha\text{"}}{2g}=\frac{v_1^2\sin^2(\pi-\theta_1)}{2g}=\frac{v_1^2\sin^2\theta_1}{2g}$$

$$=\frac{2g\,(h-a)}{3\times 2g}\,(1-\cos^2\theta_1),\ \text{from (iv)}$$

$$=\frac{(h-a)}{3}\left[1-\frac{4\,(h-a)^2}{9a^2}\right]=\frac{(h-a)\,(5a+2h)\,(a+2h)}{27a^2}$$

$\therefore$ The required maximum height reached by the particle above A=height of B above A+max. height reached above B

$$=\frac{(a+2h)}{3}+\frac{(h-a)\,(5a-2h)\,(a+2h)}{27a^2}$$

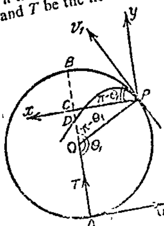
$$=\frac{(a+2h)}{27a^2}\,[9a^2+(h-a)\,(5a-2h)]$$

$$=\frac{(a+2h)}{27a^2}\,[4a^2+7ah-2h^2]=\frac{(a+2h)}{27a^2}\,[(4a-h)\,(a+2h)]$$

$=(a+2h)^2\,(4a-h)/(27a^2)$. Hence proved.

***Ex. 6 (a). A particle is projected, along the inside of a smooth fixed sphere, from the lowest point, with a velocity equal to that due to falling freely down the vertical diameter of the sphere. Show that the particle will leave the sphere and afterwards pass vertically over the point of projection at a distance equal to (25/32) of the diameter.**

Sol. Let a be the radius of the sphere, u the velocity of the projection from the lowest point A of the sphere and T be the normal reaction, at the point A, between the particle and the sphere.



(Fig. 8)

Then velocity of projection u
$$=\sqrt{(2g\,2a)} \qquad \ldots\text{(i)}$$
or $u^2=4ga$, which being $< 5ga$ and $> 2ga$ the particle will leave the circular path above the horizontal level of the centre O. Let it leave the sphere at P, whose angular distance from A is θ_1.

Also we can prove that velocity v and reaction R at any point, whose angular distance from the lowest point A is θ, are given by
$$v^2=u^2+2ag\cos\theta-2ga \qquad \ldots\text{(ii)}$$
and $$R=(m/a)\left[u^2+3ga\cos\theta-2ag\right] \qquad \ldots\text{(iii)}$$
At P, the circular motion ceases and therefore $R=0$.

Hence from (iii) at P, $\theta=\theta_1$ and $R=0$ gives
$$0=u^2+3ag\cos\theta_1-2ag \quad \text{or} \quad \cos\theta_1=(2ag-u^2)/(3ag)$$
or $\cos\theta_1=(2ag-4ag)/3ag$, from (i).

or $\cos\theta_1=-\frac{2}{3}$ or $\tan\theta_1=-\frac{1}{2}\sqrt{5}$. $\qquad \ldots$(iv)

Also at P let velocity be v_1, then from (ii)
$$v_1^2=4ag+2ag\left(-\tfrac{2}{3}\right)-4ag, \text{ from (i) and (iv)}$$
or $$v_1^2=\tfrac{2}{3}ag.$$

The parabolic path begins from P with velocity of projection v_1 and angle of projection $(\pi-\theta_1)$, given by (iv) and (v).

Referred to P as origin, horizontal line PC as x-axis and the vertical line through P as y-axis, the equation of the parabolic path is
$$y=x\tan(\pi-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(\pi-\theta_1)}$$
or $$y=-x\tan\theta_1-\frac{gx^2}{2v_1^2\cos^2\theta_1}$$
$$=-x\left(-\frac{\sqrt{5}}{2}\right)-\frac{gx^2}{2\left(\frac{2}{3}ag\right)(4/9)}, \text{ from (iv) and (v)}$$
or $$y=\frac{x\sqrt{5}}{2}-\frac{27x^2}{16a}. \qquad \ldots\text{(vi)}$$
Also $PC=OP\sin(\pi-\theta_1)=a\sin\theta_1=a.\frac{1}{3}\sqrt{5}$ $\qquad$ ...from (iv) $\sin\theta_1=\frac{1}{3}\sqrt{5}$.

$\therefore$ The equation of the diameter AB is $x=\frac{1}{3}a\sqrt{5}$.

Let the parabolic path cut the diameter AB at D, such that $CD=y_1$. (See figure)

Then the coordinates of D are $(\frac{1}{4}a\sqrt{5}, -y_1)$ and as D lies on the parabolic path, $\therefore$ from (vi) we get

$$-y_1 = \frac{\sqrt{5}a}{3} \cdot \frac{\sqrt{5}}{2} - \frac{27}{16a} \cdot \frac{5a^2}{9} = \frac{5a}{6} - \frac{15a}{16}$$

or $\qquad y_1 = (5/48)\, a = CD.$

Also $OC = OP \cos(\pi - \theta_1) = -a \cos \theta_1 = (2/3)\, a$, from (iv)

$\therefore \quad OD = OC - CD = (2/3)a - (5/48)a = (27/48)a = (9/16)\, a$

$\therefore$ The required height $= AD = AO + OD = a + (9/16)a = (25/16)a$
$= (25/32)\,(2a) = (25/32) \times$ diameter.

Hence proved.

***Ex. 6 (b). A particle is projected from the lowest point inside a smooth sphere of radius a with a velocity $2\sqrt{(ag)}$. Find the point at which it will leave the sphere and equation to subsequent path of the particle.**

Sol. Proceed as in Ex. 6 (a) above. The particle leaves the path at P, whose angular distance θ_1 from the lowest point A is given by the result (iv) of Ex. 6 (a) above. Also the required equation of the path of the particle beyond P is given by result (vi) of Ex. 6 (a) above.

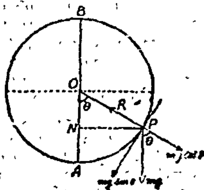
Ex. 7. Show that a particle projected with velocity $\sqrt{(2ag)}$ from the lowest point of a vertical circle of radius a and moving inside it will just reach the end of the horizontal diameter; while if projected with velocity $\sqrt{(5ag)}$, it will just reach the highest point. Prove that the reaction at any point in the first case is proportional to the depth below the horizontal diameter and in the second case to the depth below the highest point.

Sol. Let after time t, the particle be at P, whose angular distance from the lowest point A be θ. If v and R be the velocity of the particle and reaction between the circle and the particle at P, then we can show that

$$v^2 = u^2 + 2ag \cos\theta - 2ag \quad \ldots(i)$$

and $$R = \frac{m}{a}\left[u^2 + 3ag \cos\theta - 2ag\right], \quad \ldots(ii)$$

where u is the velocity of projection.



(Fig. 9)

$\therefore$ From (i) when $v=0$, we have $\quad 0 = u^2 + 2ag \cos\theta - 2ag$

or $\quad 0 = 2ag + 2ag \cos\theta - 2ag, \qquad \because u = \sqrt{(2ag)}$

or $\qquad \cos\theta = 0$ or $\theta = \frac{1}{2}\pi$

i.e. the particle will just reach the end of the horizontal diameter.

Again if $u=\sqrt{(5ag)}$ we have from (ii) when $R=0$

$0=(m/a)[5ag+3ag\cos\theta-2ag]$ or $\cos\theta=-1$ or $\theta=\pi$

i.e. the particle will just reach the highest point in this case.

Again if $u=\sqrt{(2ag)}$, we have from (ii)

$$R=(m/a)[2ag+3ag\cos\theta-2ag]=3(m/a)g(a\cos\theta)$$

$$=3(m/a)g.(ON), \quad \because ON=OP\cos\theta=a\cos\theta$$

$\therefore$ R varies as ON *i.e.* the depth below the horizontal diameter.

Also if $u=\sqrt{(5ag)}$, we have from (ii).

$$R=(m/a)[5ag+3ag\cos\theta-2ag]=(m/a)[3ag(1+\cos\theta)]$$

$$=3(m/a)g[a+a\cos\theta]=3(m/a)g[BN], \quad \because BN=BO+ON=a+a\cos\theta.$$

$\therefore$ R varies as BN *i.e.* the depth below the highest point B.

Ex. 8. A heavy particle of weight W, attached to a fixed point by a light inextensible string describes a circle in a vertical plane. The tension in the string has the values mW and nW respectively when the particle is at the highest and lowest points in the path. Show that $n=m+6$. *(Lucknow 90, 86)*

Sol. Let a be the length of the string. We can show as in § 3 Page 2, that tension T at a point, whose angular distance from the lowest point is θ, is given by

$$T=(M/a)[u^2+3ag\cos\theta-2ag], \quad \text{...(i)}$$

where M is the mass of the particle and as such $M=W/g$.

$\therefore$ From (i) we get $T=(W/ag)[u^2+3ag\cos\theta-2ag]$. ...(ii)

At the highest point $T=mW$ (given) and $\theta=\pi$.

$\therefore$ From (ii), $mW=(W/ag)[u^2-3ag-2ag]$

or $mag=u^2-5ag$. ...(iii)

At the lowest point, $T=nW$ (given) and $\theta=0$.

$\therefore$ From (ii), $nW=(W/g)[u^2+3ag-2ag]$

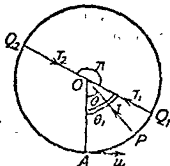
or $nag=u^2+ag$ ...(iv)

Subtracting (iv) from (iii), we get $mag-nag=-5ag-ag$

or $m-n=-6$; or $m+6=n$. Hence proved.

***Ex. 9 (a). A heavy particle hanging vertically from a point by a light inextensible string of length l is started so as to make a complete revolution in a vertical plane. Prove that the sum of the tensions at the ends of any diameter is constant.**

Sol. If m be the mass and u the velocity of projection of the particle than we can show as in § 3 Page 2 that tension T in the string at a point P whose angular distance is θ from the lowest position is given by $T=(m/l)(u^2+3lg\cos\theta-2lg)$ ...



(Fig. 10)

Take a diameter $Q_1 O Q_2$ of the circle, which the particle makes during its motion making an angle θ_1 with vertical diameter. Then at one end Q_1 of this diameter $\theta=\theta_1$ and at the other end Q_2 we have $\theta=\pi+\theta_1$ (Note)

Let T_1 and T_2 be the tensions in the string at these two ends Q_1 and Q_2.

Then from (i),

$T_1=(m/l)\ [u^2+3lg\cos\theta_1-2lg]$ ...(ii)

and $T_2=(m/l)\ [u^2+3lg\cos(\pi+\theta_1)-2lg]$

or $T_2=(m/l)\ [u^2-3lg\cos\theta_1-2lg]$ ...(iii)

Adding (ii) and (iii) we have

$T_1+T_2=(m/l)\ [2u^2-4lg]=$ constant,

as it is independent of θ. Hence proved.

Ex. 9 (b). A heavy particle is constrained to move in a vertical circle of radius r. Show that the sum of the reactions at the end of a diameter is constant.

Sol. Proceed as in Ex. 9 (a) above. Here, replace tensions T, T_1 and T_2 of Ex. 9 (a) above by normal reactions R, R_1 and R_2 respectively and prove $R_1+R_2=$constant.

***Ex. 9 (c). A heavy particle of mass m is suspended by a string of length r and hangs vertically. It is then projected with velocity u so that it just makes a complete revolution, show that $u^2=5\ gr$ and tension in the lowest position is $6\ mg$.** *(Kumaun 88)*

Sol. For the first part see § 3 Case III Page 4.

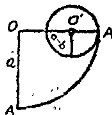
Also we know tension $T=[m/r]\ [u^2+3rg\cos\theta-2rg]$

...See § 3 (iii) Page 3.

At the lowest point $\theta=0$ and here $u^2=5rg$ (proved above), so required tension at the lowest point

$=(m/r)\ [5rg+3rg\cos 0-2rg]=6mg.$ Hence proved.

****Ex. 10. A particle is hanging from a fixed point O by means of a string of length a. There is a small smooth nail O′ in the same horizontal line with O at a distance $b\ (< a)$ from O. Find the minimum velocity with which the particle should be projected from its lowest position in order that it may make a complete revolution round the nail without the string becoming slack.**



(Fig. 11)

Sol. Initially the string is in the position OA with the particle at A. Let us suppose that [illegible] en OA turns [illegible] position OA'. [illegible] the velocity of the particle at A so that it may completely go round O' in a circle of radius $(a-b)$. (In the figure 11, change A to A' on the horizontal line OO' produced).

We know that if a particle attached to a string of length l is projected with velocity $\sqrt{(5gl)}$, it will move in a complete circle and its velocity at any point is given by

$$v^2=u^2+2gl\cos\theta-2gl, \quad \because u^2=5gl$$
$$=5gl+2gl\cos\theta-2gl,$$

or $v^2=3gl+2gl\cos\theta$.

$\therefore$ At $\theta=\frac{1}{2}\pi$, we have $v^2=3gl$ or $v=\sqrt{(3gl)}$

$\therefore$ If the particle has a velocity $\sqrt{(3gl)}$ when it is at $\theta=\frac{1}{2}\pi$, it will go round in a complete circle.

Hence for our problem, the velocity of the particle at A' should be $\sqrt{\{3g(a-b)\}}$, so that it goes round O' in a complete circle of radius $(a-b)$.

Now our problem is what should be the velocity of the particle at A, so that it may acquire a velocity $\sqrt{\{3g(a-b)\}}$ at A' i.e. at $\theta=\frac{1}{2}\pi$.

Hence from "$v^2=u^2+2ga\cos\theta-2ag$", we have at A'

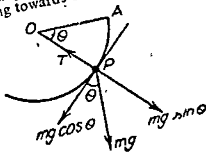
$3g(a-b)=u^2+2ga\cos\frac{1}{2}\pi-2ga$ or $u^2=(5a-3b)g$.

Hence required velocity $=\sqrt{\{(5a-3b)g\}}$. **Ans.**

Ex. 11 (a). A particle attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with horizontal, the resultant acceleration is $g\sqrt{(1+3\sin^2\theta)}$. *(Rohilkhand III 90)*

(b). If the particle is let fall from a point in the horizontal line through O at a distance $l\cos\theta$ from O, show that its velocity when it is vertically below O is $\sqrt{[2gl(1-\sin^3\theta)]}$.

Sol. (a) The particle starts from A, where $OA=l$ and A is in the horizontal level of O. Let P be the position of the particle after time t and θ be the angle which the string then makes with the horizontal line through O. The forces acting on the particle are its weight mg and the tension T of the string towards O.



(Fig. 12)

$\therefore$ The equations of motion in the tangential and the inward drawn normal senses are

$$m\frac{d^2s}{dt^2}=mg\cos\theta \quad \text{...(i)}$$

and $$m\frac{v^2}{\rho}=T-mg\sin\theta \quad \text{...(ii)}$$

Also arc $AP=s=l\theta$

or $$\frac{d^2s}{dt^2}=l\frac{d^2\theta}{dt^2} \text{ and } \rho=l \quad \text{...(ii)}$$

$\therefore$ From (i), $l\frac{d^2\theta}{dt^2}=g\cos\theta$

Multiplying both sides by $2l(d\theta/dt)$ and integrating we get

$[l(d\theta/dt)]^2=2lg\sin\theta+C$, where C is constant of integration.

Initially when $\theta=0$, $l(d\theta/dt)=0$, $\therefore C=0$

Hence $[l(d\theta/dt)]^2=v^2=2lg.\sin\theta \quad \because v=l(d\theta/dt)$

Substituting this value of v^2 in (ii), we have

$$m \cdot \frac{2lg \sin \theta}{l} = T - mg \sin \theta$$

or $\quad T = 2mg \sin \theta + mg \sin \theta = 3\, mg \sin \theta$, if T is to be calculated.

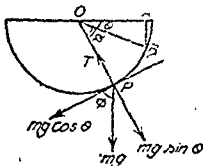
$\therefore$ From (ii), normal acceleration $= v^2/\rho = 2g \sin \theta$ ...(iv)

$\therefore$ Resultant acceleration $= \sqrt{\left\{\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right\}}$

$= \sqrt{\{(g \cos \theta)^2 + (2g \sin \theta)^2\}}$, from (i) and (iv)

$= g\sqrt{(1 + 3 \sin^2 \theta)}$.

(b) Let the particle fall from A where $OA = l \cos \theta$. The particle falls freely from A to B, at B the string becomes taut and the motion starts along a vertical circle. Also as $OA = l \cos \theta$ and $OB = l$, $\therefore \angle BOA = \theta$.



(Fig. 15)

Let P be the position of the particle after time t of its start from B and ϕ be the angle which the string makes at P with the horizontal *i.e.* $\angle POA = \phi$. The forces acting on the particle are its weight mg and the tension T of the string towards O.

$\therefore$ The equation of motion in the tangential sense is

$$m \frac{d^2s}{dt^2} = mg \cos \phi \quad \text{...(i)}$$

Also arc $AP = s = l\phi$, $\therefore \frac{d^2s}{dt^2} = l \frac{d^2\phi}{dt^2}$

$\therefore$ from (i) we get $l \frac{d^2\phi}{dt^2} = g \cos \phi$

Multiplying both sides by $2l\,(d\phi/dt)$ and integrating, we get

$\left(l \frac{d\phi}{dt}\right)^2 = 2lg \sin \phi + C$, where C is constant of integration.

Initially *i.e.* at B, $\phi = \theta$ and $l\,(d\phi/dt)$ = resolved part of velocity of the particle at B along the tangent to the curve at B (Note) $= \sqrt{(2g\, AB)} \cos \theta$, where $\sqrt{(2g\, AB)}$ is the velocity acquired by the particle after falling freely from A to B

$= \sqrt{(2gl \sin \theta)} . \cos \theta$, since $AB = l \sin \theta$

$\therefore \quad [\sqrt{(2gl \sin \theta)} . \cos \theta]^2 = 2lg \sin \theta + C$, at B

or $\quad 2gl \sin \theta \cos^2 \theta = 2lg \sin \theta + C$

or $\quad C = -2gl \sin \theta + 2gl \sin \theta \cos^2 \theta = -2gl \sin \theta\,(1 - \cos^2 \theta)$

$= -2gl \sin^3 \theta$

$\therefore \quad [l\,(d\phi/dt)]^2 = 2lg \sin \phi - 2gl \sin^3 \theta$ ...(ii)

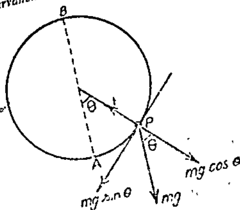
At the lowest point $\phi = \frac{1}{2}\pi$ and let velocity $l\,(d\phi/dt) = V$ (say)

Then from (ii) we get $V^2 = 2lg - 2gl \sin^3 \theta = 2gl\,(1 - \sin^3 \theta)$

$\therefore$ The required velocity of the particle when it is vertically below $O = V = \sqrt{[2gl(1-\sin^3\theta)]}$. Hence proved.

****Ex. 12.** A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that reaction between the particle and the wire is zero after time $\sqrt{(a/g)}\log(\sqrt{5}+\sqrt{6})$. *(Agra 92, 86; Bundelkhand 92; Meerut III' 90; Purvanchal 89; Rohilkhand III' 92, 90)*

Sol. Let O be the centre of the circular wire and A be the lowest poit. Let P be any position of the particle when OP makes an angle θ with OA.



(Fig. 14)

Then we can show as in § 3 Page 2 that the velocity v and reaction R at P are given by

$$v^2 = u^2 + 2ag\cos\theta - 2ag \quad \text{...(i)}$$

$$R = (m/a)\,[u^2 + 3ag\cos\theta - 2ag] \quad \text{...(ii)}$$

At the height point B, $\theta = \pi$ and velocity $v = 0$ (given)

$\therefore$ From (i),

$$0 = u^2 + 2ag\cos\pi - 2ag \quad \text{or} \quad u^2 = 4ag \quad \text{...(iii)}$$

Let $\theta = \theta_1$ when reaction R vanishes, then from (ii) we have

$$0 = (m/a)\,[u^2 + 3ag\cos\theta_1 - 2ag]$$

or $\cos\theta_1 = \dfrac{2ag - u^2}{3ag} = \dfrac{2ag - 4ag}{3ag}$, from (iii)

or $\cos\theta_1 = -\frac{2}{3}$.

Now the equation of motion in the tangential sense is

$m\dfrac{d^2s}{dt^2} = -mg\sin\theta$ or $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$, since arc $AP = s = a\theta$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating we get

$$\left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + C, \text{ where } C \text{ is constant of integration.}$$

At B, $\theta = \pi$ and $a\,d\theta/dt = \text{velocity} = 0$

$\therefore$ $0 = 2ag\cos\pi + C$ or $C = 2ag$, $\because \cos\pi = -1$

$\therefore$ $\left(a\dfrac{d\theta}{dt}\right)^2 = 2ag\cos\theta + 2ag = 2ag(1+\cos\theta) = 2ag(2\cos^2\tfrac{1}{2}\theta)$

or $a\,(d\theta/dt) = 2\sqrt{(ag)}\cos\tfrac{1}{2}\theta$ or $\tfrac{1}{2}\sec\tfrac{1}{2}\theta\,d\theta = \sqrt{(g/a)}\,dt$

or $dt = \sqrt{(a/g)}\cdot\tfrac{1}{2}\sec\tfrac{1}{2}\theta\,d\theta$.

Integrating, $t = \sqrt{(a/g)}\cdot\tfrac{1}{2}\displaystyle\int_{\theta=0}^{\theta_1}\sec\tfrac{1}{2}\theta\,d\theta$, where t is the required time from $\theta = 0$ to $\theta = \theta_1$.

or $t=\sqrt{(a/g)}\log(\sec\frac{1}{2}\theta+\tan\frac{1}{2}\theta)]_{\theta=0}^{\theta_1}$

or $t=\sqrt{(a/g)}\log(\sec\frac{1}{2}\theta_1+\tan\frac{1}{2}\theta_1)$ ...(v)

$\therefore$ From (iv), $\cos\theta_1=-\frac{2}{3}$ or $2\cos^2\frac{1}{2}\theta_1-1=-\frac{2}{3}$

or $\cos^2(\frac{1}{2}\theta_1)=\frac{1}{6}$ or $\cos(\frac{1}{2}\theta_1)=1/\sqrt{6}$

or $\sec(\frac{1}{2}\theta_1)=\sqrt{6}$ and $\tan(\frac{1}{2}\theta_1)=\sqrt{[\sec^2(\frac{1}{2}\theta_1)-1]}=\sqrt{[6-1]}=\sqrt{5}$

From (v), required time$=\sqrt{(a/g)}\log(\sqrt{6}+\sqrt{5})$.

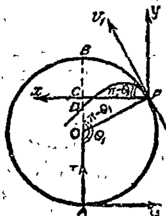
Ex. 13. A particle is projected from the lowest point of a smooth ve
to that it
that the
then returns to the point of projection.

Sol. Let a be the radius of the sphere; u the velocity of the projection from the lowest point A of the circle and T be the normal reaction at the point A between the particle and the circle.

Then the velocity of projection
$=u=\sqrt{[2g.(7/4)a]}=\sqrt{(7ag/2)}$

or $u^2=\frac{1}{2}(7ag)$. ...(i)

Let the particle leave the circle at P, whose angular distance from A is θ_1.



(Fig. 15)

Also we can prove that velocity v and reaction R at any point whose angular distance from the lowest point A is θ, are given by $v^2=u^2+2ag\cos\theta-2ag$ ...(ii)

and $R=(m/a)[u^2+3ag\cos\theta-2ag]$ ...(iii)

At P, the circular motion ceases and therefore $R=0$.

Hence, from (iii) at P *i.e.* at $\theta=\theta_1$, we get

$0=(m/a)\ [u^2+3ag\cos\theta_1-2ag]$

or $0=\frac{1}{2}(7ag)+3ag\cos\theta_1-2ag$, from (i)

or $\cos\theta_1=-\frac{1}{2}$ (iv) or $\tan\theta_1=-\sqrt{3}$ ...(v)

Also at P let velocity be v_1, then from (ii) we get

$v_1^2=u^2+2ag\cos\theta_1-2ag=\frac{7}{2}ag+2ag(-\frac{1}{2})-2ag$, from (i) and (iv)

or $v_1^2=\frac{1}{2}ag$.

The parabolic path begins from P with velocity of projection v_1 and angle of projection $(\pi-\theta_1)$ given by (iv) and (vi).

Referred to P as origin, horizontal line PC as x-axis and the vertical line through P as y-axis, the equation of the parabolic path

is $$y=x\tan(\pi-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(\pi-\theta_1)}$$

or $$y=-x\tan\theta_1-\frac{gx^2}{2v_1^2\cos^2\theta_1}$$

$$y=-x(-\sqrt{3})-\frac{gx^2}{2(\frac{1}{2}ag)(\frac{1}{4})}, \text{ from (iv), (v) and (vi)}$$

or $y=x\sqrt{3}-4x^2/a$ ...(viii)

or Also $PC=OP \sin(\pi-\theta_1)=OP \sin\theta_1=a(\frac{1}{2}\sqrt{3})=\frac{1}{2}(a\sqrt{3})$, as from (iv), $\sin\theta_1=(\sqrt{3})/2$.

$\therefore$ The equation of diameter AB is $x=\frac{1}{2}a\sqrt{3}$.

Let the parabolic path meet the diameter AB at D, such that $CD=y_1$.

Then the coordinates of D are $(\frac{1}{2}a\sqrt{3}, -y_1)$ and as D lies on (vii), so we get

$$-y_1=(3/2)a-(4/a)(\tfrac{3}{4}a^2)=(3/2)a-3a=-(3/2)a$$

or $CD=y_1=3a/2$

Also $AC=AO+OC=AO+OP\cos(\pi-\theta_1)$

$=a+a(-\cos\theta_1)=a+\frac{1}{2}a$, from (iv)

or $AC=(3/2)a$ *i.e.* the height of P above A is $3a/2$

$\therefore$ $CD=3a/2=AC$.

Hence D and A coincide or the parabolic path meets the diameter AB at A, the point of projection.

***Ex. 14 (a).** A particle is projected along the inside of a smooth vertical circle of radius a from the lowest point. Show that the velocity of projection required in order that after leaving the circle, the particle may pass through the centre is $\sqrt{(\frac{1}{2}ag)}(\sqrt{3}+1)$. *(Garhwal 88, 87 : Gorakhpur 91)*

Sol. (Figure same as in last example)

Let u be the velocity of projection from the lowest point A of circle and T be the normal reaction at A between the particle and the circle.

We can show that the velocity v and reaction R at any point, whose angular distance is θ from the lowest point A are given by

$v^2=u^2+2ag\cos\theta-2ag$ ...(i)

and $R=(m/a)[u^2+3ag\cos\theta-2ag]$. ...(ii)

Let the circular motion cease *i e.* the particle leave the circle at P, whose angular distance from A is θ_1.

Then at P *i.e.* at $\theta=\theta_1$ the reaction $R=0$

$\therefore$ from (ii), $0=u^2+3ag\cos\theta_1-2ag$

or $\cos\theta_1=-[(u^2-2ag)/(3ag)]$. ...(iii)

And if v_1 be the velocity of the particle at P, we have from (i)

$v_1^2=u^2+2ag\cos\theta_1-2ag$

or $v_1^2=u^2+2ag\left(-\frac{(u^2-2ag)}{3ag}\right)-2ag$ or $v_1^2=\frac{1}{3}(u^2-2ag)$ ...(iv)

$\therefore$ The parabolic path begins from P with velocity of projection v_1 and angle of projection $(\pi-\theta_1)$ given by (iii) and (iv). (See fig.

Referred to P as origin, horizontal line PC as x-axis and the vertical line through P as y-axis, the equation of the parabolic path

is $$y=x\tan(\pi-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(\pi-\theta_1)}$$

or $$y=-x\tan\theta_1-\frac{gx^2}{2v_1^2\cos^2\theta_1} \quad ...(v)$$

Also $PC=OP\sin(\pi-\theta_1)=a\sin\theta_1$

and $OC=OP\cos(\pi-\theta_1)=-a\cos\theta_1$.

$\therefore$ The coordinates of the centre O are $(a\sin\theta_1, a\cos\theta_1)$. Since O $(a\sin\theta_1, a\cos\theta_1)$ lies on the parabolic path in this problem, so we have $\cos\theta_1=-a\sin\theta_1\tan\theta_1-\dfrac{ga^2\sin^2\theta_1}{2v_1^2\cos^2\theta_1}$, from (v)

or $$\cos\theta_1+\sin\theta_1\tan\theta_1=-\frac{ga^2\sin^2\theta_1}{2v_1^2\cos\theta_1}$$

or $$\cos^2\theta_1+\sin^2\theta_1=-\frac{ga\sin^2\theta_1}{2v_1^2\cos\theta_1}$$

or $$2v_1^2\cos\theta_1=-ga(1-\cos^2\theta_1)$$

or $$-2\left(\frac{u^2-2ag}{3}\right)\left(\frac{u^2-2ag}{3ag}\right)=-ga\left[1-\left(\frac{u^2-2ag}{3ag}\right)^2\right]$$

or $2(u^2-2ag)^2=9a^2g^2-(u^2-2ag)^2$ or $3(u^2-2ag)^2=9a^2g^2$

or $u^2-2ag=ag\sqrt{3}$ or $u^2=ag(\sqrt{3}+2)=\frac{1}{2}ag(\sqrt{3}+1)^2$ (Note)

or $u=\sqrt{(\frac{1}{2}ag)}(\sqrt{3}+1)$.

Ex. 14 (b). A particle tied to a string of a length a is projected from its lowest point, so that after leaving the circular path it describes a free path passing through the lowest point. Prove that velocity of projection is $[\sqrt{(7/2)\,ag}]$. *(Avadh 89)*

Sol. Proceeding as in Ex. 14 (a) Page 18 we can find that the equation of the parabolic path, referred to P as origin, horizontal line PC as x-axis and the vertical line through P (where P is the point where the particle leaves the circle) as y-axis is

$$y=-x\tan\theta_1-\frac{gx^2}{2v_1^2\cos^2\theta_1} \quad ...(v)$$

Also in Ex. 14 (a) Page 18 we have proved that

$PC=a\sin\theta_1$, $OC=-a\cos\theta_1$

$\therefore$ $AC=OC+OA=-a\cos\theta_1+a$

$\therefore$ The coordinates of the lowest point A are

$(a\sin\theta_1, -a+a\cos\theta_1)$ (Note)

$\because$ A lies on the parabolic path in this problem, so from (v)

we have $$-a+a\cos\theta_1=-a\sin\theta_1\tan\theta_1-\frac{ga^2\sin^2\theta_1}{2v_1^2\cos^2\theta_1}$$

or $-1+\cos\theta_1+\sin\theta_1\tan\theta_1=-(ga\sin^2\theta_1)/(2v_1^2\cos^2\theta_1)$

or $-1+(\cos^2\theta_1+\sin^2\theta_3)/(\cos\theta_1)=-(ga\sin^2\theta_1)/(2v_1^2\cos^2\theta_1)$

or $2(1-\cos\theta_1)v_1^2\cos\theta_1=-ga\sin^2\theta_1=-ga(1-\cos^2\theta_1)$

or $2\left[1+\frac{u^2-2ag}{3ag}\right]\left(\frac{u^2-2ag}{3ag}\right)\left[-\frac{u^2-2ag}{3ag}\right]$

$=-ga\left[1-\left(\frac{u^2-2ag}{3ag}\right)^2\right]$, from (ii) and (iv) of Ex. 14 (a) Page 18.

or $\frac{2}{3}(u^2+ag)(u^2-2ag)^2=ga[9a^2g^2-(u^2-2ag)^2]$

or $(u^2-2ag)^2[\frac{2}{3}(u^2+ag)+ag]=9a^3g^3$

or $(u^2-2ag)^2(2u^2+5ag)-27a^3g^3=0$

or $2u^6-3agu^4-12a^2g^2u^2-7a^3g^3=0,$

which is a cubic equation in u^2 and is satisfied by $u^2=(7/2)/ag$ (students to show it)..

i.e. $u=\sqrt{[(7/2)\,ag]}$. Hence proved.

Ex. 15.** ***A particle is projected from the lowest point inside a smooth circular wire of radius a ***with a velocity due to a height*** h ***above the centre. Find the point where it leaves the circle and show that it will afterwards pass through (a) the centre if*** $h=\frac{1}{2}a\sqrt{3}$ ***and (b) the lowest point if*** $h=\frac{3}{2}a$. *(Rohilkhand 87)*

Sol. Let u be the velocity of projection from the lowest point A of the circle and T be the normal reaction at A between the particle and the circle.

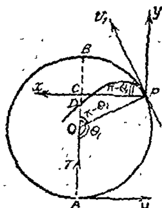
Also height h above centre means height $(h+a)$ above the lowest point. Hence the velocity of projection at A

$=u=\sqrt{\{2g(a+h)\}}$

or $u^2=2g(a+h)$ ...(i)

Also we can show that velocity v and reaction R at any point, whose angular distance is θ from the lowest point A are given by

$v_1^2=u^2+2ag\cos\theta-2ag$ ...(ii)



(Fig. 16)

and $R=(m/a)[u^2+3ag\cos\theta-2ag]$ ...(iii)

Let the particle leave the circle at P, whose angular distance from A is θ_1. Then at P the reaction $R=0$ and from (iii) we have

$0=(m/a)[u^2+3ag\cos\theta_1-2ag]$

or $\cos\theta_1=-\left(\frac{u^2-2ag}{3ag}\right)=-\left[\frac{2g(a+h)-2ag}{3ag}\right]$, from (i)

or $\cos\theta_1=-(2h/3a)$. ...(iv)

Let v_1 be the velocity of the particle at P, then from (ii) we get

$v_1^2=u^2+2ag\cos\theta_1-2ag$

$=2g(a+h)+2ag(-2h/3a)-2ag$, from (i) and (iv)

or $v_1^2=2gh-(4/3)gh=(2/3)gh$ *i.e.* $v_1^2=(2/3)gh$ ...(v)

From P draw PC perpendicular to the vertical diameter AB. Then height of P above the centre O of the circle

$$=OC=OP\cos(\pi-\theta_1)=-OP\cos\theta_1$$

$$=-a(-2h/3a)=\tfrac{2}{3}h \quad \ldots(\text{iv})$$

The parabolic path begins from P with velocity of projection v_1 and angle of projection $(\pi-\theta_1)$ (see figure) given by (iv) and (v).

$\therefore$ Referred to P as origin, PC as x-axis and the vertical line through P as y-axis the equation of the parabolic path is

$$y=x\tan(\pi-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(\pi-\theta_1)}$$

or $$y=-x\tan\theta_1-\frac{gx^2}{2v_1^2\cos^2\theta_1} \quad \ldots(\text{vii})$$

(a) $PC=OC\sin(\pi-\theta_1)=a\sin\theta_1$,

and $OC=OP\cos(\pi-\theta_1)=-a\cos\theta_1$.

$\therefore$ Coordinates of the centre O are $(a\sin\theta_1, a\cos\theta_1)$ (Note)

$\therefore$ If the parabolic path passes through O $(a\sin\theta_1, a\cos\theta_1)$

from (vii) we have $a\cos\theta_1=-a\sin\theta_1\tan\theta_1-\dfrac{ga^2\sin^2\theta_1}{2v_1^2\cos^2\theta_1}$

or $$\cos\theta_1+\frac{\sin^2\theta_1}{\cos\theta_1}=-\frac{ga\sin^2\theta_1}{2v_1^2\cos^2\theta_1}$$

or $2v_1^2\cos\theta_1=-ga\sin^2\theta_1$, after simplifications

or $2v_1^2\cos\theta_1=-ga(1-\cos^2\theta_1)$

or $$2\left(\frac{2gh}{3}\right)\left(-\frac{2h}{3a}\right)=-ga\left[1-\frac{4h^2}{9a^2}\right] \text{ or } 8h^2=9a^2-4h^2$$

or $12h^2=9a^2$ or $h^2=\tfrac{3}{4}a^2$ or $h=\tfrac{1}{2}a\sqrt{3}$ **Ans.**

(b) We have proved $PC=a\sin\theta_1$ and $OC=-a\cos\theta_1$

$\therefore$ $CA=CO+OA=-a\cos\theta_1+a$.

$\therefore$ Coordinates of the lowest point A are $(a\sin\theta_1, -a+a\cos\theta_1)$

$\therefore$ If the parabolic path passes through A $(a\sin\theta_1, -a+a\cos\theta_1)$ we have from (vii)

$$-a+a\cos\theta_1=-a\sin\theta_1\tan\theta_1-\frac{ga^2\sin^2\theta_1}{2v_1^2\cos^2\theta_1}$$

or $$-a+a\left[\cos\theta_1+\frac{\sin^2\theta_1}{\cos\theta_1}\right]=-\frac{ga^2\sin^2\theta_1}{2v_1^2\cos^2\theta_1}$$

or $$-1+\left[\frac{1}{\cos\theta_1}\right]=-\frac{ga(1-\cos^2\theta_1)}{2v_1^2\cos^2\theta_1}$$

or $$-1-\frac{3a}{2h}=-\frac{ag[1-(4h^2/9a^2)]}{2(\tfrac{2}{3}gh)(4h^2/9a^2)}=\frac{-3ag(9a^2-4h^2)}{16gh^2}$$

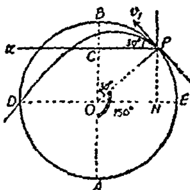
or $8(3a+2h)h^2-3a(9a^2-4h^2)=0$

or $(3a+2h)[8h^2-3a(3a-2h)]=0$

or $8h^2+6ah-9a^2=0$, since $3a+2h\neq 0$

or $h=\dfrac{-6a+\sqrt{(36a^2+288a^2)}}{16}=\dfrac{-6a\pm 18a}{16}=\dfrac{12a}{16}=\dfrac{3a}{4}$. Ans.

Ex. 16. Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius a from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter. *(Avadh 89)*



(Fig. 17)

Sol. Let u be the velocity of projection from the lowest point A of the circle.

We can show that velocity v and reaction R at any point whose angular distance is θ from the lowest point A are given by

$$v^2=u^2+2ag\cos\theta-2ag \quad ...(i)$$

and $$R=\frac{m}{a}(u^2+3ag\cos\theta-2ag) \quad ...(ii)$$

Let the particle leave the circle at P, whose angular distance from the vertical is 30°. Since the particle leaves the circle at P, therefore P must be at a level higher than the centre O. Hence $\angle BOP=30°$, where AB is the vertical distance of the circle.

$\therefore$ the angular distance of P from the lowest point A
$$=\angle AOP=150°=5\pi/6.$$

At P the particle leaves the circle, hence at P, *i.e.* at $\theta=5\pi/6$ the reaction $R=0$.

$\therefore$ From (ii) we get $0=(m/a)\,[u^2+3ag\cos\{5\pi/6\}-2ag]$

or $u^2-3ag\,\tfrac{1}{2}\sqrt{3}-2ag=0$, $\because$ $\cos\{5\pi/6\}=-\tfrac{1}{2}\sqrt{3}$

or $u^2=\tfrac{1}{2}(3\sqrt{3}+4)\,ag$. Ans.

Also if v_1 be the velocity at P, we have from (i)

$$v_1^2=u^2+2ag\cos\{5\pi/6\}-2ag$$
$$=\tfrac{1}{2}(3\sqrt{3}+4)\,ag-2ag\,(\tfrac{1}{2}\sqrt{3})-2ag=\tfrac{1}{2}\sqrt{3}ag. \quad ...(iii)$$

The parabolic path begins from P with velocity of projection v_1 given by (iii) and angle of projection 30° (as shown in the figure).

$\therefore$ Referred to P as origin and Px and Py as axes as shown in the figure, the equation of the parabolic path is

$$y=x\tan 30°-\frac{gx^2}{2v_1^2\cos^2 30°}$$

or $$y=\frac{x}{\sqrt{3}}-\frac{gx}{2(\tfrac{1}{2}\sqrt{3}ag)\,(3/4)}=\frac{x}{\sqrt{3}}-\frac{4x^2}{3\sqrt{3}a}$$

or $$3\sqrt{3}ay=3ax-4x^2$$

From P draw PN perpendicular to the horizontal diameter DC.

Then $ND = NO + OD = PC + OD = a \sin 30° + a = 3a/2$

and $OC = OP . \cos 30° = \frac{1}{2} a \sqrt{3}$.

$\therefore$ The coordinates of D are $[3a/2, -\frac{1}{2} a \sqrt{3}]$.

If $D\ [3a/2, -\frac{1}{2} a \sqrt{3}]$ lies on (iv), then the coordinates of D must satisfy (iv). Substituting the coordinates of D in (iv) we have $3\sqrt{3} a\ (-\frac{1}{2} a \sqrt{3}) = 3a\ [3a/2] - 4\ [3a/2]^2$

or $-(9/2) a^2 = -(9/2) a^2$. Hence (iv) is satisfied.

$\therefore$ The parabolic path passes through D, one end of the horizontal distance DE. Hence proved.

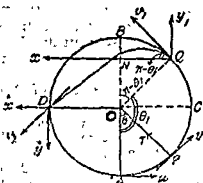
****Ex. 17. A heavy particle is attached to a fixed point by a fine string of length a; the particle is projected horizontally from the lowest point with velocity $\sqrt{[ag\{2+(3\sqrt{3}/2)\}]}$. Prove that the string would first become slack when inclined to the upward vertical at an angle of 30°, will become tight again, when horizontal, and slack again when inclined to the upward vertical at an angle $\cos^{-1}\{(3/8)\sqrt{3}\}$.**

Sol. Given that $\sqrt{[ag\ \{2+(3\sqrt{3}/2)\}]} = u$ (say) is the velocity of projection of the particle of mass m (say) from the lowest point A of the circle.

We can show in the usual way (see § 3 P. 2) that velocity v and tension T in the string at any position P, of the particle whose angular distance from A is θ are given by

$$v^2 = u^2 + 2ag \cos \theta - 2ag$$

and $$T = \frac{m}{a}\ (u^2 + 3ag \cos \theta - 2ag)$$



(Fig. 18)

Here $u^2 = ag\ [2 + (3\sqrt{3}/2)]$, so above two results reduce to $v^2 = (3/2)\sqrt{3} ag + 2ag \cos \theta$ ...(i)

and $T = \frac{m}{a}\ [(3/2)\sqrt{3} ag + 3ag \cos \theta]$ ...(ii)

[illegible] the circular path [illegible] v_1 be the velocity [illegible] string is slack at Q.

$\therefore$ From (ii) we get

$0 = \frac{m}{a}\ [(3/2)\sqrt{3} ag + 3ag \cos \theta_1]$ or $\cos \theta_1 = -\frac{\sqrt{3}}{2}$

or $\theta_1 = 150°$. Hence $\angle BOQ = \pi - \theta_1 = 30°$ ...(iii)

i.e. the string becomes slack when it is inclined at an angle 30° to the upward vertical OB. (See fig. 18 above)

$\therefore$ From (i) at Q we have

$v_1^2=(3/2)\sqrt{3}ag+2ag\cos\theta_1=(3/2)\sqrt{3}ag+2ag(-\frac{1}{2}\sqrt{3})$

or $\quad v_1^2=\frac{1}{2}ag\sqrt{3}$ ...(iv)

Beyond Q, the particle will trace out a parabolic path. Through Q draw Qx and Qy, the horizontal and vertical lines through Q lying in the plane of flight of the particle.

The parabolic path begins from Q with velocity of projection v_1 and angle of projection $\pi-\theta_1$ given by (ii) and (iv).

Referred to Qx and Qy, the equation of parabolic path is

$$y=x\tan(\pi-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(\pi-\theta_1)}$$

or $$y=x\tan 30^\circ-\frac{gx^2}{2(\frac{1}{2}ag\sqrt{3})\cos^2 30^\circ}\text{, from (iii) and (iv)}$$

$$=x(1/\sqrt{3})-\frac{x^2}{a\sqrt{3}(3/4)}=\frac{x}{\sqrt{3}}-\frac{4x^2}{3a\sqrt{3}}$$

or $$3ay\sqrt{3}=3ax-4x^2 \qquad \text{...(v)}$$

Also from the figure 18 Page 23, it is evident that in right angled triangle OQN, we have

$$ON=OQ\cos(\pi-\theta_1)=a\cos 30^\circ=\tfrac{1}{2}a\sqrt{3}$$

and $$QN=OQ\sin(\pi-\theta_1)=a\sin 30^\circ=\tfrac{1}{2}a \qquad \text{...(vi)}$$

Referred to Qx and Qy, as coordinates axes, we have

x-coordinate of $D=QN+OD$, see figure

$=\frac{1}{2}a+a=3a/2$

$=x_1$ (say) ...(vii)

And y-coordinate of $D=-ON=-\frac{1}{2}a\sqrt{3}$, from (vi)

$=-y_1$ (say) ...(viii)

Hence if $D\ (x_1, -y_1)$, where x_1 and y_1 are given by (vii) and (viii), lie on the parabolic path, then its coordinates must satisfy the equation (v) *i.e.*

$$-3ay_1\sqrt{3}=3ax_1-4x_1^2$$

or $$-3a(\tfrac{1}{2}a\sqrt{3})\sqrt{3}=3a(3a/2)-4(3a/2)^2$$

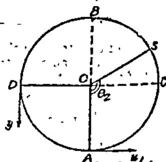
or $$-(9/2)a^2=(9/2)a^2-9a^2 \text{ or, } -(9/2)a^2=-(9/2)a^2,$$

which is true. Hence the parabolic path of the particle passes through D, the extremity of the horizontal diameter DC of the circular path of the particle.

When the particle reaches D, the length of the string $=OD=a$; and so the string is again tight in this position of the particle.

Let v_2 be the velocity of the particle when it reaches D. The direction of v_2 will be tangential to the parabolic path at D (See Fig. 18 Page 23).

Let t be the time taken by the particle in moving from Q to D. Then



(Fig. 19)

from "$x=(u \cos \alpha)\, t$" see chapter on Projectiles, we have

$$3a/2=(v_1 \cos 30°)\, t \qquad \text{(Note)}$$

or $(3a/2)=v_1\,(\frac{1}{2}\sqrt{3})\, t$ or $t=a\sqrt{3}/v_1$ ...(ix)

Also we know from the chapter on 'Projectiles', that if $\dot{x}$ and $\dot{y}$ be the components of velocity at any point on the projectile, then $\dot{y}=(u \sin \alpha)-gt$, where the symbols have their usual meanings.

Then $\dot{y}$ (the y-component or vertical component of v_2) is given by $-\dot{y}=v_1 \sin 30°-gt$, here $-\dot{y}$ has been taken as the y-component of v_2 is in the vertically downward direction (See Fig. 18 Page 23)

or $-\dot{y}=\frac{1}{2}v_1-gt$ or $\dot{y}=gt-\frac{1}{2}v_1$ ...(x)

Now when the particle reaches D, the string is tight and beyond D, the particle moves again on the circular path with O as centre and radius a. This circular path of the particle starts from D and with velocity y as given by (x) above and shown in Fig. 19 Page 24 (in this figure make $\dot{y}$ in place of y).

If now the particle reaches A with velocity u_1, then from "$v^2=u^2+2ag \cos \theta-2ag$" at D we have

$$\dot{y}^2=u_1^2+2ag \cos 270°-2ag \qquad \text{(Note)}$$

or $\dot{y}^2=u_1^2-2ag$ or $u_1^2=\dot{y}^2+2ag$ ...(xi)

Now let the particle leave the circular path *i.e.* the string becomes slack again at S, such that $\angle AOS=\theta_2$ (say).

Then from $T=\frac{m}{a}[u^2+3ag \cos \theta-2ag]$

we get $0=(m/a)[u_1^2+3ag \cos \theta_2-2ag]$

or $3ag \cos \theta_2=2ag-u_1^2=2ag-(\dot{y}^2+2ag)$, from (xi)

or $\cos \theta_2=-\dot{y}^2/(3ag)$

$\therefore \cos \angle BOS=\cos(\pi-\theta_2)=-\cos \theta_2=\dot{y}^2/(3ag)$

or $\angle BOS=\cos^{-1}[\dot{y}^2/(3ag)]$ ...(xii)

Now from (iv), (ix) and (x) we get

$$\dot{y}=g\left(\frac{a\sqrt{3}}{v_1}\right)-\frac{1}{2}v_1=\frac{2ag\sqrt{3}-v_1^2}{2v_1}=\frac{2ag\sqrt{3}-\frac{1}{2}ag\sqrt{3}}{2v_1}$$

or $\dot{y}=(3ag\sqrt{3})/4v_1$ or $\dot{y}^2=\frac{27}{16}\frac{a^2g^2}{v_1^2}=\frac{27a^2g^2}{16\,(\frac{1}{2}ag\sqrt{3})}$

or $\dot{y}^2=\frac{1}{8}[9\sqrt{3}ag]$.

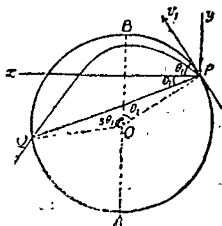
$\because$ From (xii) we get

$\therefore \angle BOS=\cos^{-1}[\{(9/8)\sqrt{3}ag\}/(3ag)]=\cos^{-1}\{(3/8)\sqrt{3}\}$.

Hence proved.

**Ex. 18. A particle of mass m is projected from the lowest point with velocity u and moves along the inside of a smooth vertical circle of radius a. The particle after leaving the circle at P meets it again at Q. Prove that the actual distance of Q is three times that of P from the highest point.

(Agra 90, 86 ; Rohilkhand III 91)



(Fig. 20)

Sol. Let the particle leave the circle at P such that $\angle BOP = \theta_1$, where B is the highest point of the circle. Let AB be the circle with centre O, then

$$\angle AOP = \pi - \theta_1.$$

We can show that the velocity v and reaction R at any point whose angular distance from the lowest point A is θ are given by

$$v^2 = u^2 + 2ag \cos \theta - 2ag \quad \text{...(i)}$$

and
$$R = (m/a)[u^2 + 3ag \cos \theta - 2ag] \quad \text{...(ii)}$$

At P the particle leaves the circle, hence $R = 0$ at P *i.e.* at $\theta = \pi - \theta_1$.

$\therefore$ from (ii) we get $0 = (m/a)[u^2 + 3ag \cos(\pi - \theta_1) - 2ag]$

or
$$u^2 = 3ag \cos \theta_1 + 2ag. \quad \text{...(iii)}$$

Let velocity of the particle at P be v_1, then from (i)

$$v_1^2 = u^2 + 2ag \cos(\pi - \theta_1) - 2ag$$
$$= u^2 - 2ag \cos \theta_1 - 2ag$$
$$= 3ag \cos \theta_1 + 2ag - 2ag \cos \theta_1 - 2ag, \text{ from (iii)}$$

or
$$v_1^2 = ag \cos \theta_1. \quad \text{...(iv)}$$

The parabolic path begins from P with velocity of projection v_1 and angle of projection θ_1 given by (iii) and (iv).

(See figure also)

Referred to P as origin and the horizontal and vertical lines Px and Py as shown in the figure as coordinate axes, the equation of the parabolic path is

$$y = x \tan \theta_1 - \frac{gx^2}{2v_1^2 \cos^2 \theta_1}, \text{ where } v_1^2 = ag \cos \theta_1$$

or
$$y = x \tan \theta_1 - x^2/(2a \cos^3 \theta_1) \quad \text{...(v)}$$

Let C be the point of intersection of Px and OB, then $OC = OP \cos \theta_1 = a \cos \theta_1$ and $PC = OP \sin \theta_1 = a \sin \theta_1$.

$\therefore$ The coordinates of the centre O are $(a \sin \theta_1, -a \cos \theta_1)$ and the radius of the circle is a.

$\therefore$ The equation of the circle is

$$(x - a \sin \theta_1)^2 + (y + a \cos \theta_1)^2 = a^2$$

or $\qquad x^2+y^2-2ax\sin\theta_1+2ay\cos\theta_1=0.$ ...(vi)

Solving (v) and (vi) we get

$$x^2+y^2-2ax\sin\theta_1+2a\cos\theta_1\left[x\tan\theta_1-\frac{x^2}{2a\cos^2\theta_1}\right]=0$$

(Note)

or $\qquad x^2+y^2-2ax\sin\theta_1+2ax\sin\theta_1-x^2\sec^2\theta_1=0$

or $\qquad y^2=x^2(\sec^2\theta_1-1)=x^2\tan^2\theta_1$ or $y=\pm x\tan\theta_1$.

Substituting $y=x\tan\theta_1$ in (v) we have $x=0$, hence $y=0$.

But these are the coordinates of P.

If we take $y=-x\tan\theta_1$, then this is a straight line PQ passing through the origin P and making an angle θ_1 below the x-axis

$\therefore\ \angle xPQ=\theta_1$, where Q is the required point.

$\therefore\ \angle v_1PQ=\angle v_1Px+\angle xPQ=\theta_1+\theta_1=2\theta_1$

$\therefore\ \angle POQ=$double of any angle subtended by the chord PQ in the segment PAQ.

$=$double of $\angle v_1PQ$ ...*by plane geometry*

$=2(2\theta_1)=4\theta_1$.

Also $\angle BOP=\theta_1$, $\therefore\ \angle BOQ=4\theta_1-\theta_1=3\theta_1$.

$\therefore$ arc $PQ=3$ times arc BP. *(Agra 90, 86)*

This gives the position of the required point Q.

Ex. 19. A heavy particle of mass m makes complete revolution in a smooth circular tube fixed in a vertical plane. Its greatest speed is n times its least speed. Prove that the pressure in the tube when the particle is moving vertically is $2mg[(n^2+1)/(n^2-1)]$ and that its speed is $\sqrt{\{\frac{1}{2}(n^2+1)\}}$ times its speed at the highest point.

Sol. Let the radius of the tube be a. The velocity is least at the highest point B and greatest at the lowest point A of the tube. Let u be the velocity of the particle at B, then the velocity at A is nu (given).

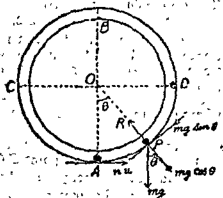
Let the particle be at P at any instant such that $\angle AOP=\theta$.

Then the equation of motion in the tangential and inward drawn normal sense are

$$m.\frac{d^2s}{dt^2}=mg\sin\theta \quad ...(i)$$

and $$m\frac{v^2}{\rho}=R-mg\cos\theta, \quad ...(ii)$$

where R is the normal reaction between the particle and the tube at P.



Let arc $AP=s$,

(Fig. 21)

then $s=a\theta$, $\therefore \frac{d^2s}{dt^2}=a\frac{d^2\theta}{dt^2}$

$\therefore$ From (i) we get $a\,(d^2\theta/dt^2)=-g\sin\theta$

Integrating $(ad\theta/dt)^2=2ag\cos\theta+k$, where k is constant of integration

At A, $\theta=0$ and $a\,(d\theta/dt)=$velocity$=nu$.

$\therefore\ n^2u^2=2ag+k$

$\therefore\ (ad\theta/dt)^2=2ag\cos\theta+n^2u^2-2ag.$ ...(iii)

Also at the highest point $\theta=\pi$ and velocity$=u$.

$\therefore$ From (iii), $u^2=2ag\cos\pi+n^2u^2-2ga$

or $u^2=-2ag+n^2u^2-2ag$ or $a=(n^2-1)\,u^2/4g.$ ...(iv)

Let CD be the horizontal diameter of the circle, then at D the particle is moving vertically. Let v_1 be the velocity of the particle at D *i.e.* at $\theta=\pi/2$.

Then from (iii), $v_1^2=2ag\cos\frac{1}{2}\pi+n^2u^2-2ga$

$=n^2u^2-2ag=n^2u^2-\frac{1}{2}(n^2-1)\,u^2$, from (iv)

or $v_1^2=\frac{1}{2}(n^2+1)\,u^2$

or $v_1=\sqrt{\{\frac{1}{2}(n^2+1)\}}\,u=\sqrt{\{\frac{1}{2}(n^2+1)\}}\times$(velocity at B).

Also from (ii) at D *i.e.* at $\theta=\frac{1}{2}\pi$, the normal reaction

$=$"$m.\frac{v^2}{\rho}+mg\cos\theta$"$=\frac{m.v_1^2}{a}+mg\cos\frac{1}{2}\pi$

$=(m/a)\,\frac{1}{2}(n^2+1)\,u^2$, from (v)

$=\frac{4mg}{(n^2-1)\,u^2}\cdot\frac{(n^2+1)\,u^2}{2}$, from (iv)

$=2mg\,[(n^2+1)/(n^2-1)].$ Hence proved.

Ex. 19 (a). A stone of weight W is tied to one end of a string and is describing a circle in a vertical plane, the other end of the string being fixed. If the maximum speed of the stone is twice the minimum speed; prove that when the string is horizontal; its tension is (10/3) W. Also find the tension when the string makes an angle of 45° with the downward vertical.

Sol. Refer Fig. 2 Page 2. Take mg as W.

Let the length of the string be a whose one end is fixed at O. Hence the radius of the circle which is being described is a. Let the stone be projected horizontally from A with velocity u, then at any point P we can show that $v^2=u^2-2ga+2ga\cos\theta$ ...(i)

and $T=(W/ga)\,[u^2-2ga+3ga\cos\theta]$, ...(ii)

where θ is the angle which the string makes with the downward vertical.

The velocity is maximum at the lowest point A (at $\theta=0$) and minimum at the highest point C (at $\theta=\pi$). Therefore from (i)

Maximum velocity$=u$ and minimum velocity$=\sqrt{(u^2-4ga)}$.

But given that maximum velocity$=2\times$(Minimum velocity)

i.e. $u=2\sqrt{(u^2-4ga)}$ or $u^2=(16/3)ga$ ...(iii)

Hence from (ii) the tension when the string is horizontal *i.e.* when $\theta=\frac{1}{2}\pi$ is given by $T=(W/ga)[(16/3)\,ag-2ag+3ag\cos\frac{1}{2}\pi]$

...putting the value of u^2 from (iii)

or $T=(10/3)\,W.$ Hence proved.

Do yourself the second part.

Second Method. Try yourself as in Ex. 19 Page 27.

**Ex. 20. Show that the greatest angle through which a person can oscillate on a swing the ropes of which can support twice the person's weight at rest is 120°.

If the ropes are strong enough and he can swing through 180° and if v is his speed at any point prove that the tension in the rope at that point is $3\,mv^2/(2l)$, where m is the mass of the person and l the length of the rope.

Sol. Let u be the velocity at the lowest point. Then we can prove that velocity v and tension T in the string at any point at an angular distance θ from the lowest point are given by

$$v^2=u^2+2lg\cos\theta-2lg \quad ...(i)$$

and

$$T=(m/l)\,(u^2+3lg\cos\theta-2lg), \quad ...(ii)$$

where m is the mass of the man.

At the lowest point $\theta=0$ and tension is maximum and equal to $2mg$ (given)

∴ At the lowest point from (ii) we have

$$2mg=(m/l)\,(u^2+3lg\cos 0-2lg)$$

or $2gl=u^2+lg$ or $u^2=gl$ ...(iii)

∴ From (i) we have $v^2=gl+2lg\cos\theta-2lg$

or $v^2=2lg\cos\theta-gl$ ...(iv)

Putting $v=0$ in (iv) we have $2\cos\theta-1=0$ or $\cos\theta=\frac{1}{2}$

or $\theta=60°$

i.e. the man can oscillate on the swing through an angle of 60° on one side of the vertical line through the lowest point. Similarly on the other side he can oscillate through an angle of 60°. Hence the greatest angle through which the man can oscillate

$$=60°+60°=120°.$$

If the man swings through 180°, then $\theta=90°$ at the extreme position of oscillation and $v=0$ also at that point.

∴ From (i), $0=u^2+2gl\cos 90°-2gl$ or $u^2=2gl$ ...(v)

Hence from (i) velocity at a point whose angular distance is θ from the lowest point we have $v^2=2gl+2gl\cos\theta-2gl$

or $v^2=2gl\cos\theta$ or $\cos\theta=v^2/2gl$

∴ from (ii), $T=(m/l)\,[2gl+3gl\,(v^2/2gl)-2gl]$, from (i)

or $T=m\,(3v^2/2l)$. Hence proved.

Exercises on § 1—§ 4

Ex. 1. A particle is projected from the lowest point of a smooth vertical circle of radius a along its inner side with a velo-

city $\frac{1}{5}\sqrt{(95ag)}$. Show that it will leave the circle at an angular distance of $\cos^{-1}(3/5)$ from the highest point and its velocity at that time will be $\sqrt{(3ag)}$.

Ex. 2. A heavy particle attached by a string to a fixed point hangs in equilibrium. Find out the least velocity with which it must be projected horizontally from the equilibrium position in order that it may describe a complete circle. Show further, that the tension of the string at any time is proportional to the depth of the particle at the moment below a certain horizontal line.

[Hint : See Ex. 7 Page 11].

*Ex. 3. A heavy particle is constrained to move in a vertical circle of radius r. Show that the sum of the reactions at the end of a diameter is constant.

[Hint : Replace tension T by reaction R in Ex. 9 (a) P. 12]

*Ex. 4. A particle is projected from the lowest point of a vertical circle with a velocity just sufficient to carry it to the highest point. Find when and where the particle will leave the circle.

[Hint : See Ex. 12 Page 16].

§ 5. *Motion on the outside of a vertical circle.*

A particle slides down the outside of a smooth vertical circle due to its weight, starting from rest at the highest point ; to discuss motion.

(Agra 91, 87 ; Bundelkhand 92 ; Kumaun 89)

O is the centre of the circle and AB is the vertical diameter. After time t, let the particle be at P, such that $\angle AOP=\theta$ and arc $AP=s$. At P, the forces acting on the particle are its weight mg acting vertically downwards and the normal reaction R acting normally to the circle at P as shown in the figure.

$\therefore$ The equation of the motion in the tangential and inward normal sense are

$$m\frac{d^2s}{dt^2}=mg\sin\theta \quad ...(i) \qquad \text{and } m.\frac{v^2}{\rho}=mg\cos\theta-R \quad ...(ii)$$

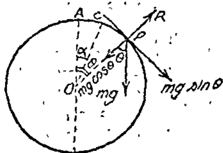
Now arc $AP=s=a\theta$, where a is radius of the circle

$$\therefore \frac{d^2s}{dt^2}=a\frac{d^2\theta}{dt^2}$$

Hence from (i) we get

$$a\,(d^2\theta/dt^2)=g\sin\theta.$$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating we get $(a\,d\theta/dt)^2=-2ag\cos\theta+C$, where C is constant of integration.



(Fig. 22)

or $v^2=-2ag\cos\theta+C$, where v is the velocity at P.

Initially at A, $v=0$, $\theta=0$ therefore $C=2ag$

$\therefore \quad v^2=2ag-2ag \cos \theta=2ag\ (1-\cos \theta)$ ...(iii)

From (ii), $R=mg \cos \theta-(mv^2/a)$, $\because \rho=a$ for the circle.

$=mg \cos \theta-2mg\ (1-\cos \theta)$, from (iii)

or $\quad R=mg\ (3 \cos \theta-2)$. ...(iv)

Equation (iii) and (iv) give the values of velocity v and reaction R at any point P at an angular distance θ from A.

Let the particle leave the circle at a point, whose angular distance from the highest point A is θ_1. Therefore putting $R=0$ and $\theta=\theta_1$ in (iv) we get $3 \cos \theta_1-2=0$ or $\cos \theta_1=\frac{2}{3}$. ...(v)

or $\quad \theta_1=\cos^{-1} \frac{2}{3}$

which gives the angular distances from A of the point where the particle leaves the circle.

Vertical distance of this point where the particle leaves contact of the circle$=a-a \cos \theta_1$. (Note)

$=a-a\ (\frac{2}{3})=\frac{1}{3}a=\frac{1}{3}$ (radius of the circle).

Therefore if a particle slides down the outside of a smooth vertical circle, starting from rest at the highest point, it will leave the circle after descending vertically a distance equal to one third of the radius. *(Agra 91 ; Kumaun 89)*

And the velocity of the particle at the point where it leaves the circle$=2ag\ (1-\cos \theta_1)$, where $\cos \theta_1=\frac{2}{3}$ from (iii), (v)

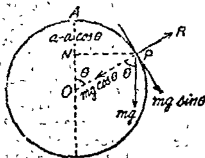
$=2ag\ [1-(\frac{2}{3})]=(2ag)/3$.

Solved Examples on § 5.

Ex. 1 (a) [illegible]

vertical distance below the point of projection is $\frac{1}{6}a$. *(Garhwal 89)*

Sol. The particle starts from the highest point A with velocity $\sqrt{(\frac{1}{3}ag)}$. Let the particle be at P after time t, such that $\angle AOP=\theta$ and arc $AP=s$.



(Fig. 23)

At P, the forces acting on the particle are its weight mg acting vertically downwards and the normal reaction R acting normally to the circle at P as shown in the figure.

$\therefore$ The equations of motion in the tangential and inward drawn normal senses are

$$m.\frac{d^2s}{dt^2}=mg \sin \theta \quad ...(i)$$

and $\quad m \frac{v^2}{\rho}=mg \cos \theta-R \quad ...(ii)$

Now arc $AP=s=a\theta$, $\therefore \frac{d^2s}{dt^2}=a\frac{d^2\theta}{dt^2}$

Hence from (i) we get $a\,(d^2\theta/dt^2)=g\sin\theta$.

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we get $(a\,d\theta/dt)^2=-2ag\cos\theta+C$, where C is constant of integration.

or $v^2=-2ag\cos\theta+C$, where v is the velocity $a\,(d\theta/dt)$ at P.

Initially at A, $\theta=0$ and $v=\sqrt{(\frac{1}{2}ag)}$, (given)

$\therefore$ $\frac{1}{2}ag=-2ag+C$ or $C=(5/2)\,ag$

$\therefore$ $v^2=-2ag\cos\theta+(5/2)\,ag$ ...(iii)

Substituting this value of v^2 in (ii) we have

$R=mg\cos\theta-(m/a)\,[-2ag\cos\theta+(5ag/2)]$

or $R=mg\,[3\cos\theta-(5/2)]$ ...(iv)

At the point where the particle leaves the circle, the reaction $R=0$ and let $\theta=\theta_1$.

Then from (iv) we get $0=mg\,[3\cos\theta_1-(5/2)]$ or $\cos\theta_1=5/6$ ...(v)

$\therefore$ The depth of this point below the highest point A

$=a-a\cos\theta_1$, where $\cos\theta_1=(5/6)$

$=a-a\,[(5/6)]=a/6$, Ans.

Ex. 1 (b). If a heavy particle slides down the outside of a smooth vertical circle, starting from the highest point A, find out the angular distance from A.

(i) of the point where the particle leaves the circle,

(ii) of the point where the pressure on the curve is half that at the highest point,

(iii) also calculate the maximum kinetic energy acquired by the particle during its motion on the circle.

Sol. Refer Fig. 22 Page 30 of this chapter.

As in § 5 Pages 30—31 of this chapter we can prove that $v^2=2ag\,(1-\cos\theta)$ ...(i) and $R=mg\,(3\cos\theta-2)$, ...(ii) where the symbols have the usual meaning as given in § 5 Pages 30—31 of this chapter.

(i) Let θ_1 be the value of angle θ at the point where the particle leaves the circle then from (ii), we get

$0=mg\,(3\cos\theta_1-2)$ or $\cos\theta_1=\frac{2}{3}$

or $\theta_1=\cos^{-1}\frac{2}{3}$, is the required angle.

(ii) At the highest point $\theta=0$

$\therefore$ From (ii) the pressure at the highest point

$=mg\,(3\cos 0-2)=mg\,(3-2)=mg$

$\therefore$ In this part according to the problem we have

$mg\,(3\cos\theta-2)=\frac{1}{2}\,(mg)$

or $2\,(3\cos\theta-2)=1$ or $6\cos\theta=5$ (Note)

or $\cos\theta=(5/6)$ or $\theta=\cos^{-1}(5/6)$ is the required angle.

(iii) Also from (i) we get $v^2=2ag\,(1-\cos\theta)$

And v is max. when the particle leaves the circle, as v increases in this case with time *i.e.* v is max. when $\cos\theta_1=\frac{2}{3}$

i.e. $(\max.\ v)^2=2ag\,(1-\frac{2}{3})=\frac{2}{3}ag$

$\therefore$ Max. value of kinetic energy $=\frac{1}{2}m\,(v^2)=\frac{1}{2}m\,(\frac{2}{3}ag)=\frac{1}{3}amg$

**Ex. 2. A particle moves under gravity in a vertical circle *sliding down the convex side of a smooth circular arc. If the initial velocity is that due to the starting point from a height h above the centre, show that it will fly off the circle when at a height $\frac{2}{3}h$ above the centre.* (Lucknow 88)

Sol. A is the highest point of the circle with centre O and C is the point of start. Let $\angle AOC = \alpha$, then if CE be the perpendicular from C on AO, we get $OE = a \cos \alpha$, where $OA = a$. ND is a line at a height h above the centre O. From C draw CD perpendicular to ND. Then $CD = (h - a \cos \alpha)$. Initial velocity of the particle

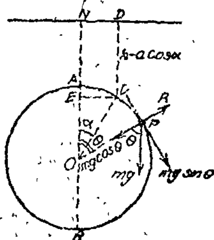
i.e. velocity at C

$= \sqrt{(2g \times DC)}$

$= \sqrt{\{2g\ (h - a \cos \alpha)\}}$

After time t, let the particle be at P, such that $\angle AOP = \theta$ and arc $AP = s$.

If m be the mass of particle, then the equations of motion in the tangential and inward drawn normal sense are



(Fig. 24)

$$m \cdot \frac{d^2s}{dt^2} = mg \sin \theta \quad \text{...(i)} \quad \text{and} \quad m \frac{v^2}{\rho} = mg \cos \theta - R \quad \text{...(ii)}$$

Also arc $AP = s = a\theta$. $\therefore \frac{d^2s}{dt^2} = a \frac{d^2\theta}{dt^2}$.

$\therefore$ from (i) we get $a\ (d^2\theta/dt^2) = g \sin \theta$

Multiplying both sides by $2a\ (d\theta/dt)$ and integrating we get

$(a\ d\theta/dt)^2 = -2ag \cos \theta + C$, where C is constant of integration

or $v^2 = -2ag \cos \theta + C$, where v is the velocity $a\ (d\theta/dt)$

At C, $v = \sqrt{\{2g\ (h - a \cos \alpha)\}}$ and $\theta = \alpha$.

So $2g\ (h - a \cos \alpha) = -2ag \cos \alpha + C$ or $C = 2gh$

$\therefore v^2 = 2gh - 2ag \cos \theta$

Substituting this value of v^2 in (ii) we have

$m\ (2gh - 2ag \cos \theta)/a = mg \cos \theta - R$, since $\rho = a$

or $R = mg \cos \theta - (m/a)\ (2gh - 2ag \cos \theta) = 3mg \cos \theta - (2mgh/a)$

At the point where the particle leaves the circle, the reaction $R = 0$ and let $\theta = \theta_1$.

$\therefore 0 = 3mg \cos \theta_1 - (2mgh/a)$ or $\cos \theta_1 = 2h/3a$.

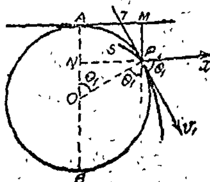
The height of this point above centre $= a \cos \theta_1 = \frac{2}{3}h$.

**Ex. 3. A particle is placed at the highest point of a smooth vertical circle of radius a and is allowed to slide down starting with

a negligible velocity. Prove that it will leave the circle after describing vertically a distance equal to one third of the radius. Find the position of the directrix and the focus of the parabola subsequently described and show that its latus rectum is (16/27) a. *(Avadh 83)*

Sol. After time t let the particle be at a point whose angular distance is θ from the highest point A. Then the equation of motion of the particle in the tangential and inward drawn normal directions

are $m.\dfrac{d^2s}{dt^2}=mg\ \sin\theta$ ...(i)



(Fig. 25)

and $m.\dfrac{v^2}{\rho}=mg\cos\theta-R$ ...(ii)

Also arc $s=a\theta$,

$\therefore\ \dfrac{d^2s}{dt^2}=a\dfrac{d^2\theta}{dt^2}$

$\therefore$ from (i) we get

$a\,(d^2\theta/dt^2)=g\ \sin\theta$

Integrating $(a\ d\theta/dt)^2$

$=-2ag\cos\theta+C$, where C is constant of integration.

At the highest point A, $\theta=0$ and velocity $a\ (d\theta/dt)=0$

$\therefore\ C=2ag$ and we have

$a\,(d\theta/dt)^2=-2ag\cos\theta+2ag$, where $a\,(d\theta/dt)=$velocity$=v$ (say)

or $$v^2=2ag\ (1-\cos\theta) \quad ...(iii)$$

Substituting this value of v^2 in (ii) we get

$R=mg\cos\theta-2mg\ (1-\cos\theta)$, $\because\ \rho=a$, radius of circle

or $$R=mg\ (3\cos\theta-2)$$

Let the particle leave the circle at P where $\angle\ AOP=\theta_1$.

Then at P, we have $R=0$ and $\theta=\theta_1$.

$\therefore$ from (iv), $0=mg\ (3\cos\theta_1-2)$. or $\cos\theta_1=\frac{2}{3}$ ...(v)

$\therefore$ vertical distance of P below $A=AN=OA-ON$

$=a-a\cos\theta_1=a-\frac{2}{3}a=\frac{1}{3}a=\frac{1}{3}$ (radius of the circle).

Let v_1 be the velocity at P *i.e.* $\theta=\theta_1$ then from (iii) we get

$v_1^2=2ag\ (1-\cos\theta_1)=2ag\ (1-\frac{2}{3})=\frac{2}{3}ag.$ ...(vi)

The velocity v_1 acts along the tangent at P (as shown in the fig.) making an angle θ_1 below the horizontal line through P. From the point P the particle starts to describe a parabolic path with velocity of projection as v_1 and angle of projection as $-\theta_1$.

Latus rectum of this parabola $=\dfrac{\text{"}2v^2\cos^2\alpha\text{"}}{g}=\dfrac{2v_1^2\cos^2\theta_1}{g}$

$=(2/g)\ (\frac{2}{3}ag)\ [(4/9)]$, from (v) and (vi)

$=(16/27)\ a.$ Hence proved.

Also we know that the velocity of a particle describing a parabolic path at any point on its path is that due to fall from the level of the directrix. Now velocity at $P = v_1 = \sqrt{(\frac{2}{3}ag)}$, from (vi).

Let depth of P below the directrix be h.

Then from the above property we have

$$\sqrt{(\tfrac{2}{3}ag)} = \sqrt{(2gh)} \quad \text{or} \quad h = \tfrac{1}{3}a = AN.$$

i.e. the directrix is a horizontal line AM through the highest point A of the circle.

From P draw PM perpendicular to the directrix.

Also PT is the tangent to the parabolic path at P.

If S be the focus of the parabola, then $\angle SPT = \angle TPM$.

Also $PS = PM = AN = \frac{1}{3}a$.

$\therefore$ From P draw a line PS making $\angle SPT = \angle TPM$ and cut off $PS = PM = \frac{1}{3}a$. This gives the position of the focus S.

Ex. 4 (a). If in Ex. 4 above, vertical circle be replaced by a sphere, prove that similar results will be obtained. *(Meerut 93)*

Sol. Do as Ex. 4 above.

Ex. 4 (b). A heavy particle is allowed to slide down a smooth vertical circle of radius $27a$ from rest at the highest point. Show that on leaving the circle it moves in a parabola of latus rectum $16a$. *(Allahabad 86 ; Kanpur 86 ; Lucknow 87)*

Sol. As in Ex. 4 above we can prove that the velocity v and reaction R of the particle at a point whose angular distance is θ from the highest point of the circle of radius r are given by

$$v^2 = 2rg\,(1 - \cos\theta) \quad \text{...(i)}$$

and $$R = mg\,(3\cos\theta - 2). \quad \text{...(ii)}$$

Let the particle leave the circle at a point P where $\theta = \theta_1$ (say) and $v = v_1$. Then at P, the reaction $R = 0$ and $\theta = \theta_1$.

$\therefore$ From (ii) we get $0 = mg\,(3\cos\theta_1 - 2)$ or $\cos\theta_1 = \frac{2}{3}$ ...(iii)

$\therefore$ from (i), $v_1^2 = 2\,(27a)\,g\,[1 - \frac{2}{3}]$, from (iii) and $r = 27a$ (given)

or $$v_1^2 = 18ag. \quad \text{...(iv)}$$

From the point P the particle starts to describe parabolic path with velocity of projection v_1 and angle of projection θ_1 (measured below the horizontal line through P). Therefore the required latus rectum of the parabola $= \dfrac{\text{"}2u^2\cos^2\alpha\text{"}}{g} = \dfrac{2v_1^2\cos^2(-\theta_1)}{g}$

$= (2/g)\,(18ag)\,[(4/9)]$, from (iii) and (iv).

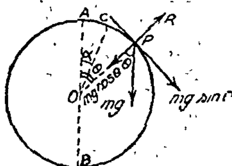
$= 16a$. **Ans.**

***Ex. 5. A particle moves along the outside of the arc of a smooth vertical circle. If the particle starts from rest down the arc from a point C, show that it leaves the circle at Q where $\cos\theta_1 = (2/3)\cos\alpha$, α and θ_1 being the angular distances of C and Q respectively from the highest point A.** *(Kanpur 89 ; Rohilkhand 88)*

Sol. AB is the vertical diameter of the circle with centre O.

The particle starts from C from rest such that $\angle AOC=\alpha$ (given). Let $OA=a$.

After time t let the particle be at P, such that $\angle AOP=\theta$ and arc $AP=s$. If m be the mass of the particle, then the equations of motion of the particle in the tangential and inward drawn normal directions are



(Fig. 26)

$$m.\frac{d^2s}{dt^2}=mg\sin\theta \quad ...(i)$$

and $$m.\frac{v^2}{\rho}=mg\cos\theta-R \quad ...(ii)$$

Also arc $AP=s=a\theta$.

$\therefore \quad (d^2s/dt^2)=a\,(d^2\theta/dt^2)=g\sin\theta$, from (i)

Integrating $(a\,d\theta/dt)^2=-2ag\cos\theta+k$, where k is constant of integration.

Initially *i.e.* at C, $\theta=\alpha$ and $a\,(d\theta/dt)=$velocity$=0$

$\therefore \quad 0=-2ag\cos\alpha+k$ or $k=2ag\cos\alpha$

$\therefore \quad (a\,d\theta/dt)^2=2ag\,(\cos\alpha-\cos\theta)$ ...(iii)

or $\quad v^2=2ag\,(\cos\alpha-\cos\theta)$, where v is the velocity at P.

Substituting this value of v in (ii) we have

$R=mg\cos\theta-2mg\,(\cos\alpha-\cos\theta)$, since $\rho=$radius a

or $\quad R=mg\,(3\cos\theta-2\cos\alpha)$. ...(iv)

Let the particle leave the circle at Q, where $\theta=\theta_1$ then from (iv) putting $R=0$ and $\theta=\theta_1$ we have $\cos\theta_1=\frac{2}{3}\cos\alpha$. Hence proved.

****Ex. 6. A heavy particle slides under gravity down the inside of a smooth vertical tube held in a vertical plane. It starts from the highest point with velocity $\sqrt{(2ag)}$, where a is the radius of the circle, prove that when in the subsequent motion the vertical component of the accelerttion is a maximum, the pressure on the curve is equal to twice the weight of the particle.**

(Bundelkhand 91, Garhwal 90 ; Meerut H 92 ; Rohilkhaud 87)

Sol. Let the particle of mass m (say) be at P after time t, such that arc $AP=s$ and $\angle AOP=\theta$, where A is the highest point of the tube. Then the equations of the motion of particle in the tangential and inward drawn normal directions are

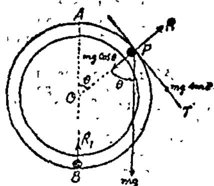
$$m.\frac{d^2s}{dt^2}=mg\sin\theta \quad ...(i) \quad \text{and } m.\frac{v^2}{a}=mg\cos\theta-R \quad ...(ii)$$

(see figure, here the ball should touch the inner circle at P).

Also as $\quad s=a\theta, \quad$ so $\quad (d^2s/dt^2)=a\,(d^2\theta/dt^2)$

Hence from (i) we get $\quad a\,(d^2\theta/dt^2)=g\sin\theta$.

Integrating, $(a\ d\theta/dt)^2=-2ag\cos\theta+C$, where C is constant of integration.



(Fig. 90)

At A, $\theta=0$ and velocity $a\ (d\theta/dt)=\sqrt{(2ga)}$

$\therefore\ 2ag=-2ag+C$

or $C=4ag$

$\therefore\ a\ (d\theta/dt)^2=-2ag\cos\theta+4ag$, where $a\ (d\theta/dt)=v$ = velocity

or $v^2=4ag-2ag\cos\theta$ ...(iii)

Substituting in (ii) we get the reaction $R=-m\ (4g-2g\cos\theta)+mg\cos\theta=mg\ (3\cos\theta-4)$ ...(iv)

We know that at P there are two components of acceleration viz. d^2s/dt^2 in the direction PT and v^2/ρ or v^2/a in the direction PO.

If f be the vertical component of acceleration at P, then

$$f=\frac{v^2}{a}\cos\theta+\frac{d^2s}{dt^2}\sin\theta \quad \text{(Note)}$$

$$=(4g-2g\cos\theta)\cos\theta+g\sin^2\theta, \text{ from (i) and (iii)}$$

or $f=4g\cos\theta-2g\cos^2\theta+g\sin^2\theta.$

$\therefore\ df/d\theta=-4g\sin\theta+4g\cos\theta\sin\theta+2g\sin\theta\cos\theta$

$=2g\sin\theta\ (3\cos\theta-2)$

If f is maximum, then $df/d\theta=0$ and $d^2f/d\theta^2$=negative.

Equating $df/d\theta$ to zero, we have $\sin\theta=0$ or $\cos\theta=\frac{2}{3}$.

Now $\sin\theta=0$ corresponds to the point of start A.

$\therefore$ At the point where f is maximum we have $\cos\theta=\frac{2}{3}$ and therefore from (iv) we have the relation R

$=mg\ [-4+3\ (\frac{2}{3})]=-2mg.$

= twice the weight of the particle.

and negative sign shows that the reaction has changd direction at that point.

***Ex. 7. A particle slides down the arc of smooth vertical circle of radius a after being slightly displaced from rest at the highest point ; find where it will leave the circle and prove that it will strike a horizontal plane through the lowest point of the circle at a distance $5\ (\sqrt{5}+4\sqrt{2})\ a/27$ from the vertical diameter.**

Sol. We can prove as in Ex. 3 Page 33 that the velocity v and reaction R between the particle and the circle at a point whose angular distance θ from the highest point A are given by

$v^2=2ag\ (1-\cos\theta)$ ...(i)

and $R=mg\ (3\cos\theta-2)$. ...(ii)

Let the particle leave the circle at P where $\theta=\theta_1$.

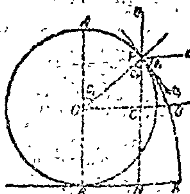
Then at P, the reaction $R=0$ and $\theta=\theta_1$ so we have from (ii)

$0=mg\ (3\cos\theta_1-2)$ or $\cos\theta_1=\frac{2}{3}$. ...(iii)

If v_1 be the velocity of the particle at P, then from (i) we get

$$v_1^2=2ag\ (1-\cos\theta_1)$$
$$=2ag\ (1-\tfrac{2}{3}),\ \text{from (iii)}$$
$$=\tfrac{2}{3}ag. \quad \text{...(iv)}$$

After leaving the circle at P the particle describes a parbolic path with velocity of projection v_1 and angle of projection θ_1 (below the horizontal line Px through P) as shown in the figure.



(Fig. 28)

Referred to P as origin and Px and Py as co-ordinate axes (as shown in figure) the equation of the parabolic path is

$$y=x\tan(-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(-\theta_1)} \quad \text{(Note)}$$

or $y=-x\tan\theta_1-\dfrac{gx^2}{2v_1^2\cos^2\theta_1}$ or $y=-x\left(\dfrac{\sqrt{5}}{2}\right)-\dfrac{gx^2}{2\,(\frac{2}{3}ag)\,[(4/9)]}$,

from (iii) and (iv). Also $\tan\theta_1=\sqrt{5}/2$, from (iii)

or $$y=-\frac{x\sqrt{5}}{2}-\frac{27x_1^2}{16a} \quad \text{...(v)}$$

Let the particle strike the horizontal plane through the lowest point B at K. Let the co-ordinates of K be $(x_1, -y_1)$.

Then $x_1=NK$ and $y_1=PN=PC+CN=a\cos\theta_1+a$

$$=\tfrac{2}{3}a+a\ [\text{...from (iii)}]=5a/3,$$

$\therefore$ K is the point $[x_1, -5a/3]$. Since K lies on (v) so we have

$$-\frac{5a}{3}=-\frac{-x_1\sqrt{5}}{2}-\frac{27x_1^2}{16a}, \text{ or } 81x^2+24\sqrt{5}ax_1-80a^2=0$$

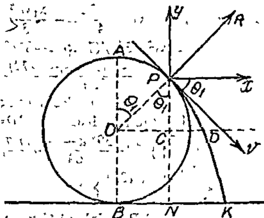
or $$x_1=\frac{-24\sqrt{5}a\pm\sqrt{(2880a^2+25920a^2)}}{162}=\frac{-4\sqrt{5}+20\sqrt{2}}{27}a$$

(x_1 being positive, negative value is inadmissible)

$\therefore$ Required horizontal distance $=BK=BN+NK$

$$=a\sin\theta_1+x_1 \text{ (see figure)}$$
$$=a.\frac{\sqrt{5}}{3}+\left(\frac{-4\sqrt{5}+20\sqrt{2}}{72}\right)a, \text{ from (iii) } \sin\theta_1=\frac{\sqrt{5}}{3}$$
$$=\left(\frac{5\sqrt{5}+20\sqrt{2}}{27}\right)a=\frac{5(\sqrt{5}+4\sqrt{2})a}{27} \quad \text{Hence proved.}$$

Ex 8 A body [illegible] radius a [illegible] of a smooth circle of It will le[illegible] $\frac{1}{2}\sqrt{(ag)}$; find where [illegible] a horizontal plane through the centre of the circle at a distance $\dfrac{9\sqrt{(39)}+7\sqrt{7}}{64}$ a, from the centre.



(Fig. 29)

Sol. Let m be the mass of the particle. Let the particle after time t be at a point, whose angular distance from the highest point A of the circle is θ. Then the equation of motion of the particle in the tangential and inward drawn normal directions are $m.(d^2s/dt^2)=mg \sin \theta$ ...(i)

and $m (v^2/\rho)=mg \cos \theta - R$, ...(ii)

where R is the normal reation between the particle and the circle.

Also arc $s=a\theta$, so $(d^2s/dt^2)=a (d^2\theta/dt^2)$

$\therefore$ from (i) we get $a (d^2\theta/dt^2)=g \sin \theta$.

Integrating, $(a\, d\theta/dt)^2=-2ag \cos \theta + C$, where C is constant of integration.

Initially at A, velocity $a\, d\theta/dt=\frac{1}{2}\sqrt{(ag)}$ and $\theta=0$

$\therefore \frac{1}{4} ga=-2ga+C$ or $C=(9/4)\, ag$

$\therefore (ad\theta/dt)^2=(9/4)ag-2ag \cos \theta$, where $ad\theta/dt=$velocity v (say)

or $v^2=(9/4)\, ag-2ag \cos \theta$ ...(iii)

Substituting this value of v^2 in (ii) we have

$R=mg \cos \theta-(m/a)\, [(9/4)\, ag-2ag \cos \theta]$, $\because$ ρ=radius a

or $R=3mg \cos \theta-\frac{9}{4}mg$ ...(iv)

Let the particle leave the circle at P, where $\theta=\theta_1$ (say). Then at P we have $R=0$ and $\theta=\theta_1$.

$\therefore$ from (iv), $0=3mg \cos \theta_1-(9/4)\, mg$ or $\cos \theta_1=\frac{3}{4}$ ...(v)

Let v_1 be the velocity of the particle at P *i.e.* at $\theta=\theta_1$ then from (iii), $v_1^2=(9/4)ag-2ag \cos \theta_1=(9/4)\, ag-2ag\, (\frac{3}{4})=\frac{3}{4}ag$ ...(vi)

Beyond P the particle traces out a parabolic path with v_1 as velocity of projection and $-\theta_1$ as the angle of projection. (Here the angle of projection is below the horizontal line through P).

$\therefore$ Referred to P as origin and horizontal and vertical lines Px and Py through P as axes, the equation of the parabolic path is

$$y=x \tan (-\theta_1)-\frac{gx^2}{2v_1^2 \cos^2 (-\theta_1)}=-x \tan \theta_1-\frac{gx^2}{2v_1^2 \cos^2 \theta_1}$$

or
$$y=-x\, (\sqrt{7}/3)-\frac{gx^2}{2\, (\frac{3}{4}ag)[(9/16)]},$$

from (v) and (vi), also from (v) $\tan \theta_1=\frac{1}{3}\sqrt{7}$

or
$$y=-\frac{x\sqrt{7}}{3}-\frac{32\times x^2}{27a} \quad \text{...(vii)}$$

Let this parabolic path meet the horizontal plane through the centre O at D. Let the co-ordinates of D be $(x_1, -y_1)$, where $x_1=CD$ and as shown in figure $y_1=PC=OP \cos \theta_1=a\, (\frac{3}{4})=\frac{3}{4}a$.

$\therefore$ D is the point $(x_1, -\frac{3}{4}a)$ and as D lies on (vii)

so we have from (vii), $\dfrac{-3a}{4}=-\dfrac{x_1\sqrt{7}}{3}-\dfrac{32x_1^2}{27a}$

or $\quad 128x_1^2+36\sqrt{(7)}\,x_1\,a-81a^2=0$

or $\quad x_1=\dfrac{-36\sqrt{7}a\pm\sqrt{\{(9072+41472)\,a^2\}}}{256}=\dfrac{-9\sqrt{(7)}\,a+9\sqrt{(39)}\,a}{64}$,

neglecting negative sign as x_1 is not negative

Now the required distance $=OD=OC+CD=a\sin\theta_1+x_1$

$=a\left(\dfrac{\sqrt{7}}{4}\right)+\dfrac{9\sqrt{(39)}\,a-9\sqrt{(7)}\,a}{64}$, from (v) $\sin\theta_1=\frac{1}{4}\sqrt{7}$

$=\left\{\dfrac{7\sqrt{7}+9\sqrt{(39)}}{64}\right\}a.$ Hence proved.

§ 6. Use of Principle of Conservation of Energy.

If the curve is smooth and the external force acting on the particle belongs to the conservative system, then the principl of conservation of energy can be used to obtain the velocity at a point.

As the normal reaction R always remains at right angles to the tangent *i.e.* to the direction of motion, so the reaction R does not work in any displacement along the curve.

Let v, v_1 be the velocities ; V, V_1 be the potential energies of the particle at two points P and P_1 on the curve, then by the principal of conservation of energy (See Chapter on Impulse, Work and Energy) viz. K.E.+P.E.=constant, we have ...(i)

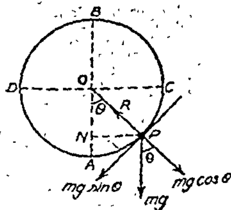
$$\tfrac{1}{2}mv^2+V=\tfrac{1}{2}mv_1^2+V_1.$$

Also we know that change in kinetic energy = work done.

$$\therefore\ \tfrac{1}{2}mv^2-\tfrac{1}{2}mv_1^2=\int_P^{P_1}T\,ds.$$

Solved Examples on § 6.

***Ex. 1.** A particle of mass m is projected with a velocity u from the lowest point A of a smooth fixed vertical circle of radius r and centre O. Find the velocity of the particle when it is at P, where $\angle AOP=\theta$, with the help of Principle of Work-Energy.



(Fig. 30)

Sol. Let v be the velocity of the particle at P.

Then K. E. at $P=\frac{1}{2}mv^2$

and K. E. at $A=\frac{1}{2}mu^2$

$\therefore$ Change in K. E.

$=\frac{1}{2}mv^2-\frac{1}{2}mu^2$ ...(i)

Also the only force which does work when the particle moves from A to P is its weight mg. (Note)

(See the chapter on Work, Energy and Impulse)

From P draw PN perpendicular to OA.

Then $ON=OP\cos\theta=r\cos\theta$, $\because OP=r$.

$\therefore AN=OA-ON=r-r\cos\theta=r\ (1-\cos\theta)$

$\therefore$ Work done by the weight mg of the particle as it moves from A to $P=-mg\ AN=-mgr\ (1-\cos\theta)$ ...(ii)

Now by the Principle of Work and Energy viz. change in K.E.=work done, from (i) and (ii) we have

$$\tfrac{1}{2}mv^2-\tfrac{1}{2}mu^2=-mgr\ (1-\cos\theta)$$

or $$v^2-u^2=-2gr\ (1-\cos\theta)=2gr\cos\theta-2gr$$

or $$v^2=u^2+2gr\cos\theta-2gr.$$ **Ans.**

Ex. 2. With the help of Principle of Conservation of energy determine the period for small amplitude of a simple pendulum of length l and mass m.

Sol. Let θ be the small angle which the string OP makes with the vertical OA at any instant.

We know that principle of conservation of energy is

K.E.+P.E.=constant,

where K.E. and P.E. stands for kinetic and potential energies.

Let A be the standard position for potential energy. Let are $AP=s$ and $OA=l$, then $s=l\theta$.

$\therefore \dfrac{ds}{dt}=l\dfrac{d\theta}{dt}$ i.e. the velocity of the particle of mass m at P

$=ds/dt=l\dot{\theta}$, where $\dot{\theta}=d\theta/dt$. ...(i)

$\therefore$ K.E. at $P=\tfrac{1}{2}m\cdot(l\dot{\theta})^2=\tfrac{1}{2}ml^2\dot{\theta}^2$

Also P.E. at P

$=mg$ (height of P above A) (Note)

$=mg\ (l-l\cos\theta)$, see last example.

$\therefore$ From Principle of conservation of energy, we have

$\tfrac{1}{2}ml^2\dot{\theta}^2+mg\ (l-l\cos\theta)=$constant.

Differentiating both sides with respect to t, we get

$\tfrac{1}{2}ml^2\, 2\dot{\theta}\,\ddot{\theta}+mg\ (l\sin\theta)\ \dot{\theta}=0$

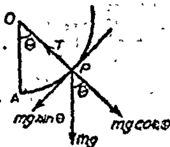
or $l\ddot{\theta}+g\theta=0$, since θ is small

so $\sin\theta$ can be taken as θ

or $\ddot{\theta}=-\left(\dfrac{g}{l}\right)\theta,$

which represents S. H. M. and hence its time period

$=\dfrac{2\pi}{\sqrt{(g/l)}}=2\pi\sqrt{\left(\dfrac{l}{g}\right)}$ **Ans.**



(Fig. 31)

§ 7. Simple Pendulum.

(Avadh 86; Gorakhpur 91, 89; Purvanchal 90)

Definition. *A heavy particle is tied to one end of a light inextensible string the other end of which is fixed and oscillating in a vertical plane through a small angle. Such a system is called a simple pendulum.*

Refer Fig. 31 Page 41.

O is the fixed point and P is the position of the particle at time t when the string OP is inclined at an angle θ with the vertical line OA, where A is the lowest position of the particle. Let arc $AP = s$, then if l be the length of the string, we have

$$s = l\theta, \text{ whence } \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2} \quad \text{...(i)}$$

The forces acting on the particle at P are its weight mg acting vertically downwards and the tension T in the string OP in the direction OP. So the equation of motion of the particle in the tangential and inward drawn normal senses are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{...(ii) and } m\frac{v^2}{\rho} = T - mg\cos\theta \quad \text{...(iii)}$$

From (i) and (ii) we have $l\,(d^2\theta/dt^2) = -g\sin\theta$;

or $l\,(d^2\theta/dt^2) = -g\theta$, since $\sin\theta = \theta$ to a first approximation

or $\dfrac{d^2\theta}{dt^2} = -\left(\dfrac{g}{l}\right)\theta$, which is of the standard form of S.H.M.

Hence the motion of the particle is simple harmonic and if T be the period of a small complete oscillation, then

$$T = 2\pi\sqrt{(l/g)}. \quad \text{...(iv)}$$

If the oscillation is not small, then from (i) and (ii) as before we get $l\,(d^2\theta/dt^2) = -g\sin\theta$.

Integrating, $[l\,(d\theta/dt)]^2 = 2gl\cos\theta + C$, where C is constant of integration.

If the pendulum oscillates through an angle α on either side of the vertical line OB, then we have $l\,(d\theta/dt) = 0$ when $\theta = \alpha$

Hence $0 = 2gl\cos\alpha + C$ or $C = -2gl\cos\alpha$

$\therefore$ $[l\,(d\theta/dt)^2] = 2gl\cos\theta - 2gl\cos\alpha$

or $d\theta/dt = \sqrt{(2g/l)}\sqrt{(\cos\theta - \cos\alpha)}$

or $dt = \sqrt{(l/2g)}\,.\,\dfrac{d\theta}{\sqrt{(\cos\theta - \cos\alpha)}}$

Integrating, $t = \sqrt{(l/2g)}\displaystyle\int_{\theta=0}^{\alpha} \frac{d\theta}{\sqrt{(\cos\theta - \cos\alpha)}}$

where t is the time taken from the lowest point B to the extreme position A.

or $$t=\sqrt{(l/2g)}\int_{\theta=0}^{\alpha}\frac{d\theta}{\sqrt{(2\sin^2\frac{1}{2}\alpha-2\sin^2\frac{1}{2}\theta)}},\quad \because \cos\theta=1-2\sin^2\tfrac{1}{2}\theta$$

$$=\tfrac{1}{2}\sqrt{(l/g)}\int_{\phi=0}^{\pi/2}\frac{2\sin\frac{1}{2}\alpha\cos\phi\,d\phi}{\cos\frac{1}{2}\theta\sin\frac{1}{2}\alpha\cos\phi},$$

putting $\sin\frac{1}{2}\theta=\sin\frac{1}{2}\alpha\sin\phi$ or $\frac{1}{2}\cos\frac{1}{2}\theta\,d\theta=\sin\frac{1}{2}\alpha\cos\phi\,d\phi$

or $$t=\sqrt{(l/g)}\int_{\phi=0}^{\pi/2}\frac{d\phi}{\cos(\theta/2)}=\sqrt{(l/g)}\int_0^{\pi/2}\frac{d\phi}{\sqrt{[1-\sin^2\frac{1}{2}\alpha\sin^2\phi]}}$$

$$=\sqrt{(l/g)}\int_0^{\pi/2}(1-\sin^2\tfrac{1}{2}\alpha\sin^2\phi)^{-1/2}\,d\phi$$

$$=\sqrt{(l/g)}\int_0^{\pi/2}\left(1+\tfrac{1}{2}\sin^2\tfrac{1}{2}\alpha\sin^2\phi+\frac{1.3}{2.4}\sin^4\tfrac{1}{2}\alpha\sin^4\phi+\ldots\right)d\phi$$

$$=\sqrt{(l/g)}\left[\left(\phi\right)_0^{\pi/2}+\tfrac{1}{2}\sin^2\tfrac{1}{2}\alpha.\tfrac{1}{2}\,\tfrac{1}{2}\pi+\frac{1.3}{2.4}\sin^4\tfrac{1}{2}\alpha\,\tfrac{3}{4}.\tfrac{1}{2}.\tfrac{1}{2}\pi+\ldots\right]$$

$$=\tfrac{1}{2}\pi\sqrt{(l/g)}\left[1+\frac{1}{2^2}\sin^2\tfrac{1}{2}\alpha+\left(\frac{1.3}{2.4}\right)^2\sin^4\tfrac{1}{2}\alpha+\ldots\right]$$

If T_1 be time of one complete oscillation *i.e.* period, then

$$T_1=4t=2\pi\sqrt{(l/g)}\left[1+\frac{1}{2^2}\sin^2\tfrac{1}{2}\alpha+\ldots\right]$$

or $$T_1=2\pi\sqrt{(l/g)}\,[1+(1/16)\,\alpha^2],$$

neglecting powers of α higher than second

$$=T\,[1+(1/16)\,\alpha^2],\text{ from (iv)}$$

Hence $T_1=T\,[1+(1/16)\,\alpha^2]$. ...(v)

Solved Examples on § 7. *(Gorakhpur 91; Purvanchal 90, 88)*

Ex. 1. A simple pendulum has a period T. When the string is lengthened by a small fraction $(1/n)$ of its length, the period becomes T'. Show that approximately $1/n=2\,(T'-T)/T$.

Sol. Let l be the length of the string. Then the period T is given by

$$T=\pi\sqrt{(l/g)} \quad ...(i)$$

When the string is lengthened by a small fraction $(1/n)$ of its length, its length becomes $l+(1/n)\,l$ *i.e.* $[1+(1/n)]\,l$ (Note)

$\therefore$ The corresponding period T' is given by

$$T'=\pi\sqrt{\left[\frac{\{1+(1/n)\}\,l}{g}\right]}=\pi\sqrt{\left(\frac{l}{g}\right)}\left(1+\frac{1}{n}\right)^{1/2}$$

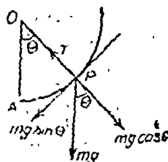
$$=T\left(1+\frac{1}{n}\right)^{1/2},\text{ from (i)}$$

$$=T\,[1+(1/2n)],\text{ approx.}$$

$\therefore$ $T'-T=T/(2n)$ or $2\,(T'-T)/T=1/n$. Hence proved.

Ex. 2. *Show that if the tension of the string when the bob is in its lowest position is k times the tension when the bob is in its highest position, the velocities in this position being u_1 and u_2, respectively; then* $u_1^2/u_2^2=(5k+1)/(k+5)$.

Sol. Let O be the fixed point, a the length of the string and m be the mass of the bob. At any time t, let the bob be at P, such that $\angle POA=\theta$, where OA is the vertical line through O. The force acting on the bob at P are it's weight mg acting vertically downwards and tension T acting towards O.



(Fig. 32)

$\therefore$ The equations of the motion of bob in the tangential and inward drawn normal directions are

$$m\, d^2s/dt^2=-mg \sin \theta.$$

or $$d^2s/dt^2=-g \sin \theta \quad \ldots(i)$$

and $$m\,(v^2/\rho)=T-mg \cos \theta \quad \ldots(ii)$$

Also if arc $AP=s$, then $s=a\theta$ or $\dfrac{d^2s}{dt^2}=a\dfrac{d^2\theta}{dt^2}$

$\therefore$ From (i) we get $a\,(d^2\theta/dt^2)=-g \sin \theta$

Integrating we get $(a\, d\theta/dt)^2=2ga \cos \theta+C$, where C is constant of integration.

When $\theta=0$ *i.e.* at the lowest point velocity $a\, d\theta/dt=u_1$ (given)

then $u_1^2=2ag+C$ or $C=u_1^2-2ga$.

$\therefore$ $(a\, d\theta/dt)^2=2ga \cos \theta+u_1^2-2ga \quad \ldots(iii)$

$\therefore$ At the highest point, velocity $a\, d\theta/dt=u_2$ (given) and $\theta=\pi$

$\therefore$ From (iii), $u_2^2=2ga \cos \pi+u_1^2-2ga$

or $$4ga=u_1^2-u_2^2. \quad \ldots(iv)$$

Also from (iii) we get

$T=m\,(v^2/\rho)+mg \cos \theta$, where $\rho=$ radius a

$=(m/a)\;(2ga \cos \theta+u_1^2-2ga)+mg \cos \theta$, from (iii) $a\,d\theta/dt=v$

or $$T=(m/a)\,(u_1^2+3ga \cos \theta-2ga). \quad \ldots(v)$$

Let T_1 and T_2 be the tensions in the string at the highest and lowest positions of the bob.

Then at the lowest point $\theta=0$, and from (v) we get

$$T_2=(m/a)\,(u_1^2+ga)$$

And at the highest point $\theta=\pi$, so we have

$$T_1=(m/a)\,(u_1^2-5ga)$$

And $\quad T_2=k\times T_1$ (given)

or $\quad u_1^2+ga=k\,(u_1^2-5ga)=k\,(u_2^2-ga)$, from (iv) (Note)

or $\quad k=\dfrac{u_1^2+ga}{u_2^2-ga}=\dfrac{4u_1^2+4ga}{4u_2^2-4ga}=\dfrac{4u_1^2+(u_1^2-u_2^2)}{4u_2^2-(u_1^2-u_2^2)}=\dfrac{5u_1^2-u_2^2}{5u_2^2-u_1^2}$

or $\quad k\,(5u_2^2-u_1^2)=5u_1^2-u_2^2$ or $u_2^2\,(5k+1)=u_1^2\,(5+k)$

or $\quad u_1^2/u_2^2=(5k+1)/(5+k)$. Hence proved.

****Ex. 3. A simple pendulum is started so as to make complete revolution in a vertical plane. Find the velocity of projection.**

In the subsequent motion w_1, w_2 are the greatest and least angular velocities, and T_1, T_2 are the greatest and least tensions. Prove that when the pendulum makes an angle θ with the vertical, the angular velocity is $[w_1^2\cos^2\frac{1}{2}\theta+w_2^2\sin^2\frac{1}{2}\theta]^{1/2}$, and that the tension is

$$T_1\cos^2\tfrac{1}{2}\theta+T_2\sin^2\tfrac{1}{2}\theta.$$ *(Bundelkhand 86)*

Sol. Refer Fig. 32 Page 44.

Let O be the fixed point, a the length of the string and m the mass of the bob of the pendulum.

At any time t, let the bob be at P, such that $\angle AOP=\theta$, where OA is the vertical line through O, the fixed point. The forces acting on the bob are its weight mg acting vertically downwards and the tension T acting towards O.

$\therefore$ The equations of motion in the tangential and inward drawn normal directions are

$$m\,(d^2s/dt)^2=-mg\sin\theta$$

or $\quad d^2s/dt^2=-g\sin\theta \quad$ ...(i)

and $\quad m\,(v^2/\rho)=T-mg\cos\theta \quad$ ...(ii)

If arc $s=a\theta$, then (i) becomes $a\,d^2\theta/dt^2=-g\sin\theta$

Integrating $\quad a\,(d\theta/dt)^2=2ga\cos\theta+C$,

where C is constant of integration.

Let velocity $a\,(d\theta/dt)=u$ at $\theta=0$ *i.e.* at the lowest position.

Then $\quad u^2=2ga+C$ or, $C=u^2-2ga$

$\therefore\ [a\,(d\theta/dt)]^2=u^2+2ga\cos\theta-2ga \quad$ ...(iii)

Substituting this value of velocity $a\,(d\theta/dt)$ *i.e.* v in (ii) we get

$(m/a)\,(u^2+2ga\cos\theta-2ga)=T-mg\cos\theta$, $\because\ \rho=$radius a

or $\quad T=(m/a)\,(u^2+3ga\cos\theta-2ga)$. $\quad$ ...(iv)

If the pendulum makes complete revolution, then tension T should not vanish before the particle reaches the highest point *i.e.* $\quad T\geqslant 0$ when $\theta=\pi$;

or $\quad u^2+3ga\cos\pi-2ga \geqslant 0 \quad$ or $\quad u^2 \geqslant 5ga$

or $\quad u \geqslant \sqrt{(5ga)}$ *i.e.* the least value of $u=\sqrt{(5ga)}$ for describing complete circle.

Also tension is greatest when $\theta=0$ *i.e.* at the lowest point and least when $\theta=\pi$ *i.e.* at the highest point.

$\therefore$ from (iv), $\quad T_1=(m/a)(u^2+ga)$, at $\theta=0$

and $\quad T_2=(m/a)(u^2-5ga)$, at $\theta=\pi$ ...(v)

Now tension T at any position is given by (iv) *i.e.*

$$T=(m/a)\ (u^2+3ga\cos\theta-2ga)$$
$$=(m/2a)\ [2u^2+6ag\cos\theta-4ag]$$
$$=(m/2a)\ [(u^2+ga)\ (1+\cos\theta)+(u^2+5ag)\ (1-\cos\theta)] \quad \text{(Note)}$$
$$=\tfrac{1}{2}\ [T_1\ (1+\cos\theta)+T_2\ (1-\cos\theta)], \text{ from (v)}$$
$$=\tfrac{1}{2}\ [T_1\ (2\cos^2\tfrac{1}{2}\theta)+T_2\ (2\sin^2\tfrac{1}{2}\theta)]$$
$$=T_1\cos^2\tfrac{1}{2}\theta+T_2\sin^2\tfrac{1}{2}\theta. \quad \text{Hence proved.}$$

Again we observe from (iii) that the angular velocity $d\theta/dt$ is greatest when $\theta=0$ and least when $\theta=\pi$.

Hence from (iii) we have

$(aw_1)^2=u^2+2ag-2ag$ *i.e.* $a^2w_1^2=u^2$ at $\theta=0$

and $\quad (aw_2)^2=u^2-2ga-2ga$ *i.e.* $a^2w_2^2=u^2-4ag$ at $\theta=\pi$ ...(vi)

If w be the angular velocity at any position P, then

$w=d\theta/dt$ and from (iii) we have

$$(aw)^2=u^2+2ga\cos\theta-2ga=\tfrac{1}{2}\ (2u^2+4ga\cos\theta-4ga)$$

or $\quad a^2w^2=\tfrac{1}{2}\ [u^2\ (1+\cos\theta)+(u^2-4ag)\ (1-\cos\theta)]$ (Note)

or $\quad =\tfrac{1}{2}\ [a^2w_1^2\ (1+\cos\theta)+a^2w_2^2\ (1-\cos\theta)]$, from (vi)

or $\quad w=[w_1^2\cos^2\tfrac{1}{2}\theta+w_2^2\sin^2\tfrac{1}{2}\theta]^{1/2}$. Hence proved.

Exercise

Choose the correct answer :—

The periodic time of simple pendulum is

(i) $2\pi\sqrt{(g/l)}$, (ii) $2\pi\sqrt{(l/g)}$,

(iii) $\sqrt{2\pi g/l}$ (iv) none of these.

(Hint : See § 7 Page 42). **Ans.** (ii)

§ 8. The Second's Pendulum.

Definition. A second's pendulum is that pendulum which oscillates from rest to rest in one second.

i.e. half of its time period $=1$ second.

In § 7 Pages 42—43 we have proved that the period of one complete oscillation in the case of simple pendulum is $2\pi\sqrt{(l/g)}$.

$\therefore$ For a second's pendulum, we have $\pi\sqrt{(l/g)}=1$.

**§ 9. To find whether the clock will go slow or fast if g and l change slightly—one or both.

In § 7 Pages 42—43 we have proved that the period of one complete oscillation is $2\pi\sqrt{(l/g)}$, so time of one beat or swing $=\pi\sqrt{(l/g)}$.

Let there be n beats or swings in a given interval of time T.

Then $T=n\pi\sqrt{(l/g)}$ or $n=(T/\pi)\sqrt{(g/l)}$ ...(i)

This shows that n depends upon g and l, so if one or both of them change n will also change.

Taking log of both sides of (i) we have,

$$\log n=\log (T/\pi)+\tfrac{1}{2}\log g-\tfrac{1}{2}\log l$$

i.e. $(1/n)\,\delta n=(1/2g)\,\delta g-(1/2l)\,\delta l$, ...(ii)

(T/π) being constant.

This relation (ii) determines how n changes when there is a slight change in g or l or both. (Lucknow 92)

Now following cases will arise—

(a) When only g changes (*i.e.* l remains constant *i.e.* $\delta l=0$)

Then from (ii) we get $(1/n)\,\delta n=(1/2g)\,\delta g$ ...(iii)

When g increases δg is positive and from (iii) δn is positive and when g decreases, δg is negative and from (iii) δn is negative. This means that there is increase or decrease in the number of beats according as the pendulum is taken to a place of more or less gravity *i.e.*, when g increases the clock goes fast and when g decreases the clock goes slow (l remaining constant).

(b) When only l changes (*i.e.* **g remains constant or $\delta g=0$**).

Then from (iii) we have $(1/n)\,\delta n=-(1/2l)\,\delta l$. ...(iv)

When l increases δl is positive and from (iv) δn is negative and when l decreases δl is negative and from (iv) δn is positive. This means that there is increase or decrease in the number of beats as the length l of the string is decreased or increased *i.e.* the clock will go fast or slow according as the length of the string is decreased or increased.

***(c) When the clock is taken to the top of the mountain.**

We know that outside the surface of the earth attraction varies inversely as square of the distance from the centre, so if r be the radius of the earth we have on the surface of the earth

$g=\mu/r^2$ or $\log g=\log \mu-2\log r$

i.e. $(1/g)\,\delta g=-(2/r)\,\delta r$.

∴ If the particle is taken from the surface of the earth to the the top of a mountain of height h (say), then $\delta r=h$, hence we have

$$(1/g)\,\delta g=-(2/r)\,h.$$

∴ from (iii) we have $(1/n)\,\delta n=-(1/r)\,h$ or $\delta n=-(n/r)\,h$, which being negative the clock will go slow. (Bundelkhand 90)

(d) **When the clock is taken to the bottom of a mine.**

We know that inside the earth attraction varies as the distance from the centre, so if r be the radius of the earth we have on the surface of the earth $g = \mu r$ or $\log g = \log \mu + \log r$

i.e. $(1/g)\ \delta g = (1/r)\ \delta r.$

$\therefore$ If the particle is taken to the bottom of a mine of depth d, we have $\delta r = -d.$

Hence we have $(1/g)\ \delta g = -(1/r)\ d$ (Note)

$\therefore$ from (iii) we get $(1/n)\ \delta n = -(1/2r)\ d$ or $\delta n = -(n/2r)\ d,$ which being negative the clock will go slow. *(Bundelkhand 91)*

Solved Examples on § 9.

Ex. 1 (a). Show that an incorrect seconds pendulum of a clock which loses x seconds in a day must be changed by $x/432$ percent of its length in order to keep correct time.

Sol. Here g remains constant, so from result (iv) Page 27 we have $(1/n)\ \delta n = -\frac{1}{2}\ (1/l)\ \delta l,$ ...(i)

where n is the number of seconds in a day, and l is the length of the pendulum

i.e. $n = 24 \times 60 \times 60\ ;\ \delta n = -x$ (Note)

$\therefore$ from (i) we have $\frac{1}{24 \times 60 \times 60} \cdot (-x) = -\frac{1}{2l}\ \delta l$

or $\frac{\delta l}{l} = \frac{x}{43200}$ or $\delta l = \frac{x}{432}$ percent of l.

Hence length must be altered by $x/432$ percent. Hence proved.

Ex. 1 (b). If a clock loses 5 seconds in a day, what alteration must be made in the pendulum ?

Sol. Proceed as in Ex. 1 (a) above.

Ex. 1 (c). A clock with a second's pendulum loses 20 seconds per day at the place where acceleration due to gravity is 33 ft./sec^2. Find what change (i) in length, (ii) in gravity is necessary to make it accurate.

Sol. (i) If g remains constant, then from result (iv) Page 47 we have $(1/n)\ \delta n = -\frac{1}{2}\ (1/l)\ \delta l,$ ...(i)

where l and n have their usual meanings.

Here 'n' $= 24 \times 60 \times 60$; 'δn' $= -20$ (Note)

$\therefore$ from (i) we get $\frac{-20}{24 \times 60 \times 60} = -\frac{1}{2l}\ \delta l$

or $\frac{\delta l}{l} = \frac{5}{108000}$ or $\delta l = \frac{5}{108}$ percent of l.

i.e. the length must be increased by $\frac{5}{108}$ percent.

(ii) If g varies (l remaining constant), then from result (iii) Page 47 we have $(1/n)\ \delta n = (1/2g)\ \delta g,$ ...(ii)

where n and g have their usual meanings.

Here $'n'=24\times 60\times 60$; $'\delta n'=20$

$\therefore$ from (ii) we get $\frac{20}{24\times 60\times 60}=\frac{1}{2g}\delta g$

or $\frac{\delta g}{g}=\frac{5}{10800}$ or $\delta g=+\frac{5}{108}$ percent of g

i.e. g must be increased by $\frac{5}{180}$ percent

i.e. the clock be carried to a place where

$$'g'=\frac{\left(100+\frac{4}{108}\right)}{100}g=\frac{10805}{10800}\times 32\cdot 2 \text{ ft./sec}^2$$

$$=32\cdot 215 \text{ ft./sec}^2 \text{ nearly.}$$

Ex. 1 (d). Find how many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900 : 901.

Sol. If l be the length of the pendulum, then its increased length$=\frac{901}{900}l$.

$\therefore$ Increase in length$='\delta l'=\frac{901}{900}l-l=\frac{1}{900}l$.

Also here $'n'$=number of seconds in a day $=24\times 60\times 60$.

Now here g remains constant; so from result (iv) Page 47 we have $(/n)\ \delta n=-\frac{1}{2}(1/l)\ \delta l$

or $$\frac{\delta n}{24\times 60\times 60}=-\frac{(l/900)}{2l}=-\frac{1}{1800}$$

or $$\delta n=-\frac{24\times 60\times 60}{1800}=-48.$$

$\therefore$ The clock would lose 48 seconds per day. Ans.

***Ex. 1 (e). If a pendulum of length l makes n complete oscillations in a given time, show that if the length be changed to $l+l'$, the number of oscillations lost is $nl'/2l$.**

Sol. Increase in length$=(l+l')-l=l'="\delta l"$ .. (i)

Also here g remains constant, so from result (iv) Page 47 we have $(1/n)\ \delta n=-\frac{1}{2}(1/l)\ \delta l$. ...(ii)

where δn is the increase in number of oscillations and n the number of oscillations when length is l.

$\therefore$ From (i) and (ii) we get $(1/n)\ \delta n=-\frac{1}{2}(1/l)\ l'$

or $\delta n=-\left(\frac{nl'}{2l}\right)$, the negative sign shows that the number of oscillations is lost.

Hence the required number of oscillations lost

$=nl'/(2l)$. Hence proved.

Ex. 1 (f). **If a pendulum of length l marks n complete oscillations in a given time, show that if g is changed to $g+g'$, the number of oscillations gained is $ng'/(2g)$.**

Sol. Increase in the value of g

$$=(g+g')-g=g'='\delta g' \quad ...(i)$$

Also here the length l remains constant, so from result (iii) Page 47 we get $(1/n)\ \delta n=(1/2g)\ \delta g$, ...(ii)

where δn is the increase in number of oscillations and n the number of oscillations.

∴ From (i) and (ii) we get $(1/n)\ \delta n=(1/2g)\ g'$

or $\delta n=(ng')/(2g)$, which being positive we conclude that the number of oscillations is gained.

Hence the required number of oscillations gained $=(ng')/(2g)$.

Hence proved.

Ex. 2 (a). **A pendulum beats seconds accurately at a place where g is 981 cm./sec². If taken to a place where the value of g is 982 cm./sec², prove that it will gain 44 seconds per day.**

Sol. Here l, the length of the pendulum remains constant, so we have $(1/n)\ \delta n=(1/2g)\ \delta g$. ...See result (iii) Page 47

where n = number of seconds in one day $=24\times60\times60$,

$g=981$ cm/sec² and $\delta g=982-981=1$ cm/sec²

$$\therefore \frac{1}{24\times60\times60}\ \delta n=\frac{1}{2\times981}\times1 \quad \text{or} \quad \delta n=44$$

Hence the pendulum gains 44 seconds per day.

Ex. 2 (b). **A pendulum oscillating seconds at one place is taken to a second place where it loses 2 seconds each day. Compare the accelerations due to gravity at the two places.**

(Gorakhpur 84)

Sol. Here l, the length of the pendulum, remains constant. Let the accelerations due to gravity at the two places be g_1 and g_2 then, at the first place we have

$(1/n)\ \delta n=\frac{1}{2}\ (1/g_1)\ \delta g_1$, ...See result (iii) Page 47

where n = number of seconds in one day

$=24\times60\times60$.

When the pendulum is carried to another place, it loses 2 seconds per day, ∴ δn = loss in $n=-2$

[∵ δn is a loss hence the negative sign]

$$\therefore \frac{1}{24\times60\times60}\times(-2)=\frac{1}{2g_1}\ \delta g_1 \quad \text{or} \quad \delta g_1=-\frac{1}{21600}\ g_1$$

i.e. at the new place gravity is decreased by $\frac{1}{21600} g_1$

$$\therefore \quad g_2 = g_1 - \frac{1}{21600} g_1 = \frac{21599}{21600} g_1 \quad \text{or} \quad \frac{g_1}{g_2} = \frac{21600}{21599} \quad \textbf{Ans.}$$

***Ex. 3. A pendulum is carried to the top of a mountain 0·8 kilometres high, how many seconds will it lose per day? By how much must its present length be shortened so that it may beat seconds at the top of the mountain.**

Sol. We know $\delta n = -(n/r)\, h$. ...See result (iii) Page 47

Here n = number of seconds in a day = $24 \times 60 \times 60$;

r = radius of the earth = 6400 kms.,

and h = height of the mountain = 0·8 kms.

$$\therefore \quad \delta n = -\frac{24 \times 60 \times 60}{6400} \times 0{\cdot}8 = -10{\cdot}8 \text{ km.}$$

i.e. the pendulum loses (negative sign shows loss) 10·8 seconds per day.

Also we have the relation

$(1/n)\,\delta n = (1/2l) \cdot \delta l - (1/2g)\,\delta g$, where notations are usual.

If the pendulum keeps correct time then $\delta n = 0$.

Hence $(1/2l)\,\delta l - (1/2l)\,\delta g = 0$

or $(1/l)\,\delta l = (1/g)\,\delta g$. ...(i)

Also $g = \mu/r^2$ or $\log g = \log \mu - 2 \log r$

i.e., $(1/g)\,\delta g = -(2/r)\,\delta r = -(2/r)\,(h)$, with usual notations.

$$\therefore \text{ from (i), } \frac{1}{l}\,\delta l = -\frac{2}{r}\,h = -\frac{2}{6400} \times 0{\cdot}8$$

or $\delta l = -(1/4000) . l.$

$\therefore$ the present length must be shortened by 1/4000 of itself.

****Ex. 4. A seconds pendulum was too long on a given day by a quantity α, it was then over-corrected so as to be too short by α during the next day, prove that l being the correct length the number of minutes gained in the two days was $1080\alpha^2/l^2$, nearly.**

(*Kanpur 88*)

Sol. If l be the correct length of the pendulum, then

$$1 = \pi \sqrt{(l/g)}. \quad \text{...(i)}$$

On the first day the pendulum was over correct by a quantity α, so the time of one swing is given by $t = \pi \sqrt{\{(l+\alpha)/g\}}$

$$= \sqrt{\{(l+\alpha)/l\}}, \text{ from (i)} \quad \text{...(ii)}$$

If n_1 be the number of swing that day, then

$$n_1 \times t = 24 \times 60 \times 60$$

or $$n_1 = \frac{24 \times 60 \times 60}{t} = 24 \times 60 \times 60 \sqrt{\left\{\frac{l}{l+\alpha}\right\}}, \text{ from (ii)}$$

$\therefore$ from (i) time of one swing on the inclined wall
$=\sqrt{[1/\sin\theta]}$.

Exercises on § 9

Ex. 1. How many oscillations will a pendulum of length 2 metres make in one day ?

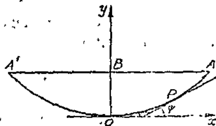
Ex. 2. A seconds pendulum which gains 10 seconds per day at one place, loses 10 seconds per day at another place, compare the acceleration due to gravity at two places.

Ex. 3. A pendulum which beats seconds at the surface of the earth, is carried to the top of a mountain 5 miles high, find the number of seconds it will lose in a day assuming the radius of the earth to be 4000 miles. [one mile$=1760\times 3$ feet, 'g'$=32$ ft./sec^2].

Ex. 4. A seconds pendulum is a carried to the summit of a mountain 2640 feet high. How many seconds will it lose per day ? By how much its present length be shortened so that it may beat seconds at the summit of the mountain. *(Gorakhpur 86)*

CYCLOIDAL MOTION

§ 10. **Cycloid.** Cycloid is a curve traced out by a point on the circumference of a circle which rolls along a fixed straight line. An arch of the cycloid is as shown in the figure. O is called the vertex, each of the points A and A' is called the cusp, the line OB is the axis and the line AA' is called the base of the cycloid. If a be the radius of the generating circle then with Ox and Oy as axes three important forms of the cycloid are—



(Fig. 33)

(a) *Parametric equations of the cycloid.*

$x=a\,(\theta+\sin\theta)$, $y=a\,(1-\cos\theta)$, where θ is the parameter,

(b) *Intrinsic equation of the cycloid*

$s=4a\sin\psi$, where s is the arcual distance measured from the vertex O and ψ is the inclination of the tangent at any point of the cycloid to Ox, the tangent at the vertex.

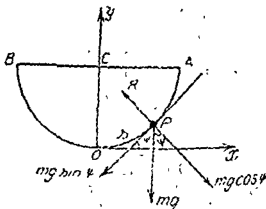
(c) *Relation between s and y.*

$s^2=8ay$, where y is the ordinate of any point P and $s=$arc OP.

Also for a cycloid we have arc $A'OA=8a$ or arc $OA=4a$: the height of A and A' above x-axis is $2a$; $\psi=0$ at O and $\psi=\frac{1}{2}\pi$ at A or A'. **(See Author's Integral Calculus)**

§11. Cycloidal Motion. *A particle slides down the arc of a smooth cycloid, whose axis is vertical and vertex downwards, to determine its motion.* *(Bundelkhand 92; Garhwal 87; Rohilkhand 88)*

Let O be the vertex of the cycloid. Let after time t the particle be at P, such that arc $OP=s$ and the tangent at P makes an angle ψ with the tangent to the cycloid at O. Then the equation of the cycloid is $s=4a \sin \psi$ ...(i)



(Fig. 34)

At P, the forces acting on the particle are its weight mg acting vertically downwards and the normal reaction R between the curve and the particle acting in the sense of the inward drawn normal at P.

$\therefore$ The equations of motion in the tangential and inward drawn normal directions are

$$m.\frac{d^2s}{dt^2}=-mg \sin \psi \quad ...(ii) \qquad m.\frac{v^2}{\rho}=R-mg\cos\psi \quad ...(iii)$$

From (ii), $\frac{d^2s}{dt^2}=-g \sin \psi$ or $\frac{d^2s}{dt^2}=-\frac{gs}{4a}$, from (i)

This is standard equation of S.H.M.

Hence the period$=\frac{``2\pi"}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{(g/4a)}}=4\pi\sqrt{\left(\frac{a}{g}\right)}$,

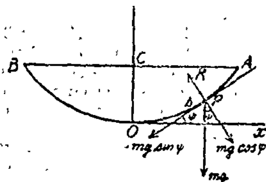
which is independent of the amplitude.

i.e. we find that from whatever point the particle may be allowed to slide down the arc of a smooth cycloid the period remains the same. This property is known as the isochronism of the cycloid.

Solved Examples on § 10 and § 11.

Ex. 1 (a). **A particle slides down a smooth cycloid whose axis is vertical and vertex downwards. Find the velocity of the particle and reaction on it at any point of the cycloid.**

(Avadh 85; Rohilkhand (IT) 90)



(Fig. 35)

Sol. Let us assume that the particle starts from the cusp at A from rest. Let the particle be at P after time t, such that arc $OP=s$, where O is the vertex of the cycloid.

The forces acting on the particle at P are its weight mg acting vertically downwards and the normal reaction R between the particle and the curve acting normally at P as shown in the figure.

Then the equation of motion in the tangential and inward drawn normal directions are

$$m.\ \frac{d^2s}{dt^2}=-mg\sin\psi \quad ...(i) \quad \text{and} \quad m.\ \frac{v^2}{\rho}=R-mg\cos\psi \quad ...(ii)$$

Also the equation of cycloid is $s=4a\sin\psi$ ...(iii)

From (i) and (iii) we get $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s.$

Integrating $(ds/dt)^2=-(g/4a)s^2+C$, where C is constant of ıtegration.

Initially at A, $s=4a$ and $ds/dt=0$, $\therefore\ C=(g/4a)\ .\ 16a^2$

$$\therefore\ \left(\frac{ds}{dt}\right)^2=\frac{g}{4a}(16a^2-s^2)=\frac{g}{4a}\ [16a^2-16a^2\sin^2\psi], \text{ from (iii)}$$

or $$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}\ .\ 16a^2\cos^2\psi, \text{ where velocity } v=\frac{ds}{dt}$$

or $$v^2=4ag\cos^2\psi \quad ...(iv)$$

Also from (iii) we get $\rho=(ds/d\psi)=4a\cos\psi$

$\therefore$ from (ii), $R=m\left(\dfrac{v^2}{\rho}\right)+mg\cos\psi=\dfrac{m\ (4ag\cos^2\psi)}{4a\cos\psi}+mg\cos\psi$

or $$R=2mg\cos\psi \quad ...(v)$$

Equations (iv) and (v) give velocity and reaction at any point P of the cycloid.

Ex. 1 (b). In Ex. 1 (a) above find the velocity and reaction at the lowest point.

Sol. At the lowest point O (See Fig. 35) above, we have $\psi=0$.

$\therefore$ from (iv) and (v) of Ex. 1 (a) Page 55 we have at the lowest point,

the velocity $=\sqrt{(4ag\cos^2 0)}=2\sqrt{(ag)}$

and the reaction $=2mg\cos 0=2mg$. **Ans.**

Ex. 1 (c). *Prove that for a particle, sliding down the arc and starting from the cusp of a smooth cycloid whose vertex is lowest, the vertical velocity is maximum when it has described half the vertical height.* *(Bundelkhand 92; Garhwal 91, 89; Kanpur 87; Meerut III' 90)*

Sol. As in Ex. 1 (a) we can prove that the velocity v at any point P is given by [See result (iv) of Ex. 1 (a) Page 55]

$$v^2=4ag\cos^2\psi \quad \text{...(i)}$$

Also from Fig. 35 Page 55, it is evident that vertical velocity

$$=v\cos(90°-\psi) \quad \text{[Note]}$$
$$=v\sin\psi=V.\text{(say)}.$$

Then $V^2=v^2\sin^2\psi=(4ag\cos^2\psi)\sin^2\psi$, from (i)

$=4ag\sin^2\psi(1-\sin^2\psi)$, $\quad\because \cos^2\psi=1-\sin^2\psi$

$=4ag(s/4a)^2[1-(s/4a)^2]$, $\quad\because s=4a\sin\psi$

$$=4ag\left(\frac{s^2}{16a^2}\right)\left[1-\frac{s^2}{16a^2}\right]=4ag\left(\frac{8ay}{16a^2}\right)\left[1-\frac{8ay}{16a^2}\right]$$

$\because s^2=8ay$, where y is the height of P above O,

or $\quad V^2=gy(2a-y)/a=(g/a)(2ay-y^2)$.

Now if V is maximum then V^2 is also maximum.

Now $\quad \dfrac{d}{dy}(V^2)=\dfrac{g}{a}\dfrac{d}{dy}(2ay-y^2)=\dfrac{g(2a-2y)}{a}$

and $\quad \dfrac{d^2}{dy^2}(V^2)=\dfrac{g}{a}(-2)=-$ve for all values of y.

$\therefore$ V^2 is max. when $\dfrac{d}{dy}(V^2)=0$ i.e. when $2a-2y=0$

i.e. when $y=a$ i.e. when the particle has described half the vertical height, remembering $OC=2a$.

**Ex. 2. *A particle slids down the arc of a smooth cycloid whose axis is vertical and vertex lowest, prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.* *(Bundelkhand 91; Garhwal 88; Kanpur 87; Kumaun 83; Lucknow 92, 86; Purvanchal 90)*

Sol. Refer Fig. 35 Page 55.

Let O be the vertex and A the cusp of the cycloid.

The equation of the cycloid is $s=4a\sin\psi$. ...(i)

Also for the cycloid $\quad s^2=8ay$ ...(ii)

After time t let the position of the particle be at P, such that arc $OP=s$ and the tangent at P make an angle ψ with x-axis.

At A, $s=4a$ and $y=$the height of A above x-axis$=2a$.

The forces acting on the particle at P are its weight mg acting vertically downwards and the normal reaction R of curve acting normally at P as shown in the figure.

$\therefore$ The equation of motion in the tangential and inward drawn normal senses are

$$m\,.\,\frac{d^2s}{dt^2}=-mg\sin\psi\ \ldots\text{(iii) and } m\,.\,\frac{v^2}{\rho}=R-mg\cos\psi.\ \ldots\text{(iv)}$$

From (i) and (ii) we have $d^2s/dt^2=-gs/4a$.

Integrating $(ds/dt)^2=-(g/4a)\,s^2+C$, where C is constant of integration.

As A, $s=4a$ and $(ds/dt)=0$. $\therefore$ $C=(g/4a).16a^2$

$$\therefore \quad (ds/dt)^2=(g/4a)\,(16a^2-s^2)$$

or $ds/dt=-\sqrt{(g/4a)}\,\sqrt{(16a^2-s^2)}$, the negative sign is due to the fact that s diminishes as t increases.

Integrating, $$dt=-\sqrt{(4a/g)}\,\frac{ds}{\sqrt{(16a^2-s^2)}} \quad \ldots\text{(v)}$$

we get $t=\sqrt{(4a/g)}\cos^{-1}(s/4a)+B$, where B is any constant.

Initially at A, $s=4a$ and $t=0$, $\therefore$ $B=0$.

$\therefore$ $t=2\sqrt{(a/g)}\cos^{-1}(s/4a)=2\sqrt{(a/g)}\cos^{-1}\{\sqrt{(8ay)}/4a\}$, from (iii)

Let t_1 and t_2 be the time taken in falling first and second halves of the vertical distance *i.e.* t_1 is time taken in falling from $y=2a$ to $y=a$ and t_2 is the time taken in falling from $y=a$ to $y=0$. (Note)

$$\therefore\ t_1=2\sqrt{(a/g)}\left[\cos^{-1}\{\sqrt{(8ay)}/4a\}\right]_{y=2a}^{a}$$

or $$t_1=2\sqrt{(a/g)}\,[\cos^{-1}(1/\sqrt{2})-\cos^{-1}(1)]$$
$$=2\sqrt{(a/g)}\,[\tfrac{1}{4}\pi-0]=\tfrac{1}{2}\pi\sqrt{(a/g)}$$

and $$t_2=2\sqrt{(a/g)}\left[\cos^{-1}\{\sqrt{(8ay/4a}\}\right]_{y=a}^{0}$$
$$=2\sqrt{(a/g)}\,[\cos^{-1}(0)-\cos^{-1}(1/\sqrt{2})]$$
$$=2\sqrt{(a/g)}\,[\tfrac{1}{2}\pi-\tfrac{1}{4}\pi]=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

$\therefore$ $t_1=t_2$. Hence proved.

**Ex. 3. A particle starts from rest to the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that when it has fallen through half the distance measured along the arc

to the vertex from the cusp, two thirds of the time of descent will have elapsed. *(Bundelkhand 90; Gorakhpur 91; Kanpur 91, 88; Kumaun 89; Lucknow 88; Rohilkhand III' 90)*

Sol. As in last example we can prove that the time t from start to any point P is given by

$$t=-\sqrt{(4a/g)}\int \frac{ds}{\sqrt{(16a^2-s^2)}} \quad \text{...[See result (v) Ex. 2 above]}$$

Let T be the total time taken in moving from the cusp to the vertex *i.e.* from $s=4a$ to $s=0$, then

$$T=-\sqrt{(4a/g)}\int_{s=4a}^{0} \frac{ds}{\sqrt{(16a^2-s^2)}}=\sqrt{(4a/g)}\left[\cos^{-1}(s/4a)\right]_{4a}^{0}$$

$$=\sqrt{(4a/g)}\,[\cos^{-1}(0)-\cos^{-1}(1)]=2\sqrt{(a/g)}\,[\tfrac{1}{2}\pi-0]=\pi\sqrt{(a/g)}.$$

Again if t_1 be the time taken in travelling half the distance measured along the arc to the vertex from the cusp *i.e.* t_1 be the time in moving from $s=4a$ to $s=2a$, then

$$t=-\sqrt{(4a/g)}\int_{s=4a}^{2a} \frac{ds}{\sqrt{(16a^2-s^2)}}=\sqrt{(4a/g)}\left[\cos^{-1}(s/4a)\right]_{4a}^{2a}$$

$$=2\sqrt{(a/g)}\,[\cos^{-1}(\tfrac{1}{2})-\cos^{-1}(1)]=2\sqrt{(a/g)}\,[\tfrac{1}{3}\pi-0]$$

$$=(\tfrac{2}{3}\pi)\sqrt{(a/g)}=(\tfrac{2}{3})\,T.$$

Hence proved.

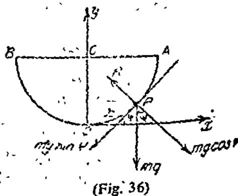
Ex. 4. A particle oscillates from cusp to cusp in a smooth cycloid whose axis is vertical and vertex lowest. Show that the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P *i.e.* $v=V\cos\psi$. *(Lucknow 91; Meerut 91 S; Rohilkhand 86)*

Sol. The equation of the cycloid is

$s=4a\sin\psi.$ ...(i)

At P, the forces acting on the particle are its weight mg acting vertically downwards and the normal reaction R acting in the direction of the inward drawn normal at P.

$\therefore$ The equation of motion in the tangential direction is



(Fig. 36)

$$m\cdot\frac{d^2s}{dt^2}=-mg\sin\psi=-mg.(s/4a), \text{ from (i)}$$

or $$d^2s/dt^2=-(g/4a)\,s.$$

Integrating, $(ds/dt)^2=-(g/4a)\,s^2+C$, where C is constant of integration.

At the cusp at A, $s=4a$, velocity$=(ds/dt)=0$.

$\therefore \quad C=(g/4a).16a^2$

$\therefore \quad (ds/dt)^2=(g/4a)\,(16a^2-s^2)$, ...(ii)

which gives the velocity at P.

Also at the vertex O, $s=0$ and let velocity be V. Then from (ii) we get $V^2=(g/4a)\,(16a^2)$ or $V=2\sqrt{(ag)}$.

Also from (i) we get $\sin\psi=(s/4a)$

$\therefore \cos\psi=\sqrt{(1-\sin^2\psi)}=\sqrt{\left(1-\frac{s^2}{16a^2}\right)}$ or $\cos\psi=\frac{\sqrt{(16a^2-s^2)}}{4a}$.

$\therefore$ Resolved part of the velocity at O along the tangent at P

$$=V\cos\psi=2\sqrt{(ag)}\,.\,\frac{\sqrt{(16a^2-s^2)}}{4a}=\sqrt{\left(\frac{g}{4a}\right)}\,\sqrt{(16a^2-s^2)}$$

$=$velocity at P, from (ii). Hence proved.

****Ex. 5. A heavy particle slides down a smooth cycloid starting from rest at the cusp; the axis being vertical and vertex downwards, prove that the magnitude of the acceleration is equal to g at every point of the path and the pressure when the particle arrives at the vertex is equal to twice the weight of the particle.**

(Agra 88, 87; Avadh 90, 87; Bundelkhand 90; Gorakhpur 88; Kanpur 86; Lucknow 89; Meerut 90 S, 87; Purvanchal 88; Rohilkhand 86)

Sol. As in last example, we have the equations of motion as $m\,\frac{d^2s}{dt^2}=-mg\sin\psi$ ...(i); $\quad m\,.\,\frac{v^2}{\rho}=R-mg\cos\psi$ ...(ii)

Also the equation of the cycloid is $s=4a\sin\psi$ ...(iii)

From (i) and (iii) we get $\frac{d^2s}{dt^2}=-\frac{g}{4a}\,s$.

Integrating, $(ds/dt)^2=-(g/4a)\,s^2+C$, where C is constant of integration.

Initially at the cusp at A, (see figure of last example)

$s=4a$ and $ds/dt=0$, $\quad\therefore \quad C=(g/4a).16a^2$

Hence $\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}\,(16a^2-s^2)=\frac{g}{4a}\,(16a^2-16a^2\sin^2\psi)$, from (iii)

so $\quad (ds/dt)=4ag\cos^2\psi$, ...(iv)

Also from (iii), $\rho=(ds/d\psi)=4a\cos\psi$. ...(v)

$\therefore$ normal acceleration$=\frac{v^2}{\rho}=\frac{4ag\cos^2\psi}{4a\cos\psi}$, from (iv) and (v)

$=g\cos\psi$.

And from (i) tangential acceleration $= d^2s/dt^2 = -g \sin \psi$.

$\therefore$ Resultant acceleration at any point

$= \sqrt{[(d^2s/dt^2)^2 + (v^2/\rho)^2]}$

$= \sqrt{[(-g \sin \psi)^2 + (g \cos \psi)^2]} = g$. Hence proved.

Also from (ii), $R = m \dfrac{v^2}{\rho} + mg \cos \psi = \dfrac{m[4ag \cos^2 \psi)}{4a \cos \psi} + mg \cos \psi$

$= mg \cos \psi + mg \cos \psi = 2mg \cos \psi$.

$\therefore$ At the vertex *i.e.* at $\psi = 0$, the normal reaction $R = 2mg$
$= 2$ (weight of the particle). Hence proved.

Ex. 6. *A particle starts from rest at any point P in the arc of a smooth cycloid whose axis is vertical and vertex A downwards, prove that the time of descent to the vertex is $\pi\sqrt{(a/g)}$, where a is radius of the generating circle.* *(Meerut 92 O, 86)*

Show also that if the particle is projected from P downwards along the curve with velocity equal to that with which it reaches A when started from rest at P, it will now reach A in half the time taken in the preceeding case. *(Meerut 92 O, 86)*

Sol. The equation of the cycloid is $s = 4a \sin \psi$ ...(i)

The particle starts from rest from a point P, whose arcual distance from the vertex A is b (say) *i.e.* $s = b$ at P, the point of start.

Then equation of motion at any point whose arcual distance from A is s, in tangential direction is

$m\,(d^2s/dt^2) = -mg \sin \psi$ (See figure of Ex. 4 Page 55)

or $d^2s/dt^2 = -g \sin \psi = -(g/4a)\,s$, from (i)

Integrating, $(ds/dt)^2 = -(g/4a)\,s^2 + C$, where C is constant of integration.

At the point of start P; $s = b$ and $(ds/dt) = 0$. $\therefore$ $C = (g/4a).b^2$

$\therefore$ $(ds/dt)^2 = (g/4a)\,(b^2 - s^2)$ ...(ii)

or $(ds/dt) = -\sqrt{(g/4a)}\sqrt{(b^2 - s^2)}$, as s decreases when t increases

or $dt = -2\sqrt{(a/g)} \cdot \dfrac{ds}{\sqrt{(b^2 - s^2)}}$

Integrating, $t = 2\sqrt{(a/g)} \cos^{-1}(s/b) + D$, where D is constant of integration.

At P, $s = b$ and $t = 0$, $\therefore$ $D = 0$.

Hence $t = 2\sqrt{(a/g)} \cos^{-1}(s/b)$

At the vertex A, $s = 0$ and let time taken from P to A be T then $T = 2\sqrt{(a/g)} \cos^{-1}(0) = \pi\sqrt{(a/g)}$ ...(iii)

Hence proved.

Also from (ii) velocity at A i.e. $s=0$ is $\sqrt{(g/4a)}.b$ ...(iv)

Second part. As in last part we have $(d^2s/dt^2)=-(g/4a).s$

Integrating, $(ds/dt)^2=-(g/4a)\,s^2+C_1$, where C_1 is constant of integration.

At P, $ds/dt=\sqrt{(g/4a)}\,b$, from (iv) and $s=b$

$\therefore$ $C_1=(g/4a)\,(2b^2)$

$\therefore$ $(ds/dt)^2=(g/4a)\,(2b^2-s^2)$ or $ds/dt=-\sqrt{(g/4a)}.\sqrt{(2b^2-s^2)}$

or $$dt=2\sqrt{(a/g)}\,\frac{ds}{\sqrt{(2b^2-s^2)}}$$

Integrating, $t=2\sqrt{(a/g)}\cos^{-1}(s/b\sqrt{2})+D_1$, where D_1 is constant of integration.

At P, $s=b$, $t=0$, so we have $0=2\sqrt{(a/g)}.\frac{1}{4}\pi+D_1$

or $$D_1=-\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

$\therefore$ $t=2\sqrt{(a/g)}\cos^{-1}(s/b\sqrt{2})-\frac{1}{2}\pi\sqrt{(a/g)}$.

At the vertex A, $s=0$ and let time taken from P to A be T_1

then $T_1=2\sqrt{(a/g)}\cos^{-1}(0)-\frac{1}{2}\pi\sqrt{(a/g)}=2\sqrt{(a/g)}\,[\frac{1}{2}\pi-\frac{1}{4}\pi]$

or $T_1=\frac{1}{2}\pi\sqrt{(a/g)}=\frac{1}{4}T$, from (iii)

or $T_1=\frac{1}{4}T$. Hence proved.

Ex. 7. If a particle slides down a smooth cycloid starting from a point whose arcual distance from the vertex is b, prove that its speed at any time t is $(2\pi b/T)\sin(2\pi t/T)$, where T is the time of complete oscillation of the particle. *(Avadh 89; Bundelkhand 92, 91; Gorakhpur 90; Lucknow 87; Meerut II 92; Purvanchal 89)*

Sol. The equation of the cycloid is $s=4a\sin\psi$ ...(i)

As in Ex. 4 Page 58, the equation of motion in the tangential sense is $$m.\frac{d^2s}{dt^2}=-mg\sin\psi=-\frac{mg}{4a}s, \text{ from (i)}$$

or $d^2s/dt^2=-(g/4a)\,s$ ...(ii)

$\therefore$ $T=$time of complete oscillation$=\dfrac{2\pi}{\sqrt{\mu}}=2\pi\sqrt{\left(\dfrac{4a}{g}\right)}$

or $\sqrt{(g/4a)}=(2\pi/T)$ ...(iii)

Integrating (ii) we have $(ds/dt)^2=-(g/4a)\,s^2+C$, where C is constant of integration.

At the point of start, $s=b$ and $ds/dt=0$, $\therefore$ $C=(g/4a)\,b^2$.

$\therefore$ $\left(\dfrac{ds}{dt}\right)^2=\dfrac{g}{4a}(b^2-s^2)$ or $\dfrac{ds}{dt}=-\sqrt{(g/4a)}\sqrt{(b^2-s^2)}$ ...(iv)

or $$\sqrt{(g/4a)}\,dt=-\frac{ds}{\sqrt{(b^2-s^2)}}.$$

Integrating, $\sqrt{(g/4a)}\,t=\cos^{-1}(s/b)+C_1$, where C_1 is constant of integration.

Initially $s=b$, $t=0$, $\therefore$ $C_1=0$

Hence $\sqrt{(g/4a)}\,t=\cos^{-1}(s/b)$ or $s=b\cos\sqrt{(g/4a)}.t$.

$\therefore$ from (iv) speed at any time t is given by

$$v=\sqrt{(g/4a)}\sqrt{[b^2-b^2\cos^2\sqrt{(g/4a)}\,t]}.$$

$$=\sqrt{\left(\frac{g}{4a}\right)}.b\sin\sqrt{(g/4a)}\,t=\frac{2\pi}{T}b\sin\frac{2\pi t}{T},\text{ from (iii)}$$

Hence proved.

Ex. 8. A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}[\sqrt{(4ag)}/V]$.

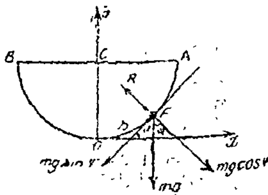
(Agra 99, 86 ; Bundelkhand 92, 86 ; Gorakhpur 92, 89 ; Kanpur 92 ; Meerut II 93, III 91 ; Purvanchal 91)

Sol. The equation of the cycloid is

$$s=4a\sin\psi \quad ...(i)$$

At any time t, let P be the position of the particle and ψ the angle which the tangent at P makes with the tangent at the vertex O.

The forces acting on the particle are its weight mg acting vertically downwards and the normal reaction R acting in the sense of the inward drawn normal.



(Fig. 37)

Then the equation of motion in the tangential sense is

$$m.\frac{d^2s}{dt^2}=-mg\sin\psi=-\frac{mg}{4a}s,\text{ from (i)}$$

or $$d^2s/dt^2=-(g/4a)\,s.$$

Integrating, $(ds/dt)^2=-(g/4a)\,s^2+C$, where C is constant of integration.

Initially at the cusp at A, $s=a$ and $(ds/dt)=V$ (given)

$\therefore$ $V^2=-(g/4a)\,(4a)^2+C$ or $C=V^2+4ag$ ...(ii)

$\therefore$ $(ds/dt)^2=C-(g/4a)\,s^2$, where C is given by (ii).

or $$\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}\left(\frac{4aC}{g}-s^2\right)$$

or $ds/dt=-\frac{1}{2}\sqrt{(g/a)}\sqrt{\{(4ac/g)-s^2\}}$, the negative sign is due to the fact that s decreases as t increases

or $$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{\{(4aC/g)-s^2\}}}.$$

If T be the time of reaching the vertex O from the cusp A, we have $T=2\sqrt{(a/g)}\int_{s=4a}^{0}\frac{-ds}{\sqrt{\{(4aC/g)-s^2\}}}$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\left\{s\sqrt{\left(\frac{g}{4aC}\right)}\right\}\right]_{4a}^{0}$$

$$=2\sqrt{(a/g)}\left[\frac{\pi}{2}-\cos^{-1}\sqrt{\left(\frac{4ag}{C}\right)}\right]$$

or $T=2\sqrt{(a/g)}\sin^{-1}\sqrt{\left(\frac{4ag}{C}\right)}$...since $\sin^{-1}x=\frac{1}{2}\pi-\cos^{-1}x$

$$=2\sqrt{(a/g)}\,\theta, \qquad \text{...(iii)}$$

where $\theta=\sin^{-1}\sqrt{\left(\frac{4ag}{C}\right)}$ or $\sin\theta=\sqrt{\left(\frac{4ag}{C}\right)}$

or $\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{\sqrt{(4ag/C)}}{\sqrt{\{1-(4ag/C)\}}}=\frac{\sqrt{(4ag)}}{\sqrt{(C-4ag)}}$

$$=\frac{\sqrt{(4ag)}}{\sqrt{(V^2+4ag-4ag)}}, \text{ since from (ii) } C=V^2+4ag$$

or $\theta=\tan^{-1}[\sqrt{(4ag)}/V]$.

$\therefore$ from (iii) required time $=T=2\sqrt{(a/g)}\tan^{-1}[\sqrt{(4ag)}/V]$.

Ex. 8 (a). Discuss in Ex. 8 above the case when $V=0$,

(Bundelkhand 86)

Hint : Do as Ex. 8 above.

****Ex. 9.** Two particles are let drop from the cusp of a cycloid down the curve at an interval of time t, prove that they will meet in time $2\pi\sqrt{(a/g)}+\frac{1}{2}t$.

(Agra 92, Bundelkhand 91 ; Kanpur 90 ; Purvanchal 89 ; Rohilkhand III 91)

Sol. Refer Fig. 37 Page 62.

The equation of the cycloid is $s=4a\sin\psi$ ...(i)

The equation of motion in the tangential sense is

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \quad\text{or}\quad \frac{d^2s}{dt^2}=-\frac{g}{4a}.s, \text{ from (i).}$$

Integrating, $(ds/dt)^2=-(g/4a)\ s^2+C$, where C is constant of integration.

At the cusp A, where the particle starts from,

$s=4a$ and $(ds/dt)=0$, $\therefore C=(g/4a).16a^2$

$$\therefore \left(\frac{ds}{dt}\right)^2=\frac{g}{4a}(16a^2-s^2)$$

or $ds/dt=-\sqrt{(g/4a)}\sqrt{(16a^2-s^2)}$,

$\because$ s decreases as t increases hence the negative sign,

or $\sqrt{(g/4a)}\,dt=-\frac{ds}{\sqrt{(16a^2-s^2)}}$.

Integrating, $\sqrt{(g/4a)}\ t=\cos^{-1}(s/4a)+D$, where D is constant of integration.

At A, $s=4a$ and $t=0$, $\therefore D=0$.

$\therefore \sqrt{(g/4a)}\ t=\cos^{-1}(s/4a)$ or $s=4a\cos\{\sqrt{(g/4a)}.t\}$. ...(ii)

Let the second particle meet the first particle after T seconds of the release of the second particle.

$\therefore$ Before meeting, the first particle was in motion for $(t+T)$ seconds.

$\therefore$ The distance s_1 moved by the first particle in $(t+T)$ sec.

$=4a\cos\{\sqrt{(g/4a)}\ (t+T)\}$, from (ii)

And the distance s_2 moved by the second particle in T sec.

$=4a\cos\{\sqrt{(g/4a)}\ T\}$.

Since the particles meet each other, $\therefore s_1=s_2$

or $4a\cos\{\sqrt{(g/4a)}\ (t+T)\}=4a\cos\{\sqrt{(g/4a)}.T\}$

or $\sqrt{(g/4a)}\ (t+T)=2\pi-\sqrt{(g/4a)}.T$ (Note)

or $t+T=2\pi\sqrt{(4a/g)}-T$

or $2T=2\pi\sqrt{(4a/g)}-t$ or $T=2\pi\sqrt{(a/g)}-\frac{1}{2}t$.

$\therefore$ Required time$=T+t=2\pi\sqrt{(a/g)}-(\frac{1}{2}t)+t$

$=2\pi\sqrt{(a/g)}+\frac{1}{2}t$, Hence proved.

*Ex. 10. If a particle starts from rest at a given point of a cycloid with its axis vertical and vertex downwards; prove that it falls $1/n$ of the vertical distance to the lowest point in time $2\sqrt{(a/g)}\sin^{-1}(1/\sqrt{n})$, where a is the radius of the generating circle. (Agra 91, 89; Kanpur 92)

Sol. The equation of the cycloid is $s=4a\sin\psi$.

Let the particle start from the point A whose vertical height above the level of the vertex is h (say) i.e. $AL=h$. We are required to find the time taken by the particle in moving from A to D where the vertical depth of D below A is $(1/n)\ (h)$.



i.e. $DM=h-(h/n)=h\ (n-1)/n$. (Fig. 38)

The equation of motion of the particle in the tangential direction is $m.\dfrac{d^2s}{dt^2}=-mg\sin\psi=-mg.\dfrac{s}{4a}$, from (i)

or $d^2s/dt^2=-(g/4a)\ s$.

Integrating, $(ds/dt)^2=-(g/4a)\ s^2+C$, where C is constant of integration.

At A, $(ds/dt)=0$ and $s=\sqrt{(8ay)}=\sqrt{(8ah)}$, $\because$ $s^2=8ay$

$\therefore$ $0=-(g/4a)(8ah)+C$ or $C=(g/4a)(8ah)$

$\therefore$ $(ds/dt)^2=(g/4a)(8ah-s^2)$

or $(ds/dt)=-\sqrt{(g/4a)}\sqrt{(8ah-s^2)}$, negative sign shows that s decreases as t increases,

or $$\sqrt{(g/4a)}\,dt=-\frac{ds}{\sqrt{(8ah-s^2)}}$$

Integrating, $\sqrt{(g/4a)}\ t=\cos^{-1}\{s/\sqrt{(8ah)}\}+D$, where D is constant of integration.

At A, $t=0$ and $s=\sqrt{(8ah)}$ as before.

$\therefore$ $0=\cos^{-1}(1)+D$ or $D=0$

Hence $t=2\sqrt{(a/g)}\cos^{-1}\{s/\sqrt{(8ah)}\}$. ...(iii)

Also for a cycloid we know $s^2=8ay$ or $s=\sqrt{(8ay)}$

$\therefore$ from (iii), $t=2\sqrt{(a/g)}\cos^{-1}\{\sqrt{(y/h)}\}$

$\therefore$ Required time from A to $D=2\sqrt{(a/g)}\cos^{-1}\left\{\sqrt{\left(\frac{h\,(n-1)/n}{h}\right)}\right\}$

since at D, $y=\{(n-1)/n\}\,h$

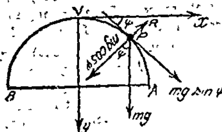
$=2\sqrt{(a/g)}\cos^{-1}[\sqrt{\{1-(1/n)\}}]=2\sqrt{(a/g)}\sin^{-1}(1/\sqrt{n})$,

since $\cos^{-1}\sqrt{\{1-(1/n)\}}=\sin^{-1}(1/\sqrt{n})$.

Ex. 11. A particle is placed very close to the vertex of a smooth cycloid whose axis is vertical and vertex upwards and is allowed to run down the curve. Show that it leaves the curve when it is moving in a direction making with the horizontal an angle of 45°. *(Avadh 86; Bundelkhand 87)*

Sol. The equation of the cycloid is $s=4a\sin\psi$, where s is measured from the vertex V.

Let the particle be at P after time t, such that arc $VP=s$ and let the tangent at P make an angle ψ with the tangent at the vertex.



(Fig. 39)

The forces acting on the particle at P are its weight mg acting vertically downwards and the normal reaction R acting normally at P as shown in the figure.

Then the equation of motion in the tangential and inward drawn normal directions are

$m \cdot \dfrac{d^2s}{dt^2} = mg \sin \psi$ ...(ii) $m \cdot \dfrac{v^2}{\rho} = mg \cos \psi - R$ ...(ii)

From (i) and (ii) we get $d^2s/dt^2 = (g/4a)\ s$.

Integrating, $(ds/dt)^2 = (g/4a)\ s^2 + C$, where C is constant of integration.

At the vertex V, $s=0$ and $ds/dt=0$, $\therefore$ $C=0$.

$$\therefore \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}\ s^2 \quad \text{or} \quad \frac{ds}{dt} = \sqrt{\left(\frac{g}{4a}\right)}\ s, \quad \text{...(iv)}$$

here +ve sign has been taken as s increases with t.

i.e. $\dfrac{ds}{dt} \propto s$ *i.e.* velocity varies as the distance measured along the arc.

Again from (iv), $\left(\dfrac{ds}{dt}\right)^2 = \dfrac{g}{4a}\ s^2 = \dfrac{g}{4a}\ (4a \sin \psi)^2$, from (i)

or $(ds/dt)^2 = 4ag \sin^2 \psi$ ...(v)

And from (i) we have $\rho = ds/d\psi = 4a \cos \psi$

$\therefore$ from (iii), $R = mg \cos \psi - \dfrac{mv^2}{\rho} = mg \cos \psi - \dfrac{m\ (4ag \sin^2 \psi)}{4a \cos \psi}$, from (v)

or $R = mg\ (\cos^2 \psi - \sin^2 \psi)/\cos \psi$

At the point where the particle leaves the cycloid, $R=0$, hence $\cos^2 \psi - \sin^2 \psi = 0$ or $\tan \psi = 1$ or $\psi = \frac{1}{4}\pi$. Hence proved.

Ex. 12. If a particle slides down from rest at the vertex of cycloid, whose axis is vertical and vertex upwards, show that the velocity at any point is due to fall from the tangent at the vertex.

Sol. As in the last example, we can prove that velocity at any point P on the cycloid is given by

$$(ds/dt)^2 = (g/4a)\ s^2.$$

But for a cycloid $s^2 = 8ay$, where y is the perpendicular distance of any point from the tangent to the cycloid at the vertex. Hence y is the depth of P below the tangent at the vertex.

$\therefore$ $(ds/dt)^2 = (g/4a)\ (8ay) = 2gy$

or $ds/dt = \sqrt{(2gy)}$

= velocity acquired by a particle when it reaches P after falling from the horizontal level of the vertex *i.e.* from the tangent at the vertex

Hence proved.

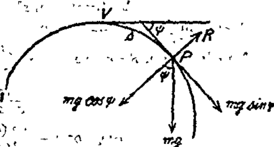
****Ex. 13. A particle is moving on a smooth curve under gravity**

and its velocity varies as the arcual distance from the highest point. Prove that the curve is a cycloid.

(Bundelkhand 92; Purvanchal 90; Rohilkhand III 91)

Also prove that the particle will leave the curve when moving in a direction making an angle of 45° with the vertical.

Sol. Let V be the highest point of the curve and P the position of the particle at time t, such that arc $VP=s$. Let the tangent at P be inclined at an angle ψ to the tangent at V. The forces acting on the particle at P are its weight mg acting vertically downwards and the normal reaction R acting normally at P as shown in the figure.



(Fig. 40)

$\therefore$ The equation of motion in the tangential direction is

$$m\frac{d^2s}{dt^2}=mg\sin\psi. \qquad ...(i)$$

And velocity at $P=(ds/dt)=ks$ (given) ...(ii)

Differentiating (ii) we have $\dfrac{d^2s}{dt^2}=k\dfrac{ds}{dt}=k(ks)$, from (ii)

or $d^2s/dt^2=k^2s$

$\therefore$ from (i), $k^2s=g\sin\psi$ or $s=4a\sin\psi$, where $4a=g/k^2$.

This equation represents a cycloid. Hence proved.

For the second part see Ex. 11, Pages 65–66.

Ex. 14. A particle is placed very near the vertex of a smooth cycloid whose axis is vertical and vertex upwards and is allowed to run down the curve. Prove that it will leave the curve when it has fallen through half the vertical height of the cycloid.

(Agra 86)

Also prove that the latus rectum of the parabola subsequently described is equal to height of the cycloid. And show that it falls upon the base of the cycloid at a distance $(\frac{1}{2}\pi+\sqrt{3})$ a from the centre of the base, a being the radius of the generating circle.

(Bundelkhand 87)

Sol. The equation of cycloid is $s=4a\sin\psi$ ...(i)

Also for the cycloid we know $s^2=8ay$ ...(ii)

The equation of motion of the particle in the tangential and

normal sense can be proved as in Ex. 11 Page 65 to be

$$m\,.\,\frac{d^2s}{dt^2}=mg\sin\psi \quad ...(iii); \quad m\,\frac{v^2}{\rho}=mg\cos\psi-R \quad ...(iv)$$

From (i) and (iii) we have $d^2s/dt^2=(g/4a)\,s$

Integrating, $(ds/dt)^2=(g/4a)\ s^2+C$, where C is constant of integration.

At the vertex $s=0$ and $(ds/dt)=0$, $\therefore C=0$.

$$\therefore \quad (ds/dt)^2=(g/4a)\,s^2. \quad ...(v)$$

Also from (i), $\rho=ds/d\psi=4a\cos\psi$ ...(vi)

Now from (iv), $R=mg\cos\psi-m\,\dfrac{v^2}{\rho}$

$$=mg\cos\psi-\frac{mgs^2}{4a.4a\cos\psi}, \text{ from (v) and (vi)}$$

$$=mg\cos\psi-\frac{mg\,(4a\sin\psi)^2}{16a^2\cos\psi}, \text{ from (i)}$$

or $\quad R=mg\,(\cos^2\psi-\sin^2\psi)/\cos\psi.$

At the point where the particle leaves the curve $R=0$, so we have $\cos^2\psi-\sin^2\psi=0$ or $\tan\psi=1$ or $\psi=\pi/4$

From (i) and (ii), $y=\dfrac{s^2}{8a}=\dfrac{(4a\sin\psi)^2}{8a}=2a\sin^2\psi.$

$\therefore$ If the depth of the point Q (say) where the particle leaves the curve be y_1 below the vertex, then

$$y_1=2a\sin^2\tfrac{1}{4}\pi, \text{ as } \psi=\tfrac{1}{4}\pi \text{ at } Q$$

$$=2a\,(\tfrac{1}{2})=a=\text{half the vertical height of cycloid.}$$

Also from (i) and (v) we have $(ds/dt)^2=4ag\sin^2\psi$ ...(vii)

$\therefore$ If v_1 be the velocity of the particle when it leaves the curve at Q, then from (vii) we have

$$v_1^2=4ag\sin^2(\tfrac{1}{4}\pi), \text{ since } \psi=\tfrac{1}{4}\pi \text{ at } Q$$

or $\quad v_1^2=2ag$ ...(viii)

Beyond Q the particle describes a parabolic path with v_1 as velocity of projection and $(-\tfrac{1}{4}\pi)$ as angle of projection.

$\therefore$ latus rectum of this parabola $=\dfrac{2u^2\cos^2\alpha}{g}$

$$=\frac{2v_1^2\cos^2(-\tfrac{1}{4}\pi)}{g}=\frac{2\times 2ag\times\tfrac{1}{2}}{g}, \text{ from (viii)}$$

$$=2a=\text{height of the cycloid.}$$

Again if Q be taken as origin, the horizontal and vertical lines through Q as x and y-axes, then the equation of the parabolic path is

$$y = x \tan(-\tfrac{1}{4}\pi) - \frac{gx^2}{2v_1^2 \cos^2(-\frac{1}{4}\pi)} = -x - \frac{gx^2}{2(2ag)(\frac{1}{2})},$$

from (vii) $v_1^2 = 2ag$

or $\quad y = -x - (x^2/2a)$. ...(ix)

Since Q is at a height above the base of the cycloid therefore the co-ordinates of the point where the particle strikes the base can be taken as $(x_1, -a)$, where x_1 is the horizontal distance of the point from Q. (Note)

$\therefore$ the point $(x_1, -a)$ lies on (ix), so we have

$$-a = -x_1 - (x_1^2/2a) \quad \text{or} \quad x_1^2 + 2ax_1 - 2a^2 = 0$$

or $\quad x_1 = \frac{1}{2}[-2a \pm \sqrt{(4a^2 + 8a^2)}] = -a \pm a\sqrt{3}$

or $\quad x_1 = a\sqrt{3} - a$, negative values of x_1 being inadmissible

or $\quad x_1 = a(\sqrt{3} - 1)$. ...(x)

Now the parametric equations of the cycloid referred to vertex as origin are

$x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, where θ is the parameter

Also from Differential Calculus we know that

$$\tan\psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\tfrac{1}{2}\theta$$

or $\quad \psi = \frac{1}{2}\theta \quad$ or $\quad \theta = 2\psi$. ...(xi)

$\therefore$ The horizontal distance of Q from the vertex

$= \text{"}a(\theta + \sin\theta)\text{"} = a(2\psi + \sin 2\psi)$, from (xi)

$= a(2.\frac{1}{4}\pi + \sin\frac{1}{2}\pi)$, since $\psi = \frac{1}{4}\pi$ at Q

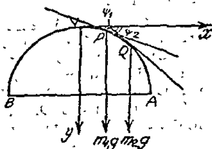
$= a(\frac{1}{2}\pi + 1)$. Hence proved.

$\therefore$ The horizontal distance from the vertex of the point where the particle strikes the base

$= x_1 + a(\frac{1}{2}\pi + 1) = a(\sqrt{3} - 1) + a(\frac{1}{2}\pi + 1)$, from (x)

$= a(\sqrt{3} + \frac{1}{2}\pi)$. Hence proved.

Ex. 15. Two particles connected by a fine string are constrained to move in a cycloidal tube in a vertical plane, the axis of the cycloid being vertical and vertex upwards; prove that the tension of the string is constant throughout the motion.



(Fig. 41)

Sol. The equation of the cycloid is $s = 4a \sin\psi$, ...(i) where s is measured from the vertex V.

After time t, let the particles be at P and Q such that the

tangents at P and Q to the cycloid make angles ψ_1 and ψ_2 with the tangent at the vertex. Let the masses of the particles at P and Q be m_1 and m_2 respectively. Let arc $VP=s$ and arc $VQ=s+l$ so that arc $PQ=l=$length of the string. Let T be the tension in the string.

The equation of motion of the particle of mass m_1 in the tangential direction is $m_1 \dfrac{d^2s}{dt^2}=T+m_1 g \sin \psi_1$

or $$\frac{d^2s}{dt^2}=\frac{T}{m_1}+g \sin \psi_1=\frac{T}{m_1}+\frac{gs}{4a}, \text{ from (i)}$$

or $$\frac{d^2s}{dt^2}=\frac{T}{m_1}+\frac{gs}{4a}. \quad \text{...(ii)}$$

And the equation of motion of the particle of mass m_2 in the tangential direction is $m_2 \dfrac{d^2(s+l)}{dt^2}=m_2 g \sin \psi_2-T$ (Note),

or $$m_2 \frac{d^2s}{dt^2}=m_2 g\left(\frac{s+l}{4a}\right)-T, \quad \because \text{ from (i) at } Q$$

we have $s+l=4a \sin \psi_2$

or $$\frac{d^2s}{dt^2}=\frac{g(s+l)}{4a}-\frac{T}{m_2}. \quad \text{...(iii)}$$

Subtracting (iii) from (ii), we have

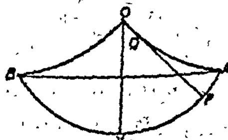
$$0=-\frac{gl}{4a}+\frac{T}{m_1}+\frac{T}{m_2} \quad \text{or} \quad T\left(\frac{1}{m_1}+\frac{1}{m_2}\right)=\frac{gl}{4a}$$

or $$T=\frac{gl}{4a}\left(\frac{m_1 m_2}{m_1+m_2}\right)=\text{constant}.$$

Hence proved.

Exercise on § 10 – § 11

Ex. A particle starts from rest from the cusp of a smooth cycloid down its arc, axis being vertical and vertex lowest. Show that the particle will reach the lowest point in time $\pi\sqrt{(a/g)}$ and pressure on the curve will be twice the weight of the particle at that time.



(Fig. 42)

*§ 12. Cycloidal Pendulum. (Gorakhpur 86 ; Lucknow 90)

Let arc AVB be an arc of a cycloid and let arc OA and arc OB be the evolutes of AVB. Arcs OA and OB are also cycloids such that arc

OA = arc OB = arc $AV = 4a$, if a be the radius of generating circle of the cycloid AVB.

Let $OV = 4a$ and the upper end of the string of length $4a$ be attached at O and P be the bob of mass m (say). The string is wrapt round OA or OB. The bob is then allowed to fall and as it falls the string wraps or unwraps itself about OA and OB. Then straight portion PQ of the string always remaining tangent to the evolute OA or OB and normal to the involute AVB. Hence P describes the involute AVB as QP = arc OA.

Thus we see that the bob moves on the cycloid under gravity and tension of the string which is along the normal at P. $\therefore$ The motion is oscillatory and the period is $4\pi\sqrt{(a/g)}$. This period being independent of the amplitude, the cycloidal pendulum *is perfectly isochronomous*.

In the case of simple pendulum the period $2\pi\sqrt{(l/g)}$ is obtained when we suppose that θ is small. (See § 7 Result (iv) Page 42). If θ is not small we can not take θ for $\sin\theta$ and the period will be given in terms of the amplitude of vibration.

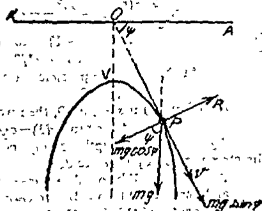
Thus Simple pendulum *is approximately isochronomous.*

Hence cycloidal pendulum is perfectly isochronomous whereas the simple pendulum is approximately so.

Motion on curves other than circle and cycloid.

Ex. 1. A particle falls from rest from the vertex of an inverted catenary. Prove that the particle will leave the curve when the path described is $\sqrt{3}$ times the vertical distance through which it has fallen.

Sol. Let V be the vertex of the catenary and $A'OA$ be its directrix. Let the axis of the catenary meet its directrix at O. Let $VO = c$.



(Fig. 43)

Let P be the position of the particle at time t. Let arc $VP = s$ and the angle which the tangent at P makes with the directrix $A'OA$ be ψ. Then the intrinsic equation of the catenary can

be written as $s=c\tan\psi$. ...(i)

Let v be the velocity of the particle at P. The forces acting on the particle at P are (i) its weight mg acting vertically downwards and (ii) the normal reaction R between the particle and the catenary.

The equations of the motion in the tangential and inward drawn normal sense at P are $m.\dfrac{d^2s}{dt^2}=mg\sin\psi$. ...(ii)

or $m.\dfrac{v^2}{\rho}=mg\sin\psi-R$. ...(iii)

From (i) we have $\tan\psi=s/c$ which gives

$\sin\psi=s/\sqrt{(s^2+c^2)}$ and $\cos\psi=c/\sqrt{(s^2+c^2)}$ ...(iv)

Now from (ii) we get $\dfrac{d^2s}{dt^2}=g\sin\psi=\dfrac{gs}{\sqrt{(s^2+c^2)}}$, from (iv).

Multiplying both sides by 2 (ds/dt) and integrating, we have

$(ds/dt)^2=g\sqrt{(s^2+c^2)}+K$, ...(v)

where K is constant of integration.

Initially the particle was at the vertex V, where $s=0$ and $ds/dt=0$ (given).

∴ From (v) we get $0=2gc+K$ or $K=-2gc$.

∴ From (v) we have

$(ds/dt)^2=2g\sqrt{(s^2+c^2)}-2gc=2g\,[\sqrt{(s^2+c^2)}-c]$

$=2g\,[\sqrt{(c^2\tan^2\psi+c^2)}-c]=2gc\,(\sec\psi-1)$ ...(vi)

Also from (i) we have $\rho=ds/d\psi=c\sec^2\psi$ ...(vii)

Now the particle will leave the curve at the point where $R=0$ and if that point be Q (say), then at Q let $\psi=\psi_1$ and so from (iii), (vi) and (vii) at Q we have

$m\left[\dfrac{2gc\,(\sec\psi_1-1)}{c\sec^2\psi_1}\right]=mg\cos\psi_1-0$, putting $R=0$, $\psi=\psi_1$

or $2(\sec\psi_1-1)=\sec\psi_1$ or $\sec\psi_1=2$

or $\cos\psi_1=1/2$ or $\psi_1=\pi/3$.

∴ At Q, where the particle leaves the curve, we have

$\psi=\psi_1=\pi/3$. ...(viii)

Also from (i) if $s=s_1$ at Q, then we have

$s_1=c\tan(\pi/3)=c\sqrt{3}$. ...(ix)

Again we know depth of any point on the catenary below directrix is given by $y=c\sec\psi$.

∴ depth of V below directrix $=c\sec 0=c$

and depth of Q below directrix $=c\sec(\pi/3)=2c$

∴ depth of Q below $V=2c-c=c$, which is the vertical distance through which the particle has fallen and it is denoted by z_1, then $z_1=c$. ...(x)

$\therefore$ From (ix) and (x) we get $s_1=\sqrt{3}\,z_1$ *i.e.* at Q, where the particle leaves the given catenary, the path described$=\sqrt{3}$ times the vertical distance fallen through. Hence proved.

Ex. 2. A particle slides down a catenary, whose plane is vertical and vertex upwards, the velocity at any point being due to the fall from the directrix, prove that the pressure at any point varies inversely as the distance of that point from the directrix.

Sol Refer Fig. 43 Page 71.

As in result (iii) of last example, we can get

$$m\,(v^2/\rho)=mg\cos\psi-R, \quad \text{...(i)}$$

where the symbols have the same meaning as in Ex. 1 above, (we are to show it in the exam.)

Also we are given that velocity v at any point is due to fall from the directrix. So if y be the depth of any position P of the particle below the directrix and ψ be the angle which the tangent at P makes with directrix of the catenary $y=c\tan\psi$, then we know y is given by $\qquad y=c\sec\psi \qquad$ ...(ii)

(See Author's Statics, chapter on Catenary)

$\therefore$ $\qquad v^2=2g\,(y)$ $\qquad$ (Note)

or $\qquad v^2=2gc\sec\psi$, from (ii) $\qquad$ ...(iii)

Also for the catenary $s=c\tan\psi$, we have

$$\rho=ds/d\psi=c\sec^2\psi \quad \text{...(iv)}$$

$\therefore$ From (i) at any position P of the particle we have pressure

$$R=mg\cos\psi-m\,(v^2/\rho)$$

or $$R=mg\cos\psi-\frac{m\,[2gc\sec\psi]}{c\sec^2\psi}, \text{ from (iii) and (iv)}$$

$$=mg\cos\psi-2mg\cos\psi=-mg\cos\psi$$

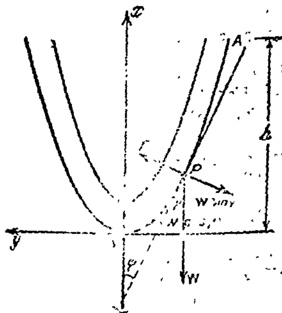
or $$R=-\frac{mg}{\sec\psi}=-\frac{mg\,c}{c\sec\psi}=-\frac{mg\,c}{y}, \text{ from (ii)}$$

i e. $\qquad R\propto 1/y$

i.e. pressure R at any point P varies inversely as the distance y of P from the directrix. Hence proved.

Ex. 3. A smooth parabolic tube placed, vertex downwards in a vertical plane, a particle of weight W slides down the tube from rest under the influence of gravity. Find in any position the reaction of the tube, h being the height above vertex from which it slides.

Sol. Let V be the vertex, Vx the axis and Vy be the tangent at the vertex of the parabolic tube. Let the equation of the parabola be $$y^2=4ax. \qquad ...(i)$$



(Fig. 44)

Let P be any position of the given particle. Let the tangent at P to the parabola make an angle ψ with x-axis. Let the particle start from A, such that height of A above V is h. Forces acting at P are the weight W and the normal reaction R.

The equations of motion in the tangential and inward drawn normal sense are $$m\frac{dv}{ds}=-W\cos\psi \qquad ...(ii)$$

and $$m\frac{v^2}{\rho}=R-W\sin\psi, \qquad ...(iii)$$

where $$mg=W \quad \text{or} \quad m=W/g. \qquad ...(iv)$$

From (ii) and (iv), $\dfrac{W}{g}\dfrac{v\,dv}{ds}=-W\dfrac{dx}{ds}$, $\because \cos\psi=\dfrac{dx}{ds}$

or $$2v\,dv=-2g\,dx.$$

Integrating, $v^2=-2gx+C$, where C is const. of integration.

Initially at A, $x=h$ and the particle starts from rest, so $v=0$ at A.

$\therefore$ we have $0=-2gh+C$ or $C=2gh$.

$\therefore$ $v^2=-2gx+2gh=2g(h-x)$. ...(v)

Again from (i), $2y\dfrac{dy}{dx}=4a$ or $\dfrac{dy}{dx}=\dfrac{2a}{y}$. ...(vi)

And $$\frac{d^2y}{dx^2}=2a\frac{d}{dx}\left(\frac{1}{y}\right)=2a\left(-\frac{1}{y^2}\frac{dy}{dx}\right)$$

$$=-\frac{2a}{y^2}\left(\frac{2a}{y}\right), \text{ from (vi)}$$

or $$d^2y/dx^2=-4a^2/y^3$$ ...(vii)

$\therefore$ radius of curvature ρ at P

$$=\frac{[1+(dy/dx)^2]^{3/2}}{(d^2y/dx^2)}=\frac{[1+(2a/y)^2]^{3/2}}{-4a^2/y^3}, \text{ from (vi), (vii)}$$

$$=\frac{(y^2+4a^2)^{3/2}}{4a^2}, \text{ numerically}$$ (Note)

$$=(4ax+4a^2)^{3/2}/4a^2, \text{ from (i)}$$

or $$\rho=2(x+a)^{3/2}/a^{1/2}$$ ...(viii)

Again from (vi) we have $\tan\psi=2a/y$

$\therefore$ $$\sin\psi=\frac{2a}{\sqrt{(y^2+4a^2)}}=\frac{2a}{\sqrt{(4ax+4a^2)}}, \text{ from (i)}$$

or $$\sin\psi=\sqrt{[a/(x+a)]}$$ ...(ix)

Substituting values of m, v^2, ρ and $\sin\psi$ from (iv), (v), (viii) and (ix) in (iii) we have

$$R=\frac{W}{g}\cdot\frac{2g(h-x)a^{1/2}}{2(x+a)^{3/2}}+W\frac{\sqrt{a}}{\sqrt{(x+a)}}$$

$$=\frac{W\sqrt{a}}{\sqrt{(x+a)}}\left[\frac{(h-x)}{(x+a)}+1\right]=\frac{W(h+a)\sqrt{a}}{(x+a)^{3/2}},$$

which gives the required reaction. **Ans.**

Exercises

Ex. 1. A particle describes a smooth curve under gravity in a vertical plane. If the arcual distance travelled in time t varies as sinh nt, find the shape of the curve.

Ex. 2. A particle slides on the curve $x^2=4a(y-a)$ with a velocity due to a fall from the horizontal x-axis, the y-axis being vertically downwards. Find the pressure on the curve at any time, and the time of sliding from $y=b$ to $y=c$.

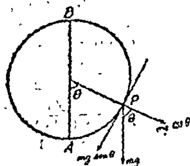
Ex. 3. A particle moves on the outside of an ellipse of eccentricity e whose major axis is vertical, starting from rest at the highest point. Show that it will leave the curve at a point whose eccentric angle is ϕ, where $e^2 \cos^3 \phi = 3 \cos \phi - 2$.

(Bundelkhand 90; Kumaun 91)

SOME IMPORTANT SOLVED EXAMPLES

****Ex. 1.** A heavy bead slides on a smooth fixed circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point, prove that the radius through the bead in time t will turn through an angle $2 \tan^{-1} [\sinh t \sqrt{(g/a)}]$ and that the bead will take an infinite time to reach the highest point.

(Agra 88; Gorakhpur 92, Meerut 92 P, 90 S, 87; Purvanchal 91)



(Fig. 45)

Sol. A is the lowest point of the bead. Let the bead be at P after time t, such that $\angle AOP = \theta$ and arc $AP = s$; O being the centre of the circular wire.

The equations of motion in the tangential direction is

$$m \frac{d^2 s}{dt^2} = -mg \sin \theta$$

or $\quad a\,(d^2\theta/dt^2) = -g \sin \theta$, since $s = a\theta$.

Integrating, $a\,(d\theta/dt)^2 = 2ga \cos \theta + C$, where C is constant of integration.

Given at the highest point B, velocity $a\,(d\theta/dt) = 0$

i.e. $\quad a\,(d\theta/dt) = 0$ at $\theta = \pi$

$\therefore \quad 0 = 2ga \cos \pi + C \quad$ or $\quad C = 2ga$

$\therefore \quad (a\, d\theta/dt)^2 = 2ga\,(1 + \cos \theta) = 4ga \cos^2 \tfrac{1}{2}\theta$

or $\quad a\,(d\theta/dt) = 2\sqrt{(ga)} \cos \tfrac{1}{2}\theta \quad$ ...(i)

or $\quad dt = \tfrac{1}{2}\sqrt{(a/g)} \sec \tfrac{1}{2}\theta \, d\theta.$

Integrating we find that time t to reach the point P from A is given by $t=\frac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta}\sec\frac{1}{2}\theta\, d\theta$

$$=\sqrt{(a/g)}\left[\log(\tan\tfrac{1}{2}\theta+\sec\tfrac{1}{2}\theta)\right]_{\theta=0}^{\theta}$$

$$=\sqrt{(a/g)}\left[\log(\tan\tfrac{1}{2}\theta+\sec\tfrac{1}{2}\theta)\right]$$

or $t=\sqrt{(a/g)}\sinh^{-1}(\tan\frac{1}{2}\theta)$, since $\sinh^{-1}x=\log\{x+\sqrt{(1+x^2)}\}$

or $\sinh\{t\sqrt{(g/a)}\}=\tan\frac{1}{2}\theta$

or $\theta=2\tan^{-1}\{\sinh(t\sqrt{(g/a)})\}$. Hence proved.

Also if T be the time to reach the highest point from A, then from (i) we get $T=\frac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\pi}\sec\frac{1}{2}\theta\, d\theta$

$$=\tfrac{1}{2}\sqrt{(a/g)}\left[\log\{\tan(\tfrac{1}{2}\theta)+\sec(\tfrac{1}{2}\theta)\}\right]_{\theta=0}^{\pi}$$

$$=\tfrac{1}{2}\sqrt{(a/g)}\left[\log(\tan\tfrac{1}{2}\pi+\sec\tfrac{1}{2}\pi)-\log(\tan 0+\sec 0)\right]$$

$$=\tfrac{1}{2}\sqrt{(a/g)}\left[\log\infty-\log 1\right]=\infty.$$ Hence proved.

****Ex. 2. The middle point of a bridge, in the form of a circular arc on a canal of width 20 metres is at a height 2·5 metres from either end. Find the maximum speed at which a car can safely pass over the bridge if height of the C.G. of the car be 0.75 metres above the point of contact of the wheels with the ground.**

Sol. ADB is the bridge (with D as its highest point) in the form of a circular arc with O as centre. Let x be the radius of this circular arc.

Given $CD=2{\cdot}5$ metres, $AB=20$ metres.

$\therefore$ $BC=10$ metres, $\because$ C is the mid-point of AB.

Now $OC=OD-CD=x-(2{\cdot}5)$ and $OB=x=OD$.

$\therefore$ In $\triangle OBC$, $OB^2=OC^2+CB^2$

i.e. $x^2=[x-(2{\cdot}5)]^2+(10)^2$

or $5x=6{\cdot}25+100=106{\cdot}25$

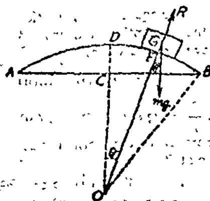
or $x=21{\cdot}25$ metres.

Let G be the C.G. of the car, then height of G above the bridge $=EG=0{\cdot}75$ metres and therefore

$OG=OE+EG=x+0{\cdot}75$

$=21{\cdot}25+0{\cdot}75=22$ metres.

The forces acting on the car of mass m (say) are its weight mg acting vertically downwards and the normal reaction R acting in the direction as shown in the figure.



(Fig. 46)

$\therefore$ The equation of motion in the inward drawn normal sense is $\dfrac{mv^2}{\rho}=mg\cos\theta-R,$

where $\rho=OG=22$ metres$=2200$ cms.

or $\dfrac{mv^2}{2200}=mg\cos\theta-R$ or $R=mg\cos\theta-\dfrac{mv^2}{2200}$.

The car will leave contact with the bridge when $R=0$ (Note) and therefore if the car passes safely over the bridge then R should not vanish *i.e.* $R\geqslant 0$

i.e. $mg\cos\theta-\dfrac{mv^2}{2200}\geqslant 0$ or $\dfrac{v^2}{2200}\leqslant g\cos\theta$

$\therefore$ Max. value of $v^2=2200\,g\cos\theta$, when $\cos\theta$ is max.

$=2200\,g$, $\because$ max. value of $\cos\theta=1$

$=2200\times 981$

or max. value of $v=\sqrt{(2200\times 981)}$ cms/sec.

$=14\cdot 7$ metres/sec. nearly. **Ans.**

Ex. 3 A cycloid is placed with its axis vertical and vertex upwards and a heavy particle is projected from the cusp up the concave side of the curve with velocity $\sqrt{(2gh)}$; prove that the latus rectum of the parabola described after leaving the arc is $(h^2/2a)$, where a is the radius of the generating circle. *(Rohilkhand 87)*

Sol. The equation of the cycloid is $s=4a\sin\psi$. ...(i)

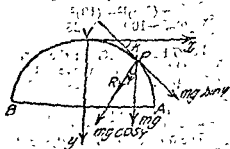
After time t let the particle be at P such that arc $VP=s$ and the tangent at P make an angle ψ with the tangent at the vertex V.

The equations of motion in the tangential and inward drawn normal senses are

$$m\frac{d^2s}{dt^2}=mg\sin\psi \quad ...(ii)$$

$$m.\frac{v^2}{\rho}=mg\cos\psi-R, \quad ...(iii)$$

where R is the normal reaction at P.



(Fig. 47)

From (i) and (ii) we get $d^2s/dt^2=(g/4a)\,s$

Integrating, $(ds/dt)^2=(g/4a)\,s^2+C$, where C is the constant of

At the cusp at A, $ds/dt=\sqrt{(2gh)}$ and $s=4a$

$\therefore$ $2gh=4ag+C$ or $C=2g\ (h-2a)$

$\therefore$ $(ds/dt)^2=(g/4a)\ s^2+2g\ (h-2a)$. ...(iv)

Also the particle leaves the cycloid when $R=0$ and let $\psi=\psi_1$ at the point. Then from (iii) we have $\dfrac{v^2}{\rho}=g\cos\psi_1$

or $\dfrac{(g/4a)\ s^2+2g\ (h-2a)}{4a\cos\psi_1}=g\cos\psi_1$, from (i) and (iv)

or $\dfrac{(g/4a)\ (4a\sin\psi_1)^2+2g\ (h-2a)}{4a\cos\psi_1}=g\cos\psi_1$ from (i)

or $4ag\sin^2\psi_1+2g\ (h-2a)=4ag\cos^2\psi_1$

or $h-2a=2a\ (\cos^2\psi_1-\sin^2\psi_1)=2a\cos 2\psi_1$

or $h=2a\ (1+\cos 2\psi_1)=4a\cos^2\psi_1$

or $\cos^2\psi_1=h/4a$ and $\sin^2\psi_1=1-\cos^2\psi_1=1-(h/4a)$ ...(v)

Also if v_1 be the velocity at the point where the particle leaves contact we have from (iv)

$v_1^2=(g/4a)\ (4a\sin\psi_1)^2+2g\ (h-2a)$, $\because$ from (i) $s=4a\sin\psi$

$=4ag\ \sin^2\psi_1+2g\ (h-2a)=4ag\ [1-(h/4a)]+2g\ (h-2a)$, from (v)

or $v_1^2=gh$. ...(v)

Beyond this point the particle describes a parabolic path with velocity of projection v_1 and angle of projection $(-\psi_1)$, hence the required latus rectum $=\dfrac{\text{"}2u^2\cos^2\alpha\text{"}}{g}=\dfrac{2v_1^2\cos^2(-\psi_1)}{g}$

$=\dfrac{2gh\ (h/4a)}{g}$, from (v) and (vi)

$=h^2/(2a)$. Hence proved.

EXERCISES ON CONSTRAINED MOTION

Ex. 1. A particle moves on the inner side of a smooth circular hoop of radius a whose plane is vertical. It is projected from the bottom with velocity insufficient to carry it round the hoop and comes off at angular distance α from the highest point. Show that it crosses the vertical diameter of the hoop at a distance $\frac{1}{4}a\sec^3\alpha$ $(1-\cos\alpha)^2\ (1+2\cos\alpha)$ below the top of the hoop.

Ex. 2. A particle at the end of a string of length l, the upper end of which is fixed, is projected horizontally with the velocity $\sqrt{(ngl)}$. If the string becomes slack before the particle reaches the top of the circle show that it does so at a height $\frac{1}{3}l\ (1+n)$ above the lowest point. Prove also that the vertex of the parabola that it afterwards describes is higher than the point of projection by

$l\ (1+n)^2\ (8-n)/54$.

Ex. 3. If a pendulum, beating seconds at the foot of a mountain loses 9 seconds a day when taken to its summit; find the height of the mountain assuming the radius of the earth to be 6400 kilometres and neglecting the attraction of the mountain.

(Hint. Apply $\delta = -(n/r)\, a$. Here $n = 24 \times 60 \times 60$; $\delta l = -9$)

Ex. 4. The radius of a circular bridge over a stream is 51 ft. Show that in order that a cyclist may cross the bridge without leaving contact with it at the highest point his speed should not exceed 30 miles per hour, it being given that the height of the centre of gravity of the rider and his machine is $2\frac{1}{2}$ ft. above the ground.

[one mile $= 1 \times 1760 \times 3$ feet; 'g' $= 32$ ft./sec^2]

(Hint. See Ex. 2 Page 77).

Central Orbits

§ 1. Definitions.

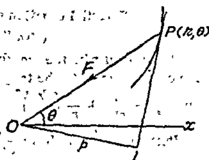
Central Force. *If a force is always directed towards a fixed point it is called a central force and this fixed point is known as the 'centre of force'*

Central Orbit. *The path described by a particle moving under the action of a central force is defined as the 'central orbit' and it is a plane curve.*

§ 2. Differential Equation of a Central Orbit, (Polar form).

(*Agra* 91, 88; *Avadh* 93, 91; *Garhwal* 91, 89; *Gorakhpur* 91, 89; *Meerut* 92; *Purvanchal* 92, 90; *Rohilkhand* 90).

Suppose a particle moves in a plane curve under an acceleration F which is always directed towards a fixed point O in the plane. Let Ox be a fixed line in the plane of the orbit.



(Fig. 1)

Let the position of a particle at time t be $P(r, \theta)$ referred to O as pole and Ox as initial line.

Since the acceleration F is always directed towards O, hence the particle has only radial acceleration (towards O) of magnitude F and no transverse acceleration. (Note)

$\therefore$ The equation of motion in the radial and transverse directions are $\ddot{r}-r\dot{\theta}^2=-F$, (i)

and $$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=0. \quad \text{...(ii)}$$

Integrating (ii), we get $r^2\dot{\theta}=\text{constant}=h$ (say). .. (iii)

Now from (i), (ii) and (iii) we are to get a relation between r and θ which will be the equation (polar) of the central orbit. This can be easily obtained by putting $r=1/u$.

$$\therefore \quad \frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

or $$\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\left(\frac{h}{r^2}\right), \quad \because \text{ from (iii) } \dot{\theta}=\frac{h}{r^2}$$

or $$\dot{r}=\frac{dr}{dt}=-h\frac{du}{d\theta}, \text{ as } r=\frac{1}{u} \quad \text{...(iv)}$$

$$\therefore \quad \ddot{r}=\frac{d^2r}{dt^2}=\frac{d}{dt}\left(-h\frac{du}{d\theta}\right)=-h\,\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$=-h\,\frac{d^2u}{d\theta^2}\frac{h}{r^2}, \quad \because \quad \dot{\theta}=h/r^2, \text{ from (iii)}$$

or $$\ddot{r}=-h^2u^2\frac{d^2u}{d\theta^2}, \quad \because \quad r=\frac{1}{u}. \qquad \ldots(v)$$

Substituting the values of $\dot{\theta}$ and $\ddot{r}$ from (iii) and (v) in (i), we get $$-h^2u^2\frac{d^2u}{d\theta^2}-r\left(\frac{h}{r^2}\right)^2=-F$$

or $$h^2u^2\frac{d^2u}{d\theta^2}+h^2u^3=F \quad \text{or} \quad \frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \qquad \ldots(vi)$$

which is the required differential equation of the central orbit in polar form as it is satisfied by the coördinates (u, θ) of a point on the orbit.

§ 3. Differential equation of a central orbit. (Pedal form).

(Avadh 92; Gorakhpur 92, 90; Lucknow 89)

We know if p be the length of the perpendicular drawn from the pole (or origin) to the tangent to a curve at any point $P(r, \theta)$, then $$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$ ...See Fig. 1 Page 1 of this Ch.

Putting $u=\frac{1}{r}$, we have $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$

$\therefore$ The above equation reduces to

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2. \qquad \ldots(i)$$

Differentiating (i) with respect to θ, we get

$$-\frac{2}{p^3}\frac{dp}{d\theta}=2u\,\frac{du}{d\theta}+2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}$$

or $$-\frac{1}{p^3}\frac{dp}{d\theta}=\left(\frac{d^2u}{d\theta^2}+u\right)\frac{du}{d\theta}$$

$$=\left(\frac{F}{h^2u^2}\right)\cdot\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right), \text{ from (vi) of § 2 above}$$

and $du/d\theta=(-1/r^2)(dr/d\theta)$.

or $$\frac{1}{p^3}\frac{dp}{d\theta}=\frac{F}{h^2}\frac{dr}{d\theta} \quad \text{or} \quad \frac{h^2}{p^3}\frac{dp}{dr}=F, \qquad \ldots(ii)$$

which is the required differential equation of the central orbit in the pedal form.

§ 4. Areal Velocity.

Let $P(r, \theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be the two neighbouring

positions of the particle, at times t and $t+\delta t$, moving along a curve.

Then in time δt, the sectorial area OPQ swept out by OP

$$=\tfrac{1}{2}\cdot OP.OQ \sin\delta\theta \quad \text{(Note)}$$

$$=\tfrac{1}{2}.r.(r+\delta r)\sin\delta\theta$$

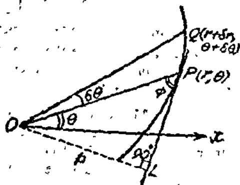
$=\tfrac{1}{2}r^2\delta\theta$, to a first approximation

$\therefore$ Rate of description of the sectorial area OPQ as radius vector passes through

$$OP=\lim_{\delta t\to 0}\left[\frac{\tfrac{1}{2}r^2\delta\theta}{\delta t}\right]$$

$=\tfrac{1}{2}r^2\dot{\theta}=\tfrac{1}{2}h$, from (iii) of § 2

$=$constant, as h is constant. ...(i)



(Fig. 2)

Hence this rate of description of the sectorial area is constant or in other words 'the sectorial area traced out by radius vector to the centre of force increases uniformly per unit of time'.

This rate of description of sectorial area is defined as the areal velocity of particle at P about the fixed point O.

Also sectorial area $OPQ=\tfrac{1}{2}$ (base PQ). perpendicular from O on PQ. (Note)

$\therefore$ Rate of description of the sectorial area OPQ

$$=\lim_{\delta t\to 0}\left[\frac{\tfrac{1}{2}.PQ.(\text{perp. from } O \text{ on } PQ)}{\delta t}\right]$$

$$=\tfrac{1}{2}\lim_{\delta t\to 0}\left[\frac{PQ}{\delta s}\cdot\frac{\delta s}{\delta t}\cdot(\text{perp. from } O \text{ on } PQ)\right], \quad \text{...(ii)}$$

(Note)

Now as $\delta t\to 0$, $Q\to P$ and secant $PQ\to$tangent at P to the curve and so (perp. from O on PQ) $\to p$ and $\dfrac{PQ}{\delta s}\to 1$.

$\therefore$ from (ii) we get rate of description of sectorial area OPQ

$$=\tfrac{1}{2}\left[1.\frac{ds}{dt}.p\right]=\tfrac{1}{2}vp. \quad \text{...(iii)}$$

$\therefore$ From (i) and (iii) we get $\tfrac{1}{2}h=\tfrac{1}{2}vp$.

or $h=vp$ or $v=h/p$ ...(iv)

i.e. the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

Cor. We know $v=h/p$ or $v^2=h^2/p^2$

or $\frac{v^2}{h^2} = \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$, where $u = \frac{1}{r}$

(See Author's Differential Calculus)

or $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] \qquad \text{...(v)}$$

Momentum and Angular Momentum

Momentum If v be the velocity at any instant t of the particle of mass m moving in a curve, then mv is called its *linear momentum or momentum.*

Angular Momentum The angular momentum of a particle at and instant t about a point O is the moment of its linear momentum at that instant about O

From Fig. 2 Page 3, we have angular momentum of the particle at P (or at the instant t)

$$= (mv)\,p, \text{ where } p = OL$$
$$= mh, \text{ from (iv)}$$
$$= \text{constant.}$$

***§ 5. Elliptic Orbit. (Centre of force being the focus).**

A particle moves in an ellipse under a force which is always directed towards its focus, to find the law of force, the velocity at any point of its path and the periodic time.

(*Agra* 89 ; *Avadh* 91, 89 ; *Gorakhpur* 92, 90, 89 ; *Purvanchal* 93 ; *Rohilkhand* 92, 91)

We know the polar equation of the ellipse referred to focus as pole is $\frac{l}{r} = 1 + e\cos\theta$, where l is the semi-latus rectum and $e < 1$

or $$u = (1 + e\cos\theta)/l, \text{ where } u = 1/r \qquad \text{...(i)}$$

$$\therefore \frac{du}{d\theta} = -\frac{e}{l}\sin\theta,\ \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta \qquad \text{...(ii)}$$

Also we know that the differential equation of central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \qquad \text{[See § 2 (vi) Page 2]}$$

or $$-\frac{e}{l}\cos\theta + \frac{1 + e\cos\theta}{l} = \frac{F}{h^2u^2}$$

or $$\frac{1}{l} = \frac{F}{h^2u^2} \text{ or } F = \frac{h^2}{l}u^2 = \frac{\mu}{r^2}, \text{ where } \mu = \frac{h^2}{l}$$

$\therefore$ The acceleration F varies inversely as the square of the distance of the particle from the focus. (*Agra* 89 ; *Avadh* 91)

Also from $\mu = h^2/l$ we get $h = \sqrt{(\mu \times l)}$. .. (iii)

Again we know that the velocity of the particle at any point in a central orbit is given by

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta} \right)^2 \right]$$ ...See Cor. § 4 Page 3

$$\therefore \quad v^2 = \mu l \left[\left(\frac{1 + e \cos \theta}{l} \right)^2 + \left(-\frac{e}{l} \sin \theta \right)^2 \right],$$

from (i), (ii), (iii)

$$= (\mu/l)\,[1 + 2e \cos \theta + e^2 \cos^2 \theta + e^2 \sin \theta]$$

$$= (\mu/l)\,[1 + 2e \cos \theta + e^2]$$

$$= (\mu/l)\,[2\,(1 + e \cos \theta) + (e^2 - 1)]$$

$$= \mu \left[2 \left(\frac{1 + e \cos \theta}{l} \right) - \frac{(1 - e^2)}{l} \right]$$

or $$v^2 = \mu \left[\frac{2}{r} - \frac{1}{a} \right],$$ .. (iv)

where $2a$ is the major axis the ellipse.

$$\left[\text{Here } l = \frac{b^2}{a} = \frac{a^2\,(1 - e^2)}{a} = a\,(1 - e^2) \quad \text{or} \quad \frac{1 - e^2}{l} = \frac{1}{a} \right]$$

Again if T be the periodic time *i.e.* the time taken by the particle in describing the whole arc of the ellipse, then $\frac{1}{2}h\ T = \pi ab$, where πab is the area of the ellipse and the rate of description of the arc is $\frac{1}{2}h$. (See § 4 Pages 2–3)

or $$T = (2\pi ab)/h = (2\pi ab)/\{\sqrt{(\mu l)}\}, \quad \because \quad h^2 = \mu l \text{ and } l = b^2/a$$

or $$T = \frac{2\pi ab}{\sqrt{[\mu\,(b^2/a^2)]}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$ (v)

i.e. time period is proportional to $a^{3/2}$.

Also $$T^2 = \frac{4\pi a^3}{\mu} = \frac{\pi (8a^3)}{2\mu} = \frac{\pi}{2\mu}\,(2a)^3$$

i.e. square of periodic time varies as the cube of the major axis. *(Gorakhpur 90)*

§ 6. **Hyperbolic and Parabolic orbits. (Centre of force being the focus).** *(Rohilkhand 92)*

(a) Hyperbolic orbit. If the path traced out by the particle be a hyperbola when it is moving under the action of a force which is always directed towards its focus, then also we shall proceed as in § 5 Pages 4-5. The only difference being that here we have $$l = \frac{b^2}{a} = \frac{a^2\,(e^2 - 1)}{a},\ e > 1.$$

Hence in the case of the hyperbolic orbit we shall have

$$F=\frac{\mu}{r^2} \text{ and } v^2=\mu\left[\frac{2}{r}+\frac{1}{a}\right]$$

(b) Parabolic orbit. In this case we shall have $e=1$ and therefore we have $v^2=\mu(2/r)$.

Thus in this case we have $F=\mu/r^2$ and $v^2=2\mu/r$.

*§ 7. *A particle moves with a central acceleration $\mu/(distance)^2$, to find the path and discuss the three possible cases.*

(Avadh 90; Gorakhpur 89)

Since the acceleration is central, so the orbit of the particle is a central one and also we are given that $F=\mu/r^2$, where r is the distance of the particle from the centre.

The differential equation of the path in the pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr}=F=\frac{\mu}{r^2}$$ ...See § 3 Page 2

or $$\frac{h^2}{p^3}dp=\frac{\mu}{r^2}dr.$$

Integrating, $-\frac{h^2}{2p^2}=-\frac{\mu}{r}+c$ or $\frac{h^2}{p^2}=\frac{2\mu}{r}+c_1$, ...(i)

where c_1 is constant of integration, is the required path.

Also from § 4 Pages 2-3 we know that $h=pv$.

$\therefore$ From (i) we have $v^2=\frac{h^2}{p^2}=\frac{2\mu}{r}+c_1$ ...(ii)

Now we know that the pedal equations referred to focus as pole of ellipse, parabola and that branch of hyperbola which is nearer to the pole (*i.e.* focus) are

$$\frac{b^2}{p^2}=\frac{2a}{r}-1;\ p^2=ar \text{ and } \frac{b^2}{p^2}=\frac{2a}{r}+1$$ (Note)

respectively, where $2a$, $2b$ are the lengths of the semi-axes in the case of ellipse, $4a$ is the length of latus rectum in the case of parabola and $2a$, $2b$ are the lengths of transverse and conjugate axes in the case of hyperbola.

Now three cases arise :

Case I. Elliptic Path.

Comparing the equation (ii) with the pedal equation of the ellipse viz. $\frac{b^2}{p^2}=\frac{2a}{r}-1$, we get

$$\frac{h^2}{b^2}=\frac{2\mu}{2a}=\frac{c_1}{-1} \Rightarrow h^2=\frac{\mu b^2}{a},\ c_1=-\frac{\mu}{a}$$

$\therefore$ From (ii), for the elliptic path, we have

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left[\frac{2}{r} - \frac{1}{a}\right] \quad \text{...(iii)}$$

Also from (iii) it is evident that $v^2 < (2\mu/r)$, for the elliptic path.

Case II. Parabolic Path

Comparing the equation (ii) with the pedal equation of the parabola viz. $p^2 = ar$, or $\frac{1}{p^2} = \frac{(1/a)}{r}$ **(Note)**

we get $\frac{h^2}{1} = \frac{2\mu}{(1/a)}$ and $c_1 = 0$ *i.e.* $h^2 = 2a\mu$, $c_1 = 0$.

$\therefore$ From (ii), for the parabolic path, we have

$$v^2 = \frac{h^2}{p^2} = \frac{2a\mu}{ar}, \quad \because h^2 = 2a\mu;\ p^2 = ar$$

or $v^2 = 2\mu/r$, for the parabolic path. ...(iv)

Case III. Hyperbolic Path.

Comparing the equation (ii) with the pedal equation of the hyperbola viz. $b^2/p^2 = (2a/r) + 1$, we get

$$\frac{h^2}{b^2} = \frac{2\mu}{2a} = \frac{c_1}{1} \Rightarrow h^2 = \frac{\mu b^2}{a},\ c_1 = \frac{\mu}{a}$$

$\therefore$ From (ii), for the hyperbolic path, we have

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + \frac{\mu}{a} = \mu\left[\frac{2}{r} + \frac{1}{a}\right]. \quad \text{...(v)}$$

From here it is evident that $v^2 > (2\mu/r)$ for the hyperbolic path.

From above three cases we conclude that $\frac{h^2}{p^2} = \frac{2\mu}{r} + c_1$. always represents a conic section referred to the focus as pole (Here pole is also the centre of force) and it represents an ellipse, parabola or hyperbola according as $c_1 >, =$ or < 0, *i.e.* according as c_1 is positive, zero or negative.

Also we conclude here that

(I) If $v^2 < (2\mu/r)$, then the path is elliptic,

(II) If $v^2 = (2\mu/r)$, then the path is parabolic,

and (III) If $v^2 > (2\mu/r)$, then the path is hyperbolic.

Solved Examples on § 1 to § 7.

***Ex. 1 (a). A particle moves in a conic $l/r = 1 + e\cos\theta$, under a force which is always directed towards the focus. Find the law of force and the velocity of the particle at any time**

Solution. Do as in § 5 and § 6 Pages 4-6.

Ex. 1 (b). A particle moves in an elliptic orbit $r=a(1-e^2)/(1+e\cos\theta)$ with one of the foci as force centre. Find the law of force. *(Bundelkhand 92)*

Solution. The given equation can be rewritten as

$$(1+e\cos\theta)/[a(1-e^2)]=u, \text{ where } u=1/r$$

$$\therefore \quad \frac{du}{d\theta}=\frac{-e\sin\theta}{a(1-e^2)},\ \frac{d^2u}{d\theta^2}=\frac{-e\cos\theta}{a(1-e^2)}.$$

Also we know that differential equation of central orbit is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2} \qquad \text{...See § 2 (vi) Page 2}$$

or
$$\frac{-e\cos\theta}{a(1-e^2)}+\frac{1+e\cos\theta}{a(1-e^2)}=\frac{F}{h^2u^2}.$$

or
$$\frac{1}{a(1-e^2)}=\frac{F}{h^2u^2} \quad \text{or} \quad F=\frac{h^2}{a(1-e^2)}\cdot\frac{1}{r^2},\ \because u=\frac{1}{r}.$$

$\therefore F \propto 1/r^2$ i.e. the required force varies inversely as the square of the distance. Ans.

***Ex. 1 (c).** A particle describes the curve $r^n=a^n\cos n\theta$ under a force F to the pole. Find the law of force.
(Avadh 93, 92; Gorakhpur 92, 89; Kumaun 92, 90, 89)

Solution. The curve is $r^n=a^n\cos n\theta$

or
$$1=a^nu^n\cos n\theta, \quad \because u=1/r.$$

Differentiating both sides w.r.t. to θ, we get

$$0=a^n\left[u^n(-n\sin n\theta)+nu^{n-1}\frac{du}{d\theta}\cos n\theta\right]$$

or
$$\frac{du}{d\theta}=u\tan n\theta. \qquad \text{...(i)}$$

[illegible] w.r.t. to θ, we get

[illegible]

$$=un\sec^2 n\theta+u\tan^2 n\theta, \text{ from (i)}.$$

The differential equation of the path is $\dfrac{d^2u}{d\theta^2}+u=\dfrac{F}{h^2u^2}$

or
$$(un\sec^2 n\theta+u\tan^2 n\theta)+u=\frac{F}{h^2u^2}, \text{ putting the value of } \frac{d^2u}{d\theta^2}$$

or
$$u(n+1)\sec^2 n\theta=\frac{F}{h^2u^2}$$

or
$$F=h^2u^3(n+1)\sec^2 n\theta=\frac{h^2}{r^3}(n+1)\left(\frac{a^n}{r^n}\right)^2, \because \cos n\theta=\frac{r^n}{a^n}$$

or $F = \frac{h^2 (n+1) a^{2n}}{r^{2n+3}}$ or $F \propto \frac{1}{r^{2n+3}}$

i.e. the force F varies inversely as $(2n+3)$rd power of the distance from the pole.

Ex 1 (d). A particle describes the curve $r^2 = a^2 \cos 2\theta$ under a force directed to the pole, show that the force is proportional to $1/r^7$. (*Agra* 91, 89; *Gorakhpur* 91)

Solution. Do as Ex. 1 (c) above. Here $n = 2$.

Ex. 1 (e). A particle describes the path $r^4 = a^4 \cos 4\theta$, under a force which is always directed towards the pole, find the law of force and the velocity at any point.

Solution Do as Ex. 1 (c) above. Here $n = 4$.

θ under the force

or an attractive central force acting at a fixed point, describes a circular orbit passing through that fixed point. (*Bundelkhand* 91)

Solution. Do as Ex. 1 (c) above.

***Ex. 2 (a).** A particle describes a circle, pole on its circumference, under a force P to the pole. Find the law of force. (*Bundelkhand* 92)

Solution. The equation of the circle with pole on its circumference is $r = a \cos \theta$ or, $1/u = a \cos \theta$ ...(i)

or $1 = au \cos \theta$.

Differentiating both sides of above w.r. to θ, we get

$$0 = a \frac{du}{d\theta} \cos \theta + au(-\sin \theta) \text{ or } \frac{du}{d\theta} = u \tan \theta. \quad \text{...(ii)}$$

Differentiating again both sides w.r. to θ, we get

$$\frac{d^2u}{d\theta^2} = u \sec^2 \theta + \frac{du}{d\theta} \tan \theta$$

or $$\frac{d^2u}{d\theta^2} = \frac{1}{a} \sec^3 \theta + u \tan^2 \theta, \text{ from (i) and (ii)}$$

$$= \frac{1}{a} \sec^3 \theta + \frac{1}{a} \sec \theta \tan^2 \theta, \text{ from (i)}$$

$$= \frac{1}{a} \sec \theta (\sec^2 \theta + \tan^2 \theta) = \frac{1}{a} \sec \theta (\sec^2 \theta + \sec^2 \theta - 1)$$

or $$\frac{d^2u}{d\theta^2} = \frac{1}{a} \sec \theta (2 \sec^2 \theta - 1) \quad \text{...(iii)}$$

Also the differential equation of the path is $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$

or $\quad \frac{1}{a}\sec\theta\,(2\sec^2\theta-1)+u=\frac{F}{h^2u^2}$, from (iii)

or $\quad \frac{1}{a}\sec\theta\,(2\sec^2\theta-1)+\frac{1}{a}\sec\theta=\frac{F}{h^2u^2}$, from (i)

or $\quad \frac{2}{a}\sec^3\theta=\frac{F}{h^2u^2}$ or $F=\frac{2}{a}(au)^3h^2u^2$, from (i)

or $\quad F=2a^2h^2u^5=\frac{2a^2h^2}{r^5}$ or $F\propto\frac{1}{r^5}$

i.e. the force F varies inversely as the fifth power of the distance from the pole.

Ex. 2 (b). A particle describes the cardioid $r=a(1+\cos\theta)$ under a central force to the pole. Find the law of force.

(*Agra* 92; *Rohilkhand* 90)

Solution. The curve is $r=a(1+\cos\theta)$ or $u=1/[a(1+\cos\theta)]$

or $\quad u=\frac{1}{a(2\cos^2\frac{1}{2}\theta)}=\frac{1}{2a}\sec^2\frac{1}{2}\theta,$ ...(i)

where $\quad u=1/r.$

Differentiating both sides of (i) with respect to θ, we get

$$\frac{du}{d\theta}=\frac{1}{2a}.2\sec\tfrac{1}{2}\theta.\sec\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta.\tfrac{1}{2}=\frac{\sec^2\frac{1}{2}\theta\tan\frac{1}{2}\theta}{2a}.$$

Again differentiating both sides with respect to θ, we get

$$\frac{d^2u}{d\theta^2}=\frac{1}{2a}[\sec^2\tfrac{1}{2}\theta.\sec^2\tfrac{1}{2}\theta.\tfrac{1}{2}+\tan\tfrac{1}{2}\theta.2\sec^2\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta\tfrac{1}{2}]$$

$$=\frac{1}{2a}[\tfrac{1}{2}\sec^4\tfrac{1}{2}\theta+\sec^2\tfrac{1}{2}\theta\tan^2\tfrac{1}{2}\theta]. \quad ...(ii)$$

The differential equation of the path is $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$

or $\quad \frac{F}{h^2u^2}=\frac{1}{2a}[\frac{1}{2}\sec^4\frac{1}{2}\theta+\sec^2\frac{1}{2}\theta\tan^2\frac{1}{2}\theta]+\frac{1}{2a}[\sec^2\frac{1}{2}\theta]$,

from (i) and (ii)

or $\quad \frac{F}{h^2u^2}=\frac{\sec^2\frac{1}{2}\theta}{2a}[\frac{1}{2}\sec^2\frac{1}{2}\theta+\tan^2\frac{1}{2}\theta+1]$

$$=\frac{\sec^2\frac{1}{2}\theta}{2a}[\tfrac{1}{2}\sec^2\tfrac{1}{2}\theta+\sec^2\tfrac{1}{2}\theta],\ \because 1+\tan^2\tfrac{1}{2}\theta=\sec^2\tfrac{1}{2}\theta$$

$$=\frac{3}{4a}(\sec^4\tfrac{1}{2}\theta)=\frac{3}{4a}(2au)^2,\text{ from (i)}$$

or $F=3au^2(h^2u^2)=3ah^2u^4=3ah^2/r^4$, $\because u=1/r$.

$\therefore$ $F \propto (1/r^4)$ *i.e.* the force varies inversely as the fourth power of the distance from the pole. **Ans.**

Ex. 3 (a). A particle describes the equiangular spiral $r=ae^{\theta \cot \alpha}$ under a force F to the pole. Find the law of force.

(*Gorakhpur* 90; *Lucknow* 89)

Solution. The curve is $r=ae^{\theta \cot \alpha}$ or $1=aue^{\theta \cot \alpha}$, ...(i)
where $u=1/r$.

Differentiating both sides of (i) with respect to θ, we get

$$0=ae^{\theta \cot \alpha}\frac{du}{d\theta}+au \cot \alpha \, e^{\theta \cot \alpha}$$

or $$\frac{du}{d\theta}=-u \cot \alpha. \quad ...(ii)$$

$\therefore$ $$\frac{d^2u}{d\theta^2}=-\frac{du}{d\theta}\cot \alpha=-(-u \cot \alpha)(\cot \alpha), \text{ from (ii)}$$

or $$\frac{d^2u}{d\theta^2}=u \cot^2 \alpha. \quad ..(iii)$$

The differential equation of the path is $\dfrac{d^2u}{d\theta^2}+u=\dfrac{F}{h^2u^2}$

or $u \cot^2 \alpha+u=\dfrac{F}{h^2u^2}$, from (iii) putting the value of $\dfrac{d^2u}{d\theta^2}$,

or $F=h^2u^3(\cot^2 \alpha+1)=h^2u^3 \operatorname{cosec}^2 \alpha=\dfrac{h^2 \operatorname{cosec}^2 \alpha}{r^3}$, $\because u=1/r$

$\therefore$ $F\propto(1/r^3)$ *i.e* the force varies inversely as the third power of the distance from the pole. **Ans.**

Ex. 3 (b). A particle describes the angular spiral $r=ae^{m\theta}$, (a and m are constants) under a force P to the pole. Prove that P varies inversely as the cupe of the distance.

Solution. Do as Ex. 3 (a) above. Instead of $\cot \alpha$ we have m.

Ex. 4. A particle describes the curve $r^2=a^2 \sin 2\theta$ under a force F to the pole. Find the law of force.

Solution. The curve is $r^2=a^2 \sin 2\theta$ or $a^2u^2 \sin 2\theta=1$, ...(i)
where $u=1/r$.

Differentiating both sides of (i) with respect to θ, we get

$$a^2\left[2u\frac{du}{d\theta}\sin 2\theta+2u^2 \cos 2\theta\right]=0 \text{ or } \frac{du}{d\theta}=-u \cot 2\theta \quad ...(ii)$$

Differentiating again, we get $\dfrac{d^2u}{d\theta^2}=-\dfrac{du}{d\theta}\cot 2\theta+2u \operatorname{cosec}^2 2\theta$

or $$\frac{d^2u}{d\theta^2}=u\cot^2 2\theta+2u\,\text{cosec}^2 2\theta, \text{ from (ii).}$$

The differential equation of the path is $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$

or $u\cot^2 2\theta+2u\,\text{cosec}^2 2\theta+u=F/(h^2u^2)$,

or $u\,[\cot^2 2\theta+2\,\text{cosec}^2 2\theta+1]=F/(h^2u^2)$, from (iii)

or $F=h^2u^3\,[3\,\text{cosec}^2 2\theta]=3h^2u^3\,[(a^2u^2)^2]$, from (i)

$=3a^4h^2u^7=4a^4h^2/r^7$

or $F\propto(1/r^7)$ i.e. the force varies inversely as the seventh power of the distance from the pole. Ans

Ex 5. A particle describes the curve $r\cosh n\theta=a$ under a force F to the pole. Find the law of force.

Solution The curve is $r\cosh n\theta=a$ or $au=\cosh n\theta$, ...(i)

where $u=1/r$.

Differentiating both sides of (i) with respect to θ, we get

$$a\frac{du}{d\theta}=n\sinh n\theta.$$

Differentialting again, we get $a\frac{d^2u}{d\theta^2}=n^2\cosh n\theta$

Differential equation of the path is $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$

or $\frac{n^2}{a}\cosh n\theta+\frac{\cosh n\theta}{a}=\frac{F}{h^2u^2}$, from (i) and (ii)

or $\frac{1}{a}(n^2+1)\cosh n\theta=\frac{F}{h^2u^2}$ or $\frac{F}{h^2u^2}=(n^2+1)u$, from (i)

or $F=(n^2+1)h^2u^3=(n^2+1)h^2/r^3$.

$\therefore F\propto(1/r^3)$ i.e. the force varies inversely as the third power of the distance from the pole. Ans.

Ex. 6. A particle describes the curve $au=\tanh(\theta/\sqrt{2})$ under a force F to the pole. Find the law of force. (Bundelkhand 91)

Solution. The curve is $au=\tanh(\theta/\sqrt{2})$. ...(i)

Differentiating both sides of (i) with respect to θ, we get

$$a\frac{du}{d\theta}=\frac{1}{\sqrt{2}}\,\text{sech}^2\left(\frac{\theta}{\sqrt{2}}\right).$$

Differentiating again, $a\frac{d^2u}{d\theta^2}=-\text{sech}^2\frac{\theta}{\sqrt{2}}\tanh\frac{\theta}{\sqrt{2}}$. ...(ii)

The differential equation of the path is $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$.

or $-\frac{1}{a}\operatorname{sech}^2\frac{\theta}{\sqrt{2}}\tanh\frac{\theta}{\sqrt{2}}+\frac{1}{a}\tanh\frac{\theta}{\sqrt{2}}=\frac{F}{h^2u^2}$, from (i), (ii)

or $\frac{1}{a}\tanh\frac{\theta}{\sqrt{2}}\left[1-\operatorname{sech}^2\frac{\theta}{\sqrt{2}}\right]=\frac{F}{h^2u^2}$

or $u\,[\tanh^2(\theta/\sqrt{2})]=F/h^2u^2$, from (i)

or $u\,[a^2u^2]=F/h^2u^2$, from (i)

or $F=h^2a^2u^5=(h^2a^2)/r^5$

or $F\propto(1/r^5)$ *i.e.* the force varies inversely as the fifth power of the distance from the pole. **Ans.**

Ex. 7. A particle describes the curve $r^n\cos n\theta=a^n$ under a force F to the pole. Find the law of force.

Solution. The curve is $r^n\cos n\theta=a^n$ or $a^nu^n=\cos n\theta$, .. (i)

where $u=1/r$.

Taking logarithm of both sides, we get

$n\log a+n\log u=\log\cos n\theta$.

Differentiating both sides with respect to θ, we get

$$\frac{n}{u}\frac{du}{d\theta}=\frac{-n\sin n\theta}{\cos n\theta}\quad\text{or}\quad\frac{du}{d\theta}=-u\tan n\theta. \quad \text{(ii)}$$

Differentiating again, $\frac{d^2u}{d\theta^2}=-\frac{du}{d\theta}\tan n\theta-un\sec^2 n\theta$

or $\frac{d^2u}{d\theta^2}=u\tan^2 n\theta-un\sec^2 n\theta$, from (ii).

The differential equation of the path is $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$

or $(u\tan^2 n\theta-un\sec^2 n\theta)+u=\frac{F}{h^2u^2}$, putting the value of $\frac{d^2u}{d\theta^2}$

or $\frac{F}{h^2u^2}=u\,(1-n)\sec^2 n\theta=u\,(1-n)\,[1/a^nu^n]^2$, from (i)

or $F=h^2(1-n)\,u^3\,(1/a^{2n}u^{2n})=\frac{h^2(1-n)}{a^{2n}}\cdot\frac{1}{u^{2n-3}}$

or $F=\frac{h^2(1-n)}{a^{2n}}r^{2n-3}$ *i.e.* $F\propto r^{2n-3}$ **Ans.**

Ex. 8 A particle describes the curve $au=\frac{\cosh\theta-2}{\cosh\theta+1}$ under a force F to the pole. Find the law of force.

Solution The curve is $au=\frac{\cosh\theta-2}{\cosh\theta+1}$ ..(i)

or $$au = \frac{(\cosh\theta+1)-3}{(\cosh\theta+1)} = 1 - \frac{3}{\cosh\theta+1}.$$

Differentiating both sides w.r.t. to θ, we get

$$a\frac{du}{d\theta} = -3\left[\frac{-\sinh\theta}{(\cosh\theta+1)^2}\right] = \frac{3\sinh\theta}{(\cosh\theta+1)^2}$$

Differentiating again, we get

$$a\frac{d^2u}{d\theta^2} = 3\left[\frac{(\cosh\theta+1)^2\cosh\theta - \sinh\theta.2(\cosh\theta+1)\sinh\theta}{(\cosh\theta+1)^4}\right]$$

or $$\frac{d^2u}{d\theta^2} = \frac{3}{a}\left[\frac{(\cosh\theta+1)\cosh\theta - 2\sinh^2\theta}{(\cosh\theta+1)^3}\right]$$

$$= \frac{3}{a}\left[\frac{\cosh^2\theta+\cosh\theta-2(\cosh^2\theta-1)}{(\cosh\theta+1)^3}\right]$$

$$= \frac{3}{a}\left[\frac{2+\cosh\theta-\cosh^2\theta}{(\cosh\theta+1)^3}\right] = \frac{3}{a}\left[\frac{(\cosh\theta+1)(2-\cosh\theta)}{(\cosh\theta+1)^3}\right]$$

or $$\frac{d^2u}{d\theta^2} = \frac{3}{a}\left[\frac{2-\cosh\theta}{(\cosh\theta+1)^2}\right] \quad ...(ii)$$

The differential equation of the path is $\dfrac{d^2u}{d\theta^2}+u = \dfrac{F}{h^2u^2}$

or $$\frac{3}{a}\left[\frac{(2-\cosh\theta)}{(\cosh\theta+1)^2}\right] + \frac{1}{a}\left[\frac{\cosh\theta-2}{\cosh\theta+1}\right] = \frac{F}{h^2u^2}, \text{ from (ii)}$$

or $$\frac{F}{h^2u^2} = \frac{1}{a}\left[\frac{3(2-\cosh\theta)+(\cosh\theta-2)(\cosh\theta+1)}{(\cosh\theta+1)^2}\right]$$

$$= \frac{1}{a}\left[\frac{6-3\cosh\theta+\cosh^2\theta-\cosh\theta-2}{(\cosh\theta+1)^2}\right]$$

$$= \frac{1}{a}\left[\frac{\cosh^2\theta-4\cosh\theta+4}{(\cosh\theta+1)^2}\right] = \frac{1}{a}\left[\frac{\cosh\theta-2}{\cosh\theta+1}\right]^2$$

$$= (1/a)\,[au]^2, \text{ from (i)}$$

or $F = h^2au^4 = h^2a/r^4$, or $F \propto (1/r^4)$. **Ans.**

***Ex. 9 (a).** If the law of force is $P=\mu/r^5$, show that the path is a circle. *(Gorakhpur 91)*

Solution. The differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2}+u = \frac{P}{h^2u^2} \text{ and here } P = \frac{\mu}{r^5} = \mu u^5$$

$$\Rightarrow \frac{d^2u}{d\theta^2}+u = \frac{\mu u^5}{h^2u^2} = \frac{\mu}{h^2}u^3$$

$$\Rightarrow 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}+2u\frac{du}{d\theta} = \frac{\mu}{h^2}.2u^3\frac{du}{d\theta},$$

multiplying both sides by $2\dfrac{du}{d\theta}$

$\therefore$ Integrating, $\left(\frac{du}{d\theta}\right)^2+u^2=\frac{\mu}{h^2}.2\frac{u^4}{4}+C$, where C is constant of integration.

Choosing $du/d\theta=0$ when $u=0$ we have $C=0$

and then $\left(\frac{du}{d\theta}\right)^2+u^2=a^2u^4$, where $a^2=\frac{\mu}{2h^2}$

or $\left(\frac{du}{d\theta}\right)^2=u^2(a^2u^2-1)$ or $\frac{du}{d\theta}=u\sqrt{(a^2u^2-1)}$

or $d\theta=\frac{du}{u\sqrt{(a^2u^2-1)}}$

Integrating, $\theta+C_1=\int\frac{du}{u\sqrt{(a^2u^2-1)}}$

$=\int\frac{(1/a)\sec z\tan z\,dz}{(1/a)\sec z.\tan z}$, putting $au=\sec z$

$=\int dz=z=\sec^{-1}(au)$

or $au=\sec(\theta+C_1)$ or $a/r=\sec(\theta+C_1)$

or $r=a\cos(\theta+C_1)$, which represents a circle.

Ex. 9 (b). If the central force varies as the distance from a fixed point, then show that the orbit is a conic, centre being the pole. (*Kumaun* 91)

Hint : Do as Ex. 9 (a) above.

Ex. 10. In an orbit described under a force to a centre, the velocity at any point is inversely proportional to the distance of the point from the centre of force. Show that the path is an equiangular spiral.

Solution. Let v be the velocity of the particle at any point at a distance r from the centre of force. Then according to the given problem, we have $v=\frac{\mu}{r}$.

Also we know $v=\frac{h}{p}$, . (ii)

where p is the length of the perpendicular from pole to the tangent at any point of the path.

$\therefore$ From (i) and (ii), we have $\frac{\mu}{r}=\frac{h}{p}$ or $p=(h/\mu)r$

or $p=ar$, where $a=h/\mu$

But $p=ar$ is the pedal equation of an equiangular spiral.

Hence proved.

Ex. 11. A particle (of unit mass) describes an equiangular spiral of an angle α and a force which is always in the direction perpendicular to the straight line joining the particle to pole of the spiral, show that the force is $\mu r^{2 \sec^2 \alpha - 3}$ and that the rate of description of sectorial area about the pole is

$$\tfrac{1}{2}\sqrt{(\mu \sin \alpha \cos \alpha)}\, r^{\sec^2 \alpha}.$$

Solution The particle is moving under the action of a force F (say) which is always in the direction perpendicular to the radius vector, therefore the radial component of its acceleration is

$$\ddot{r} - r\dot{\theta}^2 = 0. \quad ...(i)$$

Also the equation of the equiangular spiral is

$$r = de^{\theta \cot \alpha} \quad ...(ii)$$

$$\therefore \quad \dot{r} = a \cot \alpha . e^{\theta \cot \alpha} \dot{\theta} = (r \cot \alpha)\, \dot{\theta}, \text{ from (ii)} \quad ...(iii)$$

or $\quad \dot{\theta} = \dot{r}/(r \cot \alpha)$

$\therefore$ From (i), we have $\ddot{r} = r\,[\dot{r}/(r \cot \alpha)]^2 = (\dot{r}^2 \tan^2 \alpha)/r$

or $$\frac{\ddot{r}}{\dot{r}} = \left(\frac{\dot{r}}{r}\right) \tan^2 \alpha \quad \text{(Note)}$$

Integrating, we get $\log \dot{r} = (\tan^2 \alpha) \log r + \log c$, where $\log c$ is constant of integration.

or $$\log \dot{r} = \log [(r)^{\tan^2 \alpha} c] \quad \text{or} \quad \dot{r} = c\, r^{\tan^2 \alpha} \quad ...(iv)$$

$\therefore$ From (iii), we have $\dot{\theta} = (cr^{\tan^2 \alpha})/(r \cot \alpha)$

or $$r^2 \dot{\theta} = c \tan \alpha\, r^{(\tan^2 \alpha + 1)} = c \tan \alpha . r^{\sec^2 \alpha} \quad ...(v)$$

Also the force F is always in the direction perpendicular to the radius vector, so we have

$$\frac{1}{r}\frac{d}{dt}(r^2 \dot{\theta}) = F \quad \text{(Note)}$$

or $$\frac{1}{r}\frac{d}{dt}\,[c \tan \alpha\, r^{\sec^2 \alpha}] = F$$

or $$F = \frac{1}{r}\, c \tan \alpha \sec^2 \alpha\, r^{\sec^2 \alpha - 1}\, \dot{r}$$

or $$F = c \tan \alpha \sec^2 \alpha . r^{\sec^2 \alpha - 2}\, c\, r^{\tan^2 \alpha}, \text{ from (iv)}$$

$$= c^2 \tan \alpha \sec^2 \alpha\, r^{\sec^2 \alpha - 2 + \tan^2 \alpha}$$

$$= c^2 \tan \alpha \sec^2 \alpha\, r^{2 \sec^2 \alpha - 3}, \quad \because \tan^2 \alpha = \sec^2 \alpha - 1$$

or $$F = \mu\, r^{2 \sec^2 \alpha - 3}, \text{ where } \mu = c^2 \tan \alpha \sec \alpha$$

From $\mu = c^2 \tan\alpha \sec^2\alpha$, we get $c = \sqrt{(\mu \cot\alpha \cos^2\alpha)}$. ...(vi)

Now rate of description of sectorial area

$$= \tfrac{1}{2}h = \tfrac{1}{2}r^2\dot{\theta}$$

$$= \tfrac{1}{2}c \tan\alpha\, r^{\sec^2\alpha}, \text{ from (v)}$$

$$= \tfrac{1}{2}\sqrt{(\mu \cot\alpha \cos^2\alpha)} \tan\alpha\, r^{\sec^2\alpha}, \text{ from (vi)}$$

$$= \tfrac{1}{2}\sqrt{(\mu \sin\alpha \cos\alpha)}\, r^{\sec^2\alpha}$$ Hence proved.

***Ex. 12. The velocity at any point of a central orbit is 1/nth of what it would be for a circular orbit at the same distance. Show that central force varies as $1/r^{2n^2+1}$ and that the equation of the orbit is $r^{n^2-1} = a^{n^2-1} \cos(n^2-1)\theta$.**

Solution. Let v and v_1 be the velocities at a distance r from the centre of force under the same central force F in the central orbit and circular orbit respectively.

$\therefore \quad v = v_1/n$ (given) ...(i)

Also $v_1^2/r = F$ or $v_1^2 = rF$ or $n^2v^2 = rF$, from (i) (Note)

or $$v^2 = \frac{F}{un^2}, \text{ where } u = 1/r$$ ...(ii)

or $$h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{F}{un^2}, \quad \because \text{ "}v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]\text{"}$$

or $$F/u = h^2n^2\,[(du/d\theta)^2 + u^2].$$ ...(iii)

Differentiating both sides w.r.t. θ, we get

$$\frac{1}{u}\frac{dF}{d\theta} - \frac{F}{u^2}\frac{du}{d\theta} = h^2n^2\left[2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta}\right]$$

or $$\frac{1}{u}\frac{dF}{du} - \frac{F}{u^2} = 2h^2n^2\left[\frac{d^2u}{d\theta^2} + u\right], \text{ writing } \frac{dF}{d\theta} \text{ as } \frac{dF}{du}\cdot\frac{du}{d\theta}$$ (Note)

or $$\frac{dF}{du} = \frac{F}{u} + 2h^2n^2u\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= \frac{F}{u} + 2h^2n^2u\left(\frac{F}{h^2u^2}\right); \quad \because \frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$ (Note)

or $$\frac{dF}{du} = \frac{(2n^2+1)F}{u} \text{ or } \frac{dF}{F} = (2n^2+1)\frac{du}{u}.$$

Integrating we have $\log F = (2n^2+1)\log u + \log c$, where $\log c$ is constant of integration.

or $$F = cu^{2n^2+1} = \frac{c}{r^{2n^2+1}} \text{ i.e. } F \propto 1/r^{2n^2+1}$$ Hence proved.

Again putting $F=cu^{2n^2+1}$ in (iii), we get

$$cu^{2n^2}=h^2n^2[(du/d\theta)^2+u^2]$$

or $\dfrac{cu^{2n^2}}{h^2n^2}=\left(\dfrac{du}{d\theta}\right)^2+u^2$ or $\left(\dfrac{du}{d\theta}\right)^2=u^2\left[\dfrac{cu^{2n^2-2}}{h^2n^2}-1\right]$

or $(du/d\theta)^2=u^2[a^{2n^2-2}u^{2n^2-2}-1]$, putting $c=h^2n^2a^{2n^2-2}$. (Note)

...(iv)

or $$\frac{du}{u\sqrt{[(a^{n^2-1}u^{n^2-1})^2-1]}}=d\theta.$$

Put $a^{n^2-1}u^{n^2-1}=\sec z$,

then $(n^2-1)\,a^{n^2-1}u^{n^2-2}\,du=\sec z\tan z\,dz$

or $$du=\frac{\sec z\tan z\,dz}{(n^2-1)\,a^{n^2-1}u^{n^2-2}}$$

∴ (iv) becomes $$\frac{\sec z\tan z\,dz}{(n^2-1)\,a^{n^2-1}u^{n^2-1}\sqrt{[(a^{n^2-1}u^{n^2-1})^2-1]}}=d\theta$$

or $\dfrac{\sec z\tan z\,dz}{\sec z\sqrt{[\sec^2 z-1]}}=(n^2-1)\,d\theta$ or $dz=(n^2-1)\,d\theta.$

Integrating we have $z=(n^2-1)\,\theta$, choosing z and θ in such a manner that constant of integration is zero

or $\sec z=(n^2-1)\,\theta$ or $a^{n^2-1}u^{n^2-1}=\sec(n^2-1)\,\theta$

or $a^{n^2-1}=r^{n^2-1}\sec(n^2-1)\,\theta.$

or $r^{n^2-1}=a^{n^2-1}\cos(n^2-1)\,\theta.$ Hence proved

Exercises on § 1 – § 7

Ex. 1. A particle describes the curve $r=a\sin n\theta$ under a force F to the pole, find the law of force.

(*Bundelkhand* 91 ; *Kumaun* 88)

Ans. $F\propto[(2n^2a^2/r^5)-\{(n^2-1)/r^3\}]$

Ex. 2 A particle describes the curve $r^n=A\cos n\theta+B\sin n\theta$, under a force F to the pole. Find the law of force

(*Purvanchal* 93) Ans. $F\propto 1/r^{2n+3}$

Ex 3 A particle describes a curve whose equation is $a/r=\theta^2+b$, under a force to the pole. Find the law of force.

Ans. $F\propto 1/r^3$

Ex 4 A particle describes the curve $r^2=a^2\cos 2\theta$, under a force F to the pole Find the law of force. Ans. $F\propto 1/r^7$

Ex. 5. If the central force varies as the distance from a fixed point. Find the equation of the orbit. (*Rohilkhand* 91)

§ 8. Apse, Apsidal distance and Apsidal angle.

Definition. *An apse is a point on the central orbit at which the radius vector drawn from the centre of force is a normal to the orbit.* (*Kumaun* 88)

Thus the radius vector drawn from the centre of force to an apse has a maximum or minimum value and also the direction of velocity at an apse is perpendicular to the radius vector at an apse.

From the above definition of an apse we find that at an apse the radius vector r is maximum or minimum *i.e.* u is minimum or maximum, since $u=1/r$. Also we know u is maximum or minimum if $du/d\theta=0$.

Also we know that for any curve,

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2 \quad \text{...See § 3 (i) Page 2}$$

where p is the length of the perpendicular from the origin to the tangent at any point.

$\therefore$ At an apse, we have $1/p^2=u^2$, $\because$ $du/d\theta=0$

or $1/p^2=1/r^2$, $\because$ $u=1/r$.

or $p=r$, which proves that at an apse the radius vector is perpendicular to the tangent thereat.

Apsidal distance. *The length of the radius vector at an apse is called the apsidal distance.*

Apsidal angle. *The angle between two apsidal distances is called the apsidal angle.*

Note. There are not more than two apsidal distances.

Solved Examples on § 8.

Ex. 1. A particle moves under a force

$$m\mu\,\{3au^4-2(a^2-b^2)u^5\},\ a>b$$

and is projected from an apse at a distance $(a+b)$ with velocity $\sqrt{\mu}/(a+b)$. Show that the equation of its path is $r=a+b\cos\theta$. (*Agra* 91)

Solution. We know that the differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2} \qquad \text{...(i)}$$

Here $F=\mu\,\{3au^4-2(a^2-b^2)u^5\}$,

$\therefore$ From (i) we get $\dfrac{d^2u}{d\theta^2}+u=\dfrac{\mu}{h^2}\{3au^2-2(a^2-b^2)u^3\}$.

or $$h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\mu\,\{3au^2-2\,(a^2-b^2)\,u^3\}.$$

Multiplying both sides by 2 $du/d\theta$ and integrating, we have

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\,\{2au^3-(a^2-b^2)\,u^4\}+c, \quad ...(ii)$$

where c is constant of integration.

Initially $r=a+b$ *i.e.* $u=1/(a+b)$, $v=(\sqrt{\mu})/(a+b)$, $du/d\theta=0$.

∴ From (ii),

$$\frac{\mu}{(a+b)^2}=h^2\left[0+\frac{1}{(a+b)^2}\right]=\mu\left[\frac{2a}{(a+b)^3}-\frac{(a^2-b^2)}{(a+b)^4}\right]+c$$

Hence from (ii), $\mu\,[(du/d\theta)^2+u^2]=\mu\,[2au^3-(a^2-b^2)\,u^4]$

or $$(du/d\theta)^2=2au^3-(a^2-b^2)\,u^4-u^2. \quad ...(iii)$$

Also as $u=1/r$, so $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}$

∴ From (iii), we get $\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2=\dfrac{2a}{r^3}-\dfrac{(a^2-b^2)}{r^4}-\dfrac{1}{r^2}$

or $$\left(\frac{dr}{d\theta}\right)^2=2ar-(a^2-b^2)-r^2=b^2-(r-a)^2$$

or $$\frac{dr}{d\theta}=\pm\sqrt{[b^2-(r-a)^2]} \quad \text{or} \quad \frac{dr}{-\sqrt{[b^2-(r-a)^2]}}=d\theta. \quad ...(iv)$$

Integrating, $\cos^{-1}[(r-a)/b]=\theta+c_1$.

Initially, $r=a+b$, $\theta=0$, so from (iv) we get $c_1=0$.

∴ From (iv), $\cos^{-1}[(r-a)/b]=\theta$ or $[(r-a)/b]=\cos\theta$

or $r=a+b\cos\theta$ is the required equation of the path. **Ans.**

Ex. 2. A particle moves with a central acceleration $\mu\left(r+\dfrac{a^4}{r^3}\right)$ being projected from an apse at a distance a with velocity $2\sqrt{\mu a}$. Prove that it describes the curve $r^2\,(2+\cos\sqrt{3}\theta)=3a^2$.

(Kumaun 89; Purvanchal 92)

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2} \quad ...(i)$$

Here $F=\mu\left(r+\dfrac{a^4}{r^3}\right)=\mu\left(\dfrac{1}{u}+a^4u^3\right)$

∴ From (i), we get $h^2\left[\dfrac{d^2u}{d\theta^2}+u\right]=\mu\left(\dfrac{1}{u^3}+a^4u\right)$

Multiplying both sides by 2 $du/d\theta$ and integrating, we have

$$v^2=h^2\,[(du/d\theta)^2+u^2]=2\mu\,(-\tfrac{1}{2}u^{-2}+\tfrac{1}{2}a^4u^2)+c, \quad ...(ii)$$

where c is constant of integration.

Initially $r=a$ i.e. $u=1/a$, $v=2\sqrt{\mu}a$ and $du/d\theta=0$ (at an apse)

$\therefore$ From (ii), we get

$$4\mu a^2=h^2\,[1/a^2]=2\mu\left[-\tfrac{1}{2}a^2+\tfrac{1}{2}a^2\right]+c$$

$\therefore$ $h^2=4\mu a^4$ and $c=4\mu a^2$.

$\therefore$ From (ii), we get

$$4\mu a^4\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\left[-\frac{1}{u^2}+a^4u^2\right]+4\mu a^2$$

or
$$4a^4\left(\frac{du}{d\theta}\right)^2=-\frac{1}{u^2}+a^4u^2+4a^2-4a^4u^2=\frac{4a^2u^2-3a^4u^4-1}{u^2}$$

$$=\frac{-\left(a^2u^2\sqrt{3}-\frac{2}{\sqrt{3}}\right)^2+\frac{4}{3}-1}{u^2}=\frac{\frac{1}{3}-\left(a^2u^2\sqrt{3}-\frac{2}{\sqrt{3}}\right)^2}{u^2}$$

or
$$2a^2\frac{du}{d\theta}=-\frac{1}{u}\sqrt{\left[\frac{1}{3}-\left(a^2u^2\sqrt{3}-\frac{2}{\sqrt{3}}\right)^2\right]} \qquad \text{(Note)}$$

or
$$\frac{-2\sqrt{3}a^2u\,du}{\sqrt{\left[\frac{1}{3}-\left(a^2u^2\sqrt{3}-\frac{2}{\sqrt{3}}\right)^2\right]}}=\sqrt{3}\,d\theta$$

or
$$\frac{-dt}{\sqrt{[\frac{1}{3}-t^2]}}=\sqrt{3}\,d\theta, \text{ putting } a^2u^2\sqrt{3}-\frac{2}{\sqrt{3}}=t$$

Integrating, we have $\cos^{-1}(t\sqrt{3})=\sqrt{3}\theta+c_1$, ...(iii)

where c_1 is constant of integration.

Initially $u=1/a$ i.e. $t=\sqrt{3}-(2/\sqrt{3})=1/\sqrt{3}$ and let $\theta=0$.

And from (iii), we have $\cos^{-1}(t\sqrt{3})=\theta\sqrt{3}$ $\therefore$ $c_1=0$

or $t\sqrt{3}=\cos(\theta\sqrt{3})$ or $\sqrt{3}\left(a^2u^2\sqrt{3}-\frac{2}{\sqrt{3}}\right)=\cos(\theta\sqrt{3})$

or $3a^2u^2-2=\cos(\theta\sqrt{3})$ or $3a^2u^2=2+\cos\sqrt{3}\theta$

or $3a^2=r^2\,(2+\cos\sqrt{3}\theta)$ is the required equation of the path.

**Ex. 3. A particle is moving with central acceleration $\mu(r^5-c^4r)$ being projected from an apse at a distance c with velocity $c^3\sqrt{(2\mu/3)}$, show that its path is the curve $x^4+y^4=c^4$.

(*Agra* 90, 88; *Avadh* 90; *Kumaun* 90; *Lucknow* 89; *Meerut* 90; *Purvanchal* 93, *Rohilkhand* 90)

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2} \qquad \text{...(i)}$$

Here $F=\mu\,(r^5-c^4r)=\mu\left(\frac{1}{u^5}-\frac{c^4}{u}\right)$

$\therefore$ From (i), we get $\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}\left(\frac{1}{u^7}-\frac{c^4}{u^3}\right)$

or $$h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\mu\,(u^{-7}-c^4u^{-3}).$$

Multiplying both sides by 2 $(du/d\theta)$ and integrating, we get

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=2\mu\left[-\frac{1}{6u^6}+\frac{c^4}{2u^2}\right]+c_1, \quad \text{...(ii)}$$

where c_1 is constant of integration.

Initially $r=c$ *i.e.* $u=1/c$, $v=c^3\sqrt{(\frac{2}{3}\mu)}$ and $du/d\theta=0$.

$\therefore$ From (ii), we get $c^6\,(\frac{2}{3}\mu)=h^2\left[\frac{1}{c^2}\right]=2\mu\,[-\frac{1}{6}c^6+\frac{1}{2}c^6]+c_1$

$\therefore$ $h^2=\frac{2}{3}\mu c^8$ and $c_1=0$.

$\therefore$ From (ii), we get $\frac{2}{3}\mu c^8\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=2\mu\left[-\frac{1}{6u^6}+\frac{c^4}{2u^2}\right]$

or $\frac{1}{3}c^8\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=-\frac{1}{6u^6}+\frac{c^4}{2u^2}$

or $c^8\left(\frac{du}{d\theta}\right)^2=3\left[-\frac{1}{6u^6}+\frac{c^4}{2u^2}\right]-c^8u^2$

$=\frac{1}{u^6}\,[-\frac{1}{2}+\frac{3}{2}c^4u^4-c^8u^8]=\frac{1}{u^6}\,[-\frac{1}{2}-(c^4u^4-\frac{3}{4})^2+(9/16)]$

or $c^8\left(\frac{du}{d\theta}\right)^2=\frac{1}{u^6}\,[(\frac{1}{4})^2-(c^4u^4-\frac{3}{4})^2]$

or $c^4\frac{du}{d\theta}=\pm\frac{1}{4u^3}\,[\sqrt{(1-(4c^4u^4-3)^2}]$

or $-\frac{16c^4u^3\,du}{\sqrt{[1-(4c^4u^4-3)^2]}}=4\,d\theta$

Integrating, we get $\cos^{-1}(4c^4u^4-3)=4\theta+c_2$. ...(iii)

Initially $u=1/c$, $\theta=0$, $\therefore$ $c_2=0$.

$\therefore$ From (iii), we have $\cos^{-1}(4c^4u^4-3)=4\theta$

or $4c^4u^4-3=\cos 4\theta$ or $4c^4u^4=3+\cos 4\theta$

or $4c^4=r^4\,(3+\cos 4\theta)$, $\because$ $u=1/r$

$=r^4\,[3+(2\cos^2 2\theta-1)]$, $\because$ $\cos 2A=2\cos^2 A-1$

or $4c^4=r^4\,[2+2\cos^2 2\theta]=r^4\,[2+2\,(2\cos^2\theta-1)^2]$

$=r^4\,[2+2\,(4\cos^4\theta-4\cos^2\theta+1)]$

or $4c^4=4r^4\,[2\cos^4\theta-2\cos^2\theta+1]$

$=4r^4\,[\cos^4\theta+(\cos^4\theta-2\cos^2\theta+1)]$

$=4r^4\,[\cos^4\theta+(1-\cos^2\theta)^2]$ (Note)

$=4r^4\,[\cos^4\theta+\sin^4\theta]=4\,[(r\cos\theta)^4+(r\sin\theta)^4]$

or $c^4=x^4+y^4$, $\because$ $x=r\cos\theta$, $y=r\sin\theta$. Hence proved.

*Ex. 4. A particle moves with a central acceleration [μ/(distance)^5] and projected from the apse at a distance 'a' with a velocity equal to n times that which would be acquired in falling from infinity, show that the other apsidal distance is $a/\sqrt{(n^2-1)}$.

If $n=$, and particle be projected in any direction, show that the path is a circle passing through the centre of force.

(*Rohilkhand* 91)

Solution. It is given that the particle is projected from the apse at a distance a with a velocity equal to n times that which would be acquired in falling from infinity under the given acceleration.

$\therefore$ If v be the velocity at a distance x from the centre, then $v\dfrac{dv}{dx}=-\dfrac{\mu}{x^5}$, the negative sign indicates that the particle is moving towards the centre.

$\therefore$ If V be the velocity from infinity to a distance a from the centre, then $\displaystyle\int_0^V v\,dt=-\mu\int_{x=\infty}^{a}\frac{1}{x^5}\,dx$

or $$V^2=2\mu\left[\frac{1}{4x^4}\right]_{\infty}^{a}=\frac{\mu}{2}\cdot\frac{1}{a^4}\quad\text{or}\quad V=\sqrt{\left(\frac{\mu}{2a^4}\right)}$$

$\therefore$ The velocity of the particle

$$=nV=n\sqrt{(\mu/2a^4)} \quad \text{...(i)}$$

Now the differential equation of the path of the particle is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}\ \text{or}\ \frac{d^2u}{d\theta^2}+u=\frac{\mu u^5}{h^2u^2},\quad \because F=\frac{\mu}{r^5}=\mu u^5$$

or $$h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\mu u^3.$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we get

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\frac{u^4}{2}+C, \quad \text{...(iii)}$$

which C is constant of integration.

Initially $v=n\sqrt{\left(\dfrac{\mu}{2a^4}\right)}$, $\dfrac{du}{d\theta}=0$ and $u=\dfrac{1}{a}$.

$\therefore$ From (iii), $\dfrac{n^2\mu}{2a^4}=h^2\left[\dfrac{1}{a^2}\right]=\dfrac{\mu}{2a^4}+C.$

$\therefore$ $h^2=n^2\mu/(2a^2)$ and $C=\mu(n^2-1)/(2a^4)$

$\therefore$ From (iii), we get $\frac{n^2\mu}{2a^2}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\frac{\mu(n^2-1)}{2a^4}$

or $\qquad (du/d\theta)^2+u^2=(1/a^2n^2)[a^4u^4+(n^2-1)]$

or $\qquad (du/d\theta)^2=(1/a^2n^2)[a^4u^4-a^2n^2u^2+(n^2-1)]$. ...(iv)

Also at any apse $du/d\theta=0$, so from (iv) the apsidal distances are given by $\qquad a^4u^4-a^2n^2u^2+(n^2-1)=0$

or $\qquad (n^2-1)\,r^4-a^2n^2r^2+a^4=0, \quad \because \quad u=1/r.$

Let r_1^2 and r_2^2 be the roots of this quadratic equation in r^2. Then $r_1^2\,r_2^2=$ product of the roots $=a^4/(n^2-1)$

or $\qquad r_1r_2=a^2/\sqrt{(n^2-1)}.$

But one of the apsidal distances is a. Let $r_1=a$.

$\therefore \quad ar_2=a^2/\sqrt{(n^2-1)}$ or $r_2=a/\sqrt{(n^2-1)}$. Hence proved.

If $n=1$, from (iv), we have

$(du/d\theta)^2=(1/a^2)\,[a^4u^4-a^2u^2]=a^2u^4-u^2=u^2\,(a^2u^2-1)$

or $$\frac{du}{u\sqrt{(a^2u^2-1)}}=d\theta.$$

Integrating, $\quad \sec^{-1}(au)=\theta+C_1.$

Initially $\quad u=1/a$ and $\theta=0$, $\quad \therefore \quad C_1=0.$

Hence $\sec^{-1}(au)=0$ or $au=\sec\theta$ or $a/r=\sec\theta$ or $r=a\cos\theta$, which is polar equation of a circle passing through the pole (*i.e.* the centre of force).

****Ex. 5 (a).** A particle subject to central attractive acceleration $(\mu/r^3)+f$ is projected from an apse at a distance a with a velocity $\sqrt{(\mu)}/a$. Prove that at any subsequent time t, $r=a-\frac{1}{2}ft^2$.

Solution. The path is given by $\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$ ...(i)

Here $\qquad F=(\mu/r^3)+f=\mu u^3+f.$

$\therefore$ From (i), we get

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu u^3+f}{h^2u^2} \quad \text{or} \quad h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\mu u+fu^{-2}.$$

Multiplying both sides of this by $2\,(du/d\theta)$, and integrating we get $\qquad v^2=h^2\,[(du/d\theta)^2+u^2]=\mu u^2-2fu^{-1}+c,$ ...(ii)

where c is constant of integration.

Initially $r=a$ *i.e.* $u=1/a$, $v=\sqrt{\mu}/a$ and $du/d\theta=0$ (at an apse).

$\therefore$ From (ii), we get $\frac{\mu}{a^2}=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{a^2}-2fa+c$

or $\qquad h^2=\mu \quad$ and $\quad c=2fa.$

$\therefore$ From (ii), we get $\mu\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu u^2-\frac{2f}{u}+2fa.$ .. (iii)

Also we know $r^2(d\theta/dt)=h$ or $d\theta/dt=hu^2$. ...(iv)

Now from (iii), we get

$$\mu\left[\left(\frac{du}{dt}\cdot\frac{dt}{d\theta}\right)^2+u^2\right]=\mu u^2-\frac{2f}{u}+2fa \quad \text{(Note)}$$

or $$\mu\left[\left(\frac{du}{dt}\right)^2\left(\frac{1}{hu^2}\right)^2+u^2\right]=\mu u^2-\frac{2f}{u}+2fa, \text{ from (iv)}$$

or $$\mu\left[\left(\frac{du}{dt}\right)^2\left(\frac{1}{\mu u^4}\right)+u^2\right]=\mu u^2-\frac{2f}{u}+2fa, \quad \because\ h^2=\mu$$

or $$\frac{1}{u^4}\left(\frac{du}{dt}\right)^2=2fa-\frac{2f}{u} \text{ or } \left(-\frac{1}{u^2}\frac{du}{dt}\right)^2=2fa-\frac{2f}{u} \quad \text{(Note)}$$

or $$\left(\frac{dr}{dt}\right)^2=2fa-2fr, \quad \because\ r=\frac{1}{u} \text{ and } \frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt}$$

or $$\frac{dr}{dt}=\sqrt{(2f)}\sqrt{(a-r)} \text{ or } \frac{dr}{\sqrt{(a-r)}}=\sqrt{(2f)}\,dt.$$

Integrating both sides, we get

$-2\sqrt{(a-r)}=\sqrt{(2f)}\,t+c_1$, where c_1 is constant of integration,

Initially $r=a, t=0$, $\therefore$ $c_1=0$.

Hence, we get $-2\sqrt{(a-r)}=\sqrt{(2f)}\,t$.

Squaring both sides, we get $4(a-r)=2ft^2$.

or $a-r=\frac{1}{2}ft^2$ or $r=a-\frac{1}{2}ft^2$. Hence proved.

Ex. 5 (b). A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r\cos(\theta/\sqrt{2})=a$. *(Avadh 89; Gorakhpur 92)*

Solution. Let v_1 be the velocity for a circle of radius a with a central acceleration varying inversely as (distance)^3.

Then $\frac{v_1^2}{a}=\frac{\mu}{a^3}$, $\because$ the inward drawn normal acc. $=v^2/\rho$

or $v_1^2=\mu/a^2$ or $v_1=(\sqrt{\mu})/a$.

$\therefore$ If v_2 be the velocity of projection, then $v_2=\sqrt{2}v_1$ (given)

or $$v_2=\sqrt{2}\,\frac{\sqrt{\mu}}{a}=\frac{\sqrt{(2\mu)}}{a}. \quad \text{.. (i)}$$

Now the differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \text{ where } F=\frac{\mu}{r^3}=\mu u^3, \quad \because\ u=1/r$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{\mu u^3}{h^2u^2} \text{ or } h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\mu u.$$

Multiplying both sides by 2 $du/d\theta$ and integrating, we get

$$v^2=h^2[(du/d\theta)^2+u^2]=2\mu(\tfrac{1}{2}u^2)+c. \quad ...(ii)$$

Initially $r=a$, *i.e.* $u=1/a$, $du/d\theta=0$ and $v=v_1=\sqrt{(2\mu)}/a$

∴ From (ii), we have

$$2\mu/a^2=h^2[1/a^2]=\mu(1/a^2)+c.$$

or $$h^2=2\mu \quad \text{and} \quad c=\mu/a^2.$$

∴ From (ii), we get $2\mu\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\mu\left(u^2+\frac{1}{a^2}\right)$

or $\left(\frac{du}{d\theta}\right)^2+u^2=\frac{1}{2}\left(u^2+\frac{1}{a^2}\right)$ or $\left(\frac{du}{d\theta}\right)^2=\frac{1}{2}\left(\frac{1}{a^2}-u^2\right)=\frac{1}{2}\frac{(1-a^2u^2)}{a^2}$,

or $$\frac{a\,du}{\sqrt{(1-a^2u^2)}}=\frac{1}{\sqrt{2}}\,d\theta. \quad \textit{(Note)}$$

Integrating, we have $\sin^{-1}(au)=\theta/\sqrt{2}+c_1$, ... (iii)
where c_1 is constant of integration.

Initially $u=1/a$ and $\theta=0$, ∴ $c_1=\frac{1}{2}\pi$.

∴ From (iii), we get $\sin^{-1}(au)=\frac{1}{2}\pi+(\theta/\sqrt{2})$

or $$au=\sin\left(\tfrac{1}{2}\pi+\frac{\theta}{\sqrt{2}}\right)=\cos\left(\frac{\theta}{\sqrt{2}}\right)$$

or $a=r\cos(\theta/\sqrt{2})$ is the equation of the path.

Ex. 6. A particle moves under a repulsive force $\left[=\frac{m\mu}{(\text{distance})^3}\right]$ and is projected from an apse at a distance 'a' with velocity V; show that the equation to the path is $r\cos p\theta=a$, and that the angle θ described in time 't' is $(1/p)\tan^{-1}(pVt/a)$, where $p^2=(\mu+a^2V^2)/a^2V^2$.

(*Agra* 92; *Gorakhpur* 91; *Kumaun* 88; *Rohilkhand* 87)

Solution. The differential equation of the path is

$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}$, where $F=-\frac{\mu}{r^3}=-\mu u^3$ (The force is repulsive)

or $\frac{d^2u}{d\theta^2}+u=\frac{\mu u^3}{h^2u^2}$, or $h^2\left(\frac{d^2u}{d\theta^2}+u\right)=-\mu u.$

Multiplying both sides by 2 $du/d\theta$ and integrating, we get

$$v^2=h^2[(du/d\theta)^2+u^2]=-\mu u^2+C_1, \quad ...(i)$$

where C_1 is constant of integration.

Initially $v=V$, $r=a$ *i.e.* $u=1/a$, $du/d\theta=0$.

∴ $V^2=h^2[0+(1/a^2)]=-(\mu/a^2)+C_1$

$\therefore\ h^2=a^2V^2$ and $C_1=V^2+\dfrac{\mu}{a^2}=\dfrac{a^2V^2+\mu}{a^2}$

$\therefore$ From (i), we get $a^2V^2\left[\left(\dfrac{du}{d\theta}\right)^2+u^2\right]=\dfrac{a^2V^2+\mu}{a^2}-\mu u^2$

or $$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{a^2V^2+\mu}{a^4V^2}-\frac{\mu u^2}{a^2V^2}$$

or $$\left(\frac{du}{d\theta}\right)^2=\frac{a^2V^2+\mu}{a^4V^2}-\frac{\mu u^2}{a^2V^2}-u^2$$

$$=\frac{1}{a^2}\left(\frac{\mu+a^2V^2}{a^2V^2}\right)-u^2\left(\frac{\mu+a^2V^2}{a^2V^2}\right)=\frac{p^2}{a^2}-u^2p^2$$

or $\dfrac{du}{d\theta}=-\dfrac{p}{a}\sqrt{(1-a^2u^2)}$, considering that u decreases as θ increases

or $$\frac{-a\,du}{\sqrt{(1-a^2u^2)}}=p\,d\theta.$$

Integrating we get, $\cos^{-1}(au)=p\theta+C_2$. ...(ii)

Initially $u=1/a$, $\theta=0$, $\therefore\ C_2=0$

$\therefore$ From (ii), we get $\cos^{-1}(au)=p\theta$ or $au=\cos p\theta$

or $a=r\cos p\theta$, $\because\ u=1/r$. Hence proved.

Also we know $r^2\dot{\theta}=h$ ...See § 2 (iii) Page 1

or $h\,dt=r^2\,d\theta$

or $aV\,dt=a^2\sec^2 p\theta d\theta$, $\because\ h^2=a^2V^2$ and $r\cos p\theta=a$

Integrating we get $aV\,t=\dfrac{a^2\tan p\theta}{p}+C_3$

Initially $\theta=0$, $t=0$. $\therefore\ C_3=0$

$\therefore\ pVt=a\tan p\theta$ or $p\theta=\tan^{-1}(pVt/a)$

or $$\theta=\frac{1}{p}\tan^{-1}\left(\frac{pVt}{a}\right)$$ Hence proved.

Ex. 7. A particle moves with a central acceleration $\lambda^2(8au^2+a^4u^5)$ it is projected with velocity 9λ from an apse at a distance $\frac{1}{3}a$ from the origin, show that equation to its path is

$$\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}=\cot\frac{\theta}{\sqrt{6}}$$

Solution. Here 'F'$=\lambda^2(8au^3+a^4u^5)$.

$\therefore$ The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}\quad\text{or}\quad\frac{d^2u}{d\theta^2}+u=\frac{\lambda^2(8au^3+a^4u^5)}{h^2u^2}$$

or $$h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\lambda^2(8a+a^4u^3).$$

Multiplying both sides by 2 $du/d\theta$ and integrating we get

$$v^2=h^2[(du/d\theta)^2+u^2]=2\lambda^2[8au+\tfrac{1}{4}a^4u^4]+c, \quad ...(i)$$

where c is constant of integration.

Initially $r=\frac{1}{3}a$ or $u=3/a$, $du/d\theta=0$ and $v=9\lambda$.

$\therefore$ From (i), we get $(9\lambda)^2=h^2[9/a^2]=2\lambda^2[24+\frac{1}{4}.81]=c$

$\therefore$ $h^2=9a^2\lambda^2$ and $c=-(15/2)\lambda^2$

$\therefore$ From (i), we have

$$9\lambda^2a^2[(du/d\theta)^2+u^2]=2\lambda^2[8au+\tfrac{1}{4}a^4u^4]-(15/2)\lambda^2$$

or $$9a^2(du/d\theta)^2=16au+\tfrac{1}{2}a^4u^4-(15/2)-9a^2u^2$$

or $$18a^2(du/d\theta)^2=a^4u^4-18a^2u^2+32au-15$$

$$=a^4u^4-a^3u^3+a^3u^3-a^2u^2-17a^2u^2+17au+15au-15 \text{ (Note)}$$

$$=a^3u^3(au-1)+a^2u^2(au-1)-17au(au-1)+15(au-1)$$

$$=(au-1)[a^3u^3+a^2u^2-17au+15]$$

$$=(au-1)[a^3u^3-a^2u^2+2a^2u^2-2au-15au+15]$$

$$=(au-1)[a^2u^2(au-1)+2au(au-1)-15(au-1)]$$

$$=(au-1)^2(a^2u^2+2au-15)$$

$$=(au-1)^2(au-3)(au+5)$$

or $$3\sqrt{2}a\frac{du}{d\theta}=(au-1)\sqrt{[(au-3)(au+5)]}$$

or $$\frac{a\,du}{(au-1)\sqrt{[(au-3)(au+5)]}}=\frac{d\theta}{3\sqrt{2}}$$

Now put $au+5=(au-3)z^2$ or $au=\dfrac{3z^2+5}{z^2-2}$ and then integrate and get the result.

***Ex. 8.** A particle moving with a central acceleration $\mu/(\text{distance})^3$ is projected from an apse at a distance a with velocity V, show that the path is

$$r\cosh\left[\frac{\sqrt{(\mu+a^2V^2)}}{aV}\theta\right]=a \text{ or } r\cos\left[\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta\right]=a$$

according as V is $<$ or $>$ the velocity from infinity.

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \text{ where } F=\frac{\mu}{r^3}=\mu u^3 \text{ (given)}$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{\mu u^3}{h^2u^2} \quad \text{or} \quad h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\mu u.$$

Multiplying both sides by 2 $du/d\theta$ and integrating, we have

$$v^2 = h^2 [(du/d\theta)^2 + u^2] = \mu^2 u + C, \quad \text{...(i)}$$

where C is constant of integration.

Initially $r = a$ i.e. $u = 1/a$, $du/d\theta = 0$ and $v = V$.

$\therefore$ From (i) we have $V^2 = h^2 [1/a^2] = \mu (1/a^2) + C$

$\therefore$ $h^2 = a^2 V^2$ and $C = V^2 - (\mu/a^2) = (a^2 V^2 - \mu)/a^2$

$\therefore$ From (i), we have

$$a^2 V^2 \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right] = \mu u^2 + \frac{a^2 V^2 - \mu}{a^2}$$

or
$$\left(\frac{du}{d\theta} \right)^2 + u^2 = \frac{\mu u^2}{a^2 V^2} + \frac{a^2 V^2 - \mu}{a^4 V^2}$$

or
$$\left(\frac{du}{d\theta} \right)^2 = \frac{\mu u^2}{a^2 V^2} - \frac{a^2 V^2 - \mu}{a^4 V^2} - u^2$$

or
$$\left(\frac{du}{d\theta} \right)^2 = u^2 \left(\frac{\mu}{a^2 V^2} - 1 \right) + \frac{a^2 V^2 - \mu}{a^4 V^2}$$

or
$$\left(\frac{du}{d\theta} \right)^2 = \left(\frac{a^2 V^2 - \mu}{a^2 V^2} \right) (1 - a^2 u^2) \quad \text{.. (ii)}$$

Now here the central acceleration is $\mu/(\text{distance})^3$.

$\therefore$ The acceleration at a distance x from the centre of force is given by $v \dfrac{dv}{dx} = -\dfrac{\mu}{x^3}$ or $v\, dv = -\mu x^{-3}\, dx$.

$\therefore$ $\displaystyle\int_{v=0}^{V_1} v\, dv = -\mu \int_{x=\infty}^{a} x^{-3}\, dx$, where V_1 is the velocity from infinity to $x = a$

or $\quad \frac{1}{2} V_1^2 = \mu \left[1/2x^2 \right]_{\infty}^{a} = \mu/2a^2$ or $V_1^2 = \mu/a^2$. ...(iii)

Now two cases arise according as $V <$ or $>$ velocity from infinity

i.e. according as $V <$ or $> V_1$

i.e. according as $V^2 <$ or $> V_1^2$ or $V^2 <$ or $> (\mu/a^2)$ from (iii)

i.e. according as $a^2 V^2 <$ or $> \mu$.

Case I. If $a^2 V^2 < \mu$, then from (ii), we have

$$\left(\frac{du}{d\theta} \right)^2 = \left(\frac{\mu - a^2 V^2}{a^2 V^2} \right) (a^2 u^2 - 1) \quad \textbf{(Note)}$$

or
$$\frac{a\, du}{\sqrt{(a^2 u^2 - 1)}} = \frac{\sqrt{(\mu - a^2 V^2)}}{aV}\, d\theta$$

Integrating, we have $\cosh^{-1}(au) = \dfrac{\sqrt{(\mu - a^2 V^2)}}{aV} \theta + C_1$, .. (iv)

where C_1 is constant of integration.

Initially $u=1/a$, $\theta=0$, $\because$ $C_1=0$.

$\therefore$ From (iv), we have $au=\cosh\left[\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta\right]$

or $$r\cosh\left[\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta\right]=a, \quad \because u=1/r.$$

Case II. If $a^2V^2>\mu$, then from (ii), we have

$$\left(\frac{du}{d\theta}\right)=\sqrt{\left(\frac{a^2V^2-\mu}{a^4V^2}\right)}\sqrt{(1-a^2u^2)}$$

or $$\frac{-a\,du}{\sqrt{(1-a^2u^2)}}=\frac{\sqrt{(a^2V^2-\mu)}}{aV}d\theta$$

Integrating, we have $\cos^{-1}(au)=\dfrac{\sqrt{(a^2V^2-\mu)}}{aV}\theta+C_2$, .. (v)

where C_2 is constant of integration.

Initially $u=1/a$, $\theta=0$, $\therefore$ $C_2=0$.

$\therefore$ From (v), we have $au=\cos\left[\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta\right]$

or $$r\cos\left[\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta\right]=a.$$ Hence proved.

Ex. 9. A particle moves under a central force $m\lambda\,[3a^3u^4+8au^3]$. It is projected from an apse at a distance a from the centre of force with velocity $\sqrt{(10\lambda)}$. Show that the second apsidal distance is half the first, and that the equation to the path is

$$2r=a\left[1+\text{sech}\,\frac{\theta}{\sqrt{5}}\right]$$

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \text{ where } F=\lambda\,[3a^3u^4+8au^3], \text{ given}$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{\lambda\,[3a^3u^4+8au^3]}{h^2u^2}$$

or $$h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\lambda\,[3a^3u^2+8a].$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we get

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=2\lambda\left[3a^3\frac{u^3}{3}+8au\right]+C, \quad \text{(i)}$$

where C is constant of integration.

Initially $r=a$ or $u=1/a$, $du/d\theta=0$ and $v=\sqrt{(10\lambda)}$

$\therefore$ $10\lambda=h^2\,[1/a^2]-2\lambda\,[1+8]+C$

$\therefore$ $h^2=10a^2\lambda$ and $C=-8\lambda$.

From (i), we get $10a^2\lambda\,[(du/d\theta)^2+u^2]=2\lambda\,[a^3u^3+8au]-8\lambda$

or $$5a^2\,[(du/d\theta)^2+u^2]=a^3u^3+8au-4$$

or $$\left(\frac{du}{d\theta}\right)^2=\frac{a^3u^3+8au-4}{5a^2}-u^2=\frac{1}{5a^2}\,[a^3u^3+8au-4-5a^2u^2]$$

$$=[a^3u^3-a^2u^2-4a^2u^2+4au+4au-4]/5a^2 \quad \text{(Note)}$$

$$=[a^2u^2\,(au-1)-4au\,(au-1)+4\,(au-1)]/5a^2$$

$$=\frac{(au-1)(a^2u^2-4au+4)}{5a^2}=\frac{(au-1)(au-2)^2}{5a^2}$$

or $$\frac{du}{d\theta}=\frac{(au-2)\sqrt{(au-1)}}{a\sqrt{5}} \quad \text{.. (ii)}$$

or $$\frac{a\,du}{(au-2)\sqrt{(au-1)}}=\frac{1}{\sqrt{5}}\,d\theta \quad \text{or} \quad \frac{2t\,dt}{(t^2-1)\,t}=\frac{1}{\sqrt{5}}\,d\theta,$$

putting $au-1=t^2$, $\therefore$ $a\,du=2t\,dt$

or $$\frac{2\,dt}{(t^2-1)}=\frac{1}{\sqrt{5}}\,d\theta.$$

Integrating we get $2\tanh^{-1}t=(\theta/\sqrt{5})+C_1$. ....(iii)

Initially $u=1/a$, $\therefore$ $t=0$. Also $\theta=0$, $\therefore$ $C_1=0$.

$\therefore$ From (iii) we get $2\tanh^{-1}t=\theta/\sqrt{5}$ or $t=\tanh(\theta/2\sqrt{5})$

Now $$\cosh\frac{\theta}{\sqrt{5}}=\frac{1+\tanh^2(\theta/2\sqrt{5})}{1-\tanh^2(\theta/2\sqrt{5})}=\frac{1+t^2}{1-t^2},$$

$\because$ $t=\tanh(\theta/2\sqrt{5})$

$$=\frac{1+(au-1)}{1-(au-1)}, \quad \because\ t^2=au-1$$

$$=(au)/(2-au)$$

or $\dfrac{2-au}{au}=\text{sech}\,(\theta/\sqrt{5})$. or $\dfrac{2r-a}{a}=\text{sech}\,(\theta/\sqrt{5})$, $\because$ $u=1/r$

or $2r=a+a\,\text{sech}\,(\theta/\sqrt{5})=a\,[1+\text{sech}\,(\theta/\sqrt{5})]$ is the required equation of the path.

Also we know that at an apse $du/d\theta=0$.

$\therefore$ From (ii) at an apse, we get $(au-2)\sqrt{(au-1)}=0$

or $\therefore$ $u=2/a$, $1/a$ i.e. $r=\frac{1}{2}a$, a.

Thus the other apsidal distance is $\frac{1}{2}a$ as one apsidal distance is given to be a. Hence the other apsidal distance is half of the first.

***Ex. 10.** A particle subject to a central force per unit of mass equal to $\mu\,\{2\,(a^2+b^2)\,u^5-3a^2b^2u^7\}$ is projected at the distance a with velocity $(\sqrt{\mu})/a$ in a direction at right angles to the initial distance. Show that the path is the curve $r^2=a^2\cos^2\theta+b^2\sin^2\theta$.

(Rohilkhand 90)

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \text{ where } F=\mu\{2(a^2+b^2)u^5-3a^2b^2u^7\}$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{\mu\{2(a^2+b^2)u^5+3a^2b^2u^7\}}{h^2u^2}$$

or $$h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\mu\{2(a^2+b^2)u^3-a^2b^2u^5\}$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we get

$$v^2=h^2[(du/d\theta)^2+u^2]=2\mu\{2(a^2+b^2)\tfrac{1}{4}u^4-3a^2b^2\tfrac{1}{6}u^6\}+C \quad \text{...(i)}$$

Initially $r=a$ or $u=1/a$, $du/d\theta=0$ and $v=(\sqrt{\mu})/a$

$\therefore$ From (i), we get $\frac{\mu}{a^2}=h^2\left[\frac{1}{a^2}\right]=\mu\left[\frac{a^2+b^2}{a^4}-\frac{a^2b^2}{a^6}\right]+C.$

$\therefore$ $h^2=\mu$ and $C=\frac{1}{a^2}-\left[\frac{1}{a^2}+\frac{b^2}{a^4}-\frac{b^2}{a^4}\right]=0.$

$\therefore$ From (i), we get $\mu\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\{(a^2+b^2)u^4-a^2b^2u^6\}$

or $$(du/d\theta)^2=(a^2+b^2)u^4-a^2b^2u^6-u^2. \quad \text{...(ii)}$$

Putting $u=\frac{1}{r}$ or $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$ in (ii), we get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=(a^2+b^2)\cdot\frac{1}{r^4}-\frac{a^2b^2}{r^6}-\frac{1}{r^2}$$

or $$r^2(dr/d\theta)^2=r^2(a^2+b^2)-a^2b^2-r^4$$
$$=-a^2b^2-[r^2-\tfrac{1}{2}(a^2+b^2)]^2+[\tfrac{1}{2}(a^2+b^2)]^2 \text{ (Note)}$$
$$=[\tfrac{1}{2}(a^2-b^2)]^2-[r^2-\tfrac{1}{2}(a^2+b^2)]^2$$

or $$r\frac{dr}{d\theta}=\pm\sqrt{\left[\left(\frac{a^2-b^2}{2}\right)^2-\left(r^2-\frac{a^2+b^2}{2}\right)^2\right]}$$

or $$\frac{-2r\,dr}{\sqrt{[\{\frac{1}{2}(a^2-b^2)\}^2-\{r^2-\frac{1}{2}(a^2+b^2)\}^2]}}=2\,d\theta \quad \text{(Note)}$$

or $$\frac{-dt}{\sqrt{[\{\frac{1}{2}(a^2-b^2)\}^2-t^2]}}=2\,d\theta, \text{ putting } r^2-\tfrac{1}{2}(a^2+b^2)=t,$$

$\therefore$ $2r\,dr=dt$

or $$\cos^{-1}[2t/(a^2-b^2)]=2\theta+C_1. \quad \text{...(iii)}$$

Now when $\theta=0$, $r=a$ i.e. $t=a^2-\frac{1}{2}(a^2+b^2)=\frac{1}{2}(a^2-b^2)$

$\therefore$ From (iii), we get $\cos^{-1}(1)=0+C_1$ or $C_1=0$.

$\therefore$ From (iii), we have

$\cos^{-1}[2t/(a^2-b^2)]=2\theta$ or $2t=(a^2-b^2)\cos 2\theta$

or $2[r^2-\frac{1}{2}(a^2+b^2)]=(a^2-b^2)\cos 2\theta$

or $2r^2=a^2(1+\cos 2\theta)+b^2(1-\cos 2\theta)=a^2(2\cos^2\theta)+b^2(2\sin^2\theta)$

or $r^2=a^2\cos^2\theta+b^2\sin^2\theta$ is the required equation of the path.

Ex. 11. A particle moves with a central acceleration $\mu(u^5-\frac{1}{8}a^2u^7)$, it is projected at a distance a with a velocity $\sqrt{(25/7)}$ times the velocity for a circle at that distance and at an inclination $\tan^{-1}(4/3)$ to the radius vector, show that its path is the curve

$$4r^2-a^2=3a^2/(1-\theta)^2.$$

Solution. Let v_1 be the velocity for a circle at distance a.

Then we know "$\frac{v^2}{\rho}$ = normal acceleration"

i.e. $$\frac{v_1^2}{a}=\mu\left\{\frac{1}{a^5}-\frac{a^2}{8}\cdot\frac{1}{a^7}\right\}, \text{ at a distance } a \qquad \textbf{(Note)}$$

or $$v_1^2=\frac{7\mu}{8a^4} \text{ or } v_1=\frac{1}{a^2}\sqrt{\left(\frac{7\mu}{8}\right)}$$

$\therefore$ velocity of projection of the particle $=\sqrt{(25/7)}\, v_1$ (given)

$$=\sqrt{\left(\frac{25}{7}\right)}\cdot\frac{1}{a^2}\sqrt{\left(\frac{7\mu}{8}\right)}=\frac{5}{2a^2}\sqrt{\left(\frac{\mu}{2}\right)} \qquad \ldots\text{(i)}$$

Now the differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \text{ where } F=\mu\,(u^5-\tfrac{1}{8}a^2u^7)$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{\mu\,(u^5-\frac{1}{8}a^2u^7)}{h^2u^2} \text{ or } h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\mu\,(u^3-\tfrac{1}{8}a^2u^5).$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we get

$$v^2=h^2\,[(du/d\theta)^2+u^2]=2\mu\,[\tfrac{1}{4}u^4-\tfrac{1}{8}.(1/6)a^2u^6]+C_1. \qquad \ldots\text{(ii)}$$

Initially $r=a$ i.e. $u=\frac{1}{a}$ and $v=\frac{5}{2a^2}\sqrt{\left(\frac{\mu}{2}\right)}$, from (i).

Also initially "ϕ" $=\tan^{-1}(4/3)$ (given)

or $\tan\phi=(4/3)$ so we can prove that $\sin\phi=4/5$.

$\therefore$ Initially from "$p=r\sin\phi$", we get $p_0=a.\frac{4}{5}$, $\qquad\ldots$(iii)

where p_0 is the initial value of p.

Also $$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=u^2+\left(\frac{du}{d\theta}\right)^2$$

$\therefore$ Initially $u^2+\left(\frac{du}{d\theta}\right)^2=\frac{1}{p_0^2}=\left(\frac{5}{4a}\right)^2$, from (iii)

$\therefore$ From (ii), initially we have

$$\left\{\frac{5}{2a^2}\sqrt{\left(\frac{\mu}{2}\right)}\right\}^2=h^2\left\{\left(\frac{5}{4a}\right)^2\right\}=2\mu\left[\frac{1}{4a^4}-\frac{1}{48a^4}\right]+C_1$$

$\therefore$ $h^2=\frac{\mu}{a^2}$ and $C_1=\frac{24\mu}{8a^4}-2\mu\left\{\frac{1}{4a^4}-\frac{1}{48a^4}\right\}=\frac{8\mu}{3a^4}$

$\therefore$ From (ii), we have $\frac{2\mu}{a^4}\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\mu\left\{\frac{u^4}{2}-\frac{a^2u^6}{24}\right\}+\frac{8\mu}{3a^4}$

or $$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{a^4u^4}{4}-\frac{a^6u^6}{48}+\frac{4}{3a^2} \quad ...(iv)$$

Putting $u=1/r$ or $du/d\theta=(-1/r^2)(dr/d\theta)$, in (iv), we get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2+\frac{1}{r^2}=\frac{a^4}{4r^4}-\frac{a^6}{48r^6}+\frac{4}{3a^2} \text{ or } r^2\left(\frac{dr}{d\theta}\right)^2=\frac{a^4}{4}r^2-\frac{a^6}{48}+\frac{4r^6}{3a^2}-r^4$$

or $$48a^2r^2\left(\frac{dr}{d\theta}\right)^2=12a^6r^2-a^8+64r^6-48a^2r^4$$
$$=(64r^6-a^8)+12a^2r^2(a^4-4r^2)$$
$$=(4r^2-a^2)(16r^4+4a^2r^2+a^4)-12a^2r^2(4r^2-a^2)$$
$$=(4r^2-a^2)(16r^4-8a^2r^2+a^4)$$
$$=(4r^2-a^2)(4r^2-a^2)^2=(4r^2-a^2)^3$$

or $4\sqrt{3}\,ar\dfrac{dr}{d\theta}=(4r^2-a^2)^{3/2}$, taking square root of both sides,

or $\dfrac{4\sqrt{3}\,ar\,dr}{(4r^2-a^2)^{3/2}}=d\theta$ or $\dfrac{\sqrt{3}a\,dt}{2t^{3/2}}=d\theta$,

putting $4r^2-a^2=t$ or $8r\,dr=dt$

Integrating, we get $\dfrac{\sqrt{3}a}{2}\left(-\dfrac{2}{\sqrt{t}}\right)=\theta+C_1$. ...(v)

Initially $\theta=0$, $r=a$ $\therefore$ $t=4a^2-a^2=3a^2$.

$\therefore$ From (v), $\dfrac{\sqrt{3}}{2}a\left(-\dfrac{2}{a\sqrt{3}}\right)=0+C_1$ or $C_1=-1$

$\therefore$ From (v), we have $-\dfrac{a\sqrt{3}}{\sqrt{t}}=\theta-1$ or $\dfrac{a\sqrt{3}}{\sqrt{(4r^2-a^2)}}=1-\theta$

or $3a^2=(1-\theta)^2(4r^2-a^2)$ or $4r^2-a^2=3a^2/(1-\theta)^2$,

is the required equation of the path.

Ex. 12. If the law of force be $\mu(u^4-\frac{10}{9}au^5)$ and the particle be projected from an apse at a distance $5a$ with velocity equal to $\sqrt{(5/7)}$ of that in a circle at the same distance, show that the orbit is the limacon $r=a(3+2\cos\theta)$.

Solution. As in the last example we can show that if v_1 be the velocity for a circle at the same distance, then

$$\frac{v_1^2}{5a}=\mu\left\{\frac{1}{(5a)^4}-\frac{10}{9}\frac{a}{(5a)^5}\right\}$$

or $$v_1^2=\frac{\mu}{(5a)^4}\left\{5a-\frac{10a}{9}\right\}=\frac{\mu}{(5a)^4}\cdot\frac{35a}{9}=\frac{7\mu}{9(5a)^3}$$

$\therefore$ If v be the velocity of projection of the particle, then

$$v=\sqrt{\left(\frac{5}{7}\right)}v_1=\sqrt{\left(\frac{5}{7}\right)}\sqrt{\left[\frac{7\mu}{9(5a)^3}\right]} \text{ or } v^2=\frac{\mu}{225a^3} \quad ...(i)$$

Now the differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2},\ \text{where } F=\mu\left(u^4-\tfrac{10}{9}au^5\right).$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{\mu\left(u^4-\frac{10}{9}au^5\right)}{h^2u^2}\ \text{ or }\ h^2\left\{\frac{d^2u}{d\theta^2}+u\right\}=\mu\left(u^2-\tfrac{10}{9}au^3\right).$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we get

$$v^2=h^2\left[(du/d\theta)^2+u^2\right]=2\mu\left[\tfrac{1}{3}u^3-(10/9)\,a\,\tfrac{1}{4}u^4\right]+C \quad \ldots\text{(ii)}$$

Initially $u=1/5a$, $du/d\theta=0$ and $v^2=\mu/(225a^3)$, from (i)

∴ From (ii), we get

$$\frac{\mu}{225a^3}=h^2\left[\frac{1}{25a^2}\right]=2\mu\left\{\frac{1}{375a^3}-\frac{1}{18}\times\frac{1}{125a^3}\right\}+C$$

∴ $h^2=\mu/9a$

and $$C=\frac{\mu}{225a^3}-\frac{2\mu}{125^3}\left\{\frac{1}{3}-\frac{1}{18}\right\}=\frac{\mu}{25a^3}\left[\frac{1}{9}-\frac{1}{9}\right]=0.$$

∴ From (ii), we have $\dfrac{\mu}{9a}\left\{\left(\dfrac{du}{d\theta}\right)^2+u^2\right\}=\dfrac{2\mu}{9}\left[3u^3-\tfrac{5}{2}au^4\right]$

or $(du/d\theta)^2+u^2=2a\left[3u^3-\tfrac{5}{2}au^4\right]$ or $(du/d\theta)^2=6au^3-5a^2u^4-u^2$.

Putting $u=1/r$, $du/d\theta=(-1/r^2)(dr/d\theta)$, we get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{6a}{r^3}-\frac{5a^2}{r^4}-\frac{1}{r^2}$$

or $$(dr/d\theta)^2=6ar-5a^2-r^2=4a^2-(r-3a)^2$$

or $$\frac{-dr}{\sqrt{[(2a)^2-(r-3a)^2]}}=d\theta. \qquad \text{(Note)}$$

Integrating, we get $\cos^{-1}[(r-3a)/2a]=\theta+C_1$

Initially $r=5a$, $\theta=0$, ∴ $C_1=0$

So we have $\cos^{-1}[(r-3a)/2a]=\theta$ or $r-3a=2a\cos\theta$

or $r=a(3+2\cos\theta)$ is the required equation of the path.

***Ex. 13** A particle subject to a force producing an acceleration $\mu(r+2a)/r^5$ towards the origin is projected from the point $(a, 0)$ with a velocity equal to velocity from infinity at an angle $\cos^{-1} 2$ with the initial line; show that the equation to the path is

$$r=a(1+2\sin\theta).$$

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2},\ \text{where } F=\frac{\mu(r+2a)}{r^5}=\mu(u^4+2au^5)$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{\mu(u^4+2au^5)}{h^2u^2}\ \text{ or }\ h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\mu(u^2+2au^3).$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we have

$$v^2=h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=2\mu\left[\tfrac{1}{3}u^3+\tfrac{1}{2}au^4\right]+C \quad \ldots\text{(i)}$$

Now the velocity of projection of the particle is equal to the velocity acquired by a particle in falling from infinity to the point of projection under given acceleration and let v_1 be the velocity thus acquired.

Also $$v\frac{dv}{dx}=-\mu\left\{\frac{x+2a}{x^5}\right\}$$ (Note)

or $$v\,dv=-\mu\left\{\frac{1}{x^4}+\frac{2a}{x^5}\right\}dx.$$

$$\therefore \int_{v=0}^{V_1} 2v\,dv=-2\mu\int_{a=\infty}^{a}\left(\frac{1}{x^4}+\frac{2a}{x^5}\right)dx$$

or $$v_1^2=-2\mu\left[-\frac{1}{3x^3}-\frac{2a}{4x_4}\right]_{\infty}^{a}=2\mu\left[\frac{1}{3a^3}+\frac{1}{2a^3}\right]$$

or $$v_1^2=5\mu/(3a^3).$$ ...(ii)

Also from "$p=r\sin\phi$" we have initially $p_0=a\sin\phi_0$, where p_0 and ϕ_0 are the initial values of p and ϕ which have their usual meanings. (See Authors Differential Calculus)

Now $\phi_0=\cot^{-1}2$ (given) or $\cot\phi_0=2$ $\therefore$ $\sin\phi_0=1/\sqrt{5}$.

$\therefore$ $p_0=a\sin\phi_0=a/\sqrt{5}$ ...(iii)

Also we know $$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=u^2+\left(\frac{du}{d\theta}\right)^2$$

$\therefore$ Initially $(du/d\theta)^2+u^2=\dfrac{1}{p_0^2}=\dfrac{5}{a^2}$, from (ii)

$\therefore$ From (i), initially we have

$$\frac{5\mu}{(3a^3)}=h^2\left[\frac{5}{a^2}\right]+2\mu\left[\frac{1}{3a^3}+\frac{1}{2a^3}\right]+C,$$

$\therefore$ $h^2=\mu/3a$ and $C=0$.

$\therefore$ From (i), we have $\dfrac{\mu}{3a}\left[\left(\dfrac{du}{d\theta}\right)^2+u^2\right]=2\mu\left[\tfrac{1}{3}u^3+\tfrac{1}{2}au^4\right]$

or $$(du/d\theta)^2=2au^3+3a^2u^4-u^2$$

or $$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{2a}{r^3}+\frac{3a^2}{r^4}-\frac{1}{r^2},\text{ putting } u=\frac{1}{r},\ \frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$$

or $$(dr/d\theta)^2=2ar+3a^2-r^2=4a^2-(r-a)^2$$

or $$\frac{dr}{\sqrt{[(2a^2-(r-a)^2]}}=d\theta.$$

Integrating, we get $\sin^{-1}[(r-a)/(2a)]=\theta+C_1$.

Initially, $\theta=0$, $r=0$. $\therefore$ $C_1=0$.

$\therefore$ $\sin^{-1}[(r-a)/2a]=\theta$ or $r-a=2a\sin\theta$

or $r=a(1+2\sin\theta)$ is the required equation of the path.

Ex. 14. A particle describes an orbit with a central acceleration $\mu u^3-\lambda u^5$ being projected from an apse at a distance a with velocity equal to that from infinity, show that its path is

$$r=a\cosh(\theta/n),\text{ where }\lambda(n^2+1)=2\mu a^2,$$

(Garhwal 89; Rohilkhand 92)

Prove that it will be at a distance r at the end of time

$$\sqrt{\left(\frac{a^2}{2\lambda}\right)}\left[a^2\log\frac{r+\sqrt{(r^2-a^2)}}{a}+r\sqrt{(r^2-a^2)}\right]$$

(Rohilkhand 92)

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2},\text{ where } F=\mu u^3-\lambda u^5\text{ (given)}$$

or $$h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\mu u-\lambda u^3.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu u^2-\tfrac{1}{2}\lambda u^4+C, \quad \text{...(i)}$$

where C is constant of integration.

Now the velocity of projection of the particle is equal to the velocity acquired by a particle falling from infinity to the point of projection under given acceleration and let v_1 be the velocity thus acquired.

Also acceleration at any point is given by,

$$v\frac{dv}{dx}=-\left[\frac{\mu}{x^3}-\frac{\lambda}{x^5}\right] \quad \textbf{(Note)}$$

or $$v\,dv=\left(-\frac{\mu}{x^3}+\frac{\lambda}{x^5}\right)dx$$

$$\therefore \int_{x=0}^{V_1}2v\,dv=2\int_{x=\infty}^{a}\left(-\frac{\mu}{x^3}+\frac{\lambda}{x^5}\right)dx$$

or $$v_1^2=2\left[\frac{\mu}{2x^2}-\frac{\lambda}{4x^4}\right]_{\infty}^{a}=\left[\frac{\mu}{a^2}-\frac{\lambda}{2a^4}\right] \quad \text{...(ii)}$$

Thus we have initially $r=a$ or, $u=1/a$, $du/d\theta=0$ and $v=v_1$ given by (ii).

$\therefore$ From (i), we get $v_1^2=h^2\left[0+\left(\frac{1}{a^2}\right)^2\right]=\frac{\mu}{a^2}-\frac{\lambda}{2a^4}+C$

or $$\frac{\mu}{a^2}-\frac{\lambda}{2a^4}=h^2\left(\frac{1}{a^2}\right)=\frac{\mu}{a^2}-\frac{\lambda}{2a^4}+C,\text{ from (ii)}$$

$\therefore$ $C=0$ and $h^2=\mu-(\lambda/2a^2)$

or $$h^2=\frac{2a^2\mu-\lambda}{2a^2}=\frac{\lambda(n^2+1)-\lambda}{2a^2},\quad \because \lambda(n^2+1)=2a^2\mu$$

or $h^2=\lambda n^2/(2a^2)$.

$\therefore$ From (ii), we get $\frac{\lambda n^2}{2a^2}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu u^2-\frac{1}{2}\lambda u^4$

or $\lambda n^2\,[(du/d\theta)^2+u^2]=2a^2\mu u^2-\lambda a^2 u^4$

$=\lambda\,(n^2+1)\,u^2-\lambda a^2 u^4$, $\because \lambda\,(n^2+1)=2a^2\mu$

or $\left(\frac{du}{d\theta}\right)^2+u^2=\left(\frac{n^2+1}{n^2}\right)u^2-\frac{a^2u^4}{n^2}=u^2+\frac{u^2}{n^2}-\frac{a^2u^4}{n^2}$

or $\left(\frac{du}{d\theta}\right)^2=\frac{u^2\,(1-a^2u^2)}{n^2}$

or $\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=\frac{(1/r^2)\,[1-(a^2/r^2)]}{n^2}$, $\because u=\frac{1}{r}, \frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$

or $\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{(r^2-a^2)}{n^2r^4}$ or $\frac{dr}{\sqrt{(r^2-a^2)}}=\frac{d\theta}{n}$.

Integrating, $\cosh^{-1}(r/a)=(\theta/n)+C_1$.

Initially $r=a$, $\theta=0$, so $C_1=0$.

$\therefore$ $\cosh^{-1}(r/a)=\theta/n$ or $r=a\cosh(\theta/n)$ is the required equation of the path.

Second part : Do yourself.

***Ex. 15.** In a central orbit the force is $\mu u^3\,(3+2a^2u^2)$; if the particle be projected at a distance a with a velocity $\sqrt{(5\mu/a^2)}$ in a direction making an angle $\tan^{-1}(\frac{1}{2})$ with the radius, show that the equation to the path is $r=a\tan\theta$.

Solution The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2},\quad\text{where } F=\mu u^3\,(3+2a^2u^2)$$

or $\frac{d^2u}{d\theta^2}+u=\frac{\mu u^3\,(3+2a^2u^2)}{h^2u^2}$ or $h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\mu\,(3u+2a^2u^3)$.

Multiplying both sides by $2\,du/d\theta$ and integrating, we get

$$v^2=h^2\,[(du/d\theta)^2+u^2]=\mu\,[3u^2+a^2u^4]+C. \quad\text{...(i)}$$

Also from "$p=r\sin\phi$" we have initially $p_0=a\sin\phi_0$, where p_0 and ϕ_0 are the initial values of p and ϕ which have their usual meanings. (See Author's Differential Calculus)

Now $\phi=\tan^{-1}(\frac{1}{2})$, given or $\tan\phi_0=\frac{1}{2}$ $\therefore$ $\sin\phi_0=1/\sqrt{5}$.

$$\therefore\ p_0=a\sin\phi_0=a/\sqrt{5}. \quad\text{...(ii)}$$

Also we know that $\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=u^2+\left(\frac{du}{d\theta}\right)^2$.

$\therefore$ Initially $\left(\frac{du}{d\theta}\right)^2+u^2=\frac{1}{p_0^2}=\frac{5}{a^2}$, from (ii)

Also initially $v=\sqrt{(5\mu/a^2)}$, given.

$\therefore$ From (i), we get $\mu\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\,[3u^2+a^2u^4]+\frac{\mu}{a^2}$

or $\left(\frac{du}{d\theta}\right)^2=2u^2+a^2u^4+\frac{1}{a^2}=\frac{a^4u^4+3a^2u^2+1}{a^2}$.

or $\left(\frac{du}{d\theta}\right)^2=\frac{(a^2u^2+1)^2}{a^2}$ or $\frac{du}{d\theta}=\pm\left(\frac{a^2u^2+1}{a}\right)$

or $$\frac{-a\,du}{(a^2u^2+1)}=d\theta \qquad \text{(Note)}$$

or $$\cot^{-1}(au)=\theta+C_1.$$

Initially $u=1/a$ and let $\theta=\frac{1}{4}\pi$, so that $C_1=0$. (Note)

$\therefore$ $\cot^{-1}(au)=\theta$ or $au=\cot\theta$ or $r=a\tan\theta$ is the required equation of the path.

Ex. 16. A particle is projected from an apse at a distance a with the velocity from infinity under the action of a central acceleration μ/r^{2n+3}. Prove that the equation of the path is $r^n=a^n\cos n\theta$. (*Meerut* 87)

Solution. If v be the velocity of the particle, moving under the action of the given acceleration, at a distance x, from the centre of force, then $\frac{dv}{dx}=-\frac{\mu}{x^{2n+3}}$, negative sign shows that x decreases as time increases.

$\therefore$ If V be the velocity of projection at an apse at a distance a, then $$\int_0^V v\,dv=\int_{\infty}^{a}-\frac{\mu}{x^{2n+3}}\,dx$$

or $$\tfrac{1}{2}V^2=\frac{\mu}{2n+2}\left[\frac{1}{x^{2n+2}}\right]_{\infty}^{a}=\frac{\mu}{(2n+2)\,a^{2n+2}},$$

or $$V^2=\mu/[(n+1)\,a^{2n+2}]. \qquad \text{...(i)}$$

Also the differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2} \qquad \text{...(ii)}$$

Here $$F=\mu/r^{2n+3}=\mu u^{2n+3}$$

$\therefore$ From (ii), we get $\frac{d^2u}{d\theta^2}+u=\frac{\mu u^{2n+3}}{h^2u^2}=\frac{\mu}{h^2}u^{2n+1}$

or $$h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\mu u^{2n+1}.$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we get

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=2\mu\frac{u^{2n+2}}{2n+2}+C_1, \qquad \text{...(iii)}$$

where C_1 is constant of integration.

Initially $r=a$, i.e. $u=1/a$, $du/d\theta=0$, $v=V$.

$\therefore$ From (iii), $V^2=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{n+1}\cdot\frac{1}{a^{2n+2}}+C_1$

or $$\frac{\mu}{(n+1)\,a^{2n+2}}=h^2\left(\frac{1}{a^2}\right)=\frac{\mu}{(n+1)\,a^{2n+2}}+C_1, \text{ from (i)}$$

$\therefore$ $h^2=\mu/[(n+1)\,a^{2n}]$ and $C_1=0$.

$\therefore$ From (iii), we get $\frac{\mu}{(n+1)\,a^{2n}}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\frac{\mu u^{2n+2}}{(n+1)}$

or $$(du/d\theta)^2+u^2=u^{2n+2}\,a^{2n}. \quad \text{...(iv)}$$

Now $u=\frac{1}{r}$, so $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$

$\therefore$ From (iv), $\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2+\frac{1}{r^2}=\frac{a^{2n}}{r^{2n+2}}$

or $$\left(\frac{dr}{d\theta}\right)^2=r^2\left[\frac{a^{2n}}{r^{2n}}-1\right]=\frac{a^{2n}-r^{2n}}{r^{2n-2}}.$$

or $$\frac{dr}{d\theta}=\pm\frac{\sqrt{(a^{2n}-r^{2n})}}{r^{n-1}} \text{ or } \frac{-r^{n-1}\,dr}{\sqrt{(a^{2n}-r^{2n})}}=d\theta.$$

Putting $r^n=R$, we get $nr^{n-1}\,dr=dR$.

$\therefore$ From (v), we get $\frac{-dR}{\sqrt{[(a^n)^2-R^2]}}=n\,d\theta.$

Integrating, $\cos^{-1}(R/a^n)=n\theta+c_2$ or $\cos^{-1}(r^n/a^n)=n\theta+c_2$.

Initially $r=a$, $\theta=0$, $\therefore$ $c_2=0$.

$\therefore$ $\cos^{-1}(r^n/a^n)=n\theta$ or $r^n=a^n\cos n\theta$ is the required equation of the path.

***Ex. 17.** A particle is acted on by a central repulsive force which varies as the nth power of the distance. If the velocity at any point be equal to that which would be acquired in falling from the centre to the point; show that the equation to the path is of the form $r^{(n+3)/2}\cos\frac{1}{2}(n+3)\,\theta=\text{constant}.$

Solution. According to the problem the velocity of projection is equal to the velocity which would be acquired in falling from the centre of force to the point of the projection. Let V be the velocity of projection and a be the distance of the point of projection from the centre of force.

Now the equation of motion of the particle falling from the centre of force is $v\frac{dv}{dx}=\mu x^n$

or $$v\,dv=\mu x^n\,dx \text{ or } \int_{v=0}^{V}2v\,dv=2\mu\int_{x=0}^{a}x^n\,dx$$

or $$V^2=\frac{2\mu a^{n+1}}{n+1}$$ ...(i)

Now the differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \text{ where } F=-\mu r^n=-\mu u^{-n},$$ (Note)

negative sign shows repulsion

or $$\frac{d^2u}{d\theta^2}+u=\frac{-\mu u^{-n}}{h^2u^2}=-\frac{\mu u^{-n-2}}{h^2}$$

or $$h^2\left[\frac{d^2u}{d\theta^2}+u\right]=-\mu u^{-n-2}$$

Multiplying both sides by $2\ du/d\theta$ and integrating, we have

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=-2\mu\frac{u^{-n-1}}{-n-1}+C,$$

where C is constant of integration. ..(iii)

Initially $v=V$ given by (i) and $r=a$ *i.e.* $u=1/a$.

∴ From (ii), we get $V^2=\frac{2\mu a^{n+1}}{(n+1)}+C$ (Note)

or $$\frac{2\mu a^{n+1}}{n+1}=\frac{2\mu a^{n+1}}{(n+1)}+C, \text{ from (i)}.$$

∴ $C=0$ and from (ii), we have

$$h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\frac{2\mu}{(n+1)\ u^{n+1}}$$

or $\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2\mu}{h^2(n+1)\ u^{n+1}}$ or $\left(\frac{du}{d\theta}\right)^2=\frac{2\mu}{h^2(n+1)\ u^{n+1}}-u^2$

or $$\left(\frac{du}{d\theta}\right)^2=\frac{\lambda^2}{u^{n+1}}-u^2, \text{ where } \lambda^2=\frac{2\mu}{h^2(n+1)}.$$

or $\left(\frac{du}{d\theta}\right)^2=\frac{\lambda^2-u^{n+3}}{u^{n+1}}$ or $\frac{-u^{(n+1)/2}\ du}{\sqrt{(\lambda^2-u^{n+3})}}=d\theta$ (Note) ...(iii)

or $$\frac{-dz}{\sqrt{(\lambda^2-z^2)}}=\tfrac{1}{2}(n+3)\ d\theta,$$ putting $u^{(n+3)/2}=z$

∴ $\frac{1}{2}(n+3)\ u^{(n+1)/2}\ du=dz.$

Integrating, we have $\cos^{-1}(z/\lambda)=\frac{1}{2}(n+3)\ \theta+C_1$.

Choose z and θ in such a way that $C_1=0$,

∴ $\cos^{-1}(z/\lambda)=\frac{1}{2}(n+3)\ \theta$ or $z=\lambda\cos\{\frac{1}{2}(n+3)\ \theta\}$

or $u^{(n+3)/2}=\lambda\cos\{\frac{1}{2}(n+3)\ \theta\}$, where $u=1/r$

or $r^{(n+3)/2}\cos\{\frac{1}{2}(n+3)\ \theta\}=1/\lambda=$ constant. Hence proved.

Ex. 18. A particle of mass m moves under a central force $m\mu/(\text{distance})^2$ and is projected at a distance a from the centre of

force with the velocity which at angle α to the radius would be acquired by a fall from rest at infinity to the point of projection, prove that the orbit is an equiangular spiral.

Solution. Here the velocity of projection is given to be that which would be acquired by the particle falling from rest at infinity to the point of projection. Let V be the velocity of projection.

Now the equation of motion of the particle falling from rest to infinity is $mv\dfrac{dv}{dx}=-\dfrac{m\mu}{x^3}$ or $v\,dv=-\mu x^{-3}\,dx$

$$\therefore \int_{v=0}^{V} 2v\,dv=-2\mu\int_{\infty}^{a} x^{-3}\,dx \quad \text{or} \quad V^2=\mu\left[\frac{1}{a^2}\right] \qquad \text{..(i)}$$

Now the differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \text{ where } F=\frac{\mu}{r^3}=\mu u^3 \text{ (given)}$$

or

$$h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\frac{\mu u^3}{u^2}=\mu u$$

Multiplying both sides by 2 $(du/d\theta)$ and integrating we get

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=2\mu.\frac{u^2}{2}+C, \qquad \text{...(ii)}$$

where C is constant of integration.

Initially $v=V$ and $u=1/a$, where $u=1/r$.

$\therefore$ $V^2=\dfrac{\mu}{a^2}+C$ or $\dfrac{\mu}{a^2}=\dfrac{\mu}{a^2}+C$, from (i)

$\therefore$ $C=0$ and so from (ii), we get

$$v^2=h^2\left[(du/d\theta)^2+u^2\right]=\mu u^2. \qquad \text{...(iii)}$$

Again we know $\dfrac{1}{p^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2=u^2+\left(\dfrac{du}{d\theta}\right)^2$ ...(iv)

Let p_0 be the initial value of p and also we know that

$p_0=r\sin\phi$ and initially $r=a$, $\phi=\alpha$ (given)

$\therefore$ $p_0=a\sin\alpha$.

$\therefore$ From (iv) the initial value of $\left(\dfrac{du}{d\theta}\right)^2+u^2=\dfrac{1}{p_0^2}=\dfrac{1}{a^2\sin^2\alpha}$

$\therefore$ From (iii) initially, we have

$V^2=h^2\,[1/a^2\sin^2\alpha]$ or $\mu/a^2=h^2/$ rom (i)

or $h^2=\mu\sin^2\alpha$

$\therefore$ (iii) reduc [illegible] $[(du/d\theta)^2+$

or $(du/d\theta)^2+u^2=u^2$ [illegible] $u/d\theta)^2=u^2$ [illegible] $\cot^2\alpha$

or $\dfrac{du}{d\theta}=-$ [illegible] u $=-(\text{co}$ [illegible]

Integrating, $\log u = -(\cot\alpha)\,\theta + C_1$, ...(v)
where C_1 is constant of integration.

Initially $u = 1/a$ and $\theta = 0$, $\therefore$ $C_1 = \log(1/a)$

$\therefore$ From (v), we get $\log u = -(\cot\alpha).\theta + \log(1/a)$

or $\log(au) = -(\cot\alpha).\theta$ or $au = e^{-(\cot\alpha).\theta}$

or $a/r = e^{-(\cot\alpha).\theta}$ or $r = a\,e^{\theta\cot\alpha}$, which is an equiangular spiral.

Ex. 19. A particle is projected from an apse at a distance a with a velocity from infinity, the acceleration being μu^7, show that the equation to its path is $r^2 = a^2\cos 2\theta$. (*Purvanchal* 90)

Solution. As in last example, we can have the equation of motion as $v\dfrac{dv}{dx} = -\dfrac{\mu}{x^7}$ or $2v\,dv = -2\mu x^{-7}\,dx$

$$\therefore \int_0^V 2v\,dv = -2\mu\int_{\infty}^{a} x^{-7}\,dx,$$

where V is the velocity of projection.

or $$V^2 = \tfrac{1}{3}\mu a^{-6} = \frac{\mu}{3a^6}. \quad ...(i)$$

Now the differential equation of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}, \text{ where } F = \mu u^7 \text{ (given)}.$$

or $$h^2\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{\mu u^7}{u^2} = \mu u^5.$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we get

$$v^2 = h^2\left[(du/d\theta)^2 + u^2\right] = 2\mu\left(\tfrac{1}{6}u^6\right) + C, \quad ...(ii)$$

where C is constant of integration.

Initially $v = V$ given by (i) and $u = 1/a$. Also at an apse $du/d\theta = 0$.

$\therefore$ From (ii), we have $V^2 = h^2\,[1/a^2] = \tfrac{1}{3}\mu\,(1/a^6) + C$

or $$\frac{\mu}{3a^6} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{3a^6} + C, \text{ from (i)}$$

$$\therefore h^2 = \tfrac{1}{3}\frac{\mu}{a^4} \text{ and } C = 0$$

$\therefore$ From (ii), we have $\dfrac{\mu}{3a^4}\left[\left(\dfrac{du}{d\theta}\right)^2 + u^2\right] = \dfrac{\mu u^6}{3}$

or $(du/d\theta)^2 + u^2 = a^4u^6$, or $(du/d\theta)^2 = a^4u^6 - u^2$. ...(iii)

Also $u = 1/r$, $\therefore$ $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$

Substituting these values of u and $du/d\theta$ in (iii), we get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^6} - \frac{1}{r^2} = \frac{a^4 - r^4}{r^6}$$

or $\frac{-r\,dr}{\sqrt{(a^4-r^4)}}=d\theta$ or $\frac{-dz}{\sqrt{(a^4-z^2)}}=2\theta$, where $z=r^2$. (Note)

Integrating, we have $\cos^{-1}(z/a^2)=2\theta+C_1$.

Initially $r=a$ i.e. $z=r^2=a^2$ and $\theta=0$ (say). Then $C_1=0$.

$\therefore$ $\cos^{-1}(z/a^2)=2\theta$ or $z=a^2\cos 2\theta$ or $r^2=a^2\cos 2\theta$.

Hence proved

*Ex. 20. If the acceleration at a distance r is μ/r^5 and the particle is projected at a distance a from the centre of force with velocity $\sqrt{(\mu/2a^4)}$, prove that the orbit is a circle through O of diameter $a \operatorname{cosec} \alpha$, where α is the inclination of the direction of projection to the radius vector.

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}, \text{ where } F=\frac{\mu}{r^5}=\mu u^5 \text{ (given)}$$

or
$$h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\frac{\mu u^5}{u^2}=\mu u^3.$$

Multiplying both sides by $2\,du/d\theta$ and integrating, we have

$$v^2=h^2\left[(du/d\theta)^2+u^2\right]=2\mu\left(\tfrac{1}{4}u^4\right)+C,$$

where C is constant of integration.

Initially $r=a$ i.e. $u=1/a$ and $v=\sqrt{(\mu/2a^4)}$, given

Also we know $\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=u^2+\left(\frac{du}{d\theta}\right)^2$ ...(ii)

Also let p_0 be the initial value of p. Then from (ii) we get the initial value of $\left(\frac{du}{d\theta}\right)^2+u^2=\frac{1}{p_0^2}$, where p_0=initial value of $r\sin\phi$

$=1/(a\sin\alpha)^2$, $\because$ $\phi=\alpha$ and $r=a$ (initially).

$\therefore$ From (i) initially, we get

$\frac{\mu}{2a^4}=h^2\left[\frac{1}{a^2\sin^2\alpha}\right]=\frac{\mu}{2}$ ... $^2=(\mu\sin^2\alpha)/2a$ and C

$\therefore$ From (i), we get μ ... $u^2\Big]=\frac{\mu u^4}{2}$.

or $\left(\frac{du}{d\theta}\right)^2+u^2=a^2u^4\operatorname{cosec}^2\alpha$ ... $c^2\alpha-u^2$. ...

Now $u=1$ ... $\frac{du}{d\theta}=$

Integrating, we get $\cos^{-1}[r/a \operatorname{cosec} \alpha)] = \theta + C_1$.

Initially $r = a$ and $\theta = \frac{1}{2}\pi - \alpha$ (say). Then $C_1 = 0$. (Note)

$\therefore$ The equation … centre of … through the pole (*i.e.* … Hence proved.

Ex. 2. A particle is attached to a fixed point on a smooth horizontal plane by an elastic string of natural length a; initially the particle is at rest on the plane with the string just tight and it is projected horizontally in a direction perpendicular to the string with a kinetic energy equal to the potential energy of the string, when its extension is $3a/\sqrt{2}$. Prove that the second apsidal distance is equal to $3a$.

Solution. Tension in the elastic string when its extension is $3a/\sqrt{2}$.

$$=\lambda . \frac{(3a/\sqrt{2})}{a}, \text{ by Hooke's Law}$$

$$=\frac{3\lambda}{\sqrt{2}}, \text{ where } \lambda \text{ is the modulus of elasticity.}$$

Now potential energy of the string

$=\frac{1}{2}$ (sum of the initial and final tensions) $\times$ extension produced

$=\frac{1}{2}\left[0+\frac{3\lambda}{\sqrt{2}}\right] \times \frac{3a}{\sqrt{2}}$, since initially tension $= 0$

$=(9/4)\, a\lambda$.

Let v_1 be the velocity of projection of the particle, then its kinetic energy $= \frac{1}{2} m v_1^2$, where m is the mass of the particle.

Now it is given that its kinetic energy = potential energy of the string so we have $\frac{1}{2} m v_1^2 = \frac{9}{4} a\lambda$ or $v_1^2 = \frac{9}{2} . (a\lambda/m)$. ...(i)

The differential equation of the path is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}, \text{ where } mF = \frac{\lambda}{a}(r-a) \text{ i.e. tension at the point where the extended length is } r$$ (Note)

$$= \frac{\lambda (r-a)}{h^2u^2am} = \frac{\lambda[(1/u)-1]}{hu^2am} = \frac{\lambda(1-au)}{h^2amu^3}$$

or, $$h^2\left(\frac{d^2u}{d\theta^2}+u\right) = \frac{\lambda}{am}\left(\frac{1}{u^3} - \frac{a}{u^2}\right)$$

Multiplying both sides by $2\, du/d\theta$ and integrating, we get

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{2\lambda}{am}\left[-\frac{1}{2u^2} + \frac{a}{u}\right] + C, \quad \text{...(ii)}$$

where C is constant of integration.

Initially $r=a$ i.e. $u=1/a$ and $du/d\theta=0$, since the particle is projected at right angles to the string. Also initially $v=v_1$, given by (i).

∴ From (ii) initially we have

$$v_1^2=h^2\left(\frac{1}{a^2}\right)=\frac{2\lambda}{2m}\left[-\tfrac{1}{2}a^2+a^2\right]+C$$

or $$\frac{9a\lambda}{2m}=\frac{h^2}{a^2}=\frac{\lambda a}{m}+C$$

∴ $h^2=9a^3\lambda/2m$ and $C=7a\lambda/2m$.

∴ From (ii), we have

$$\frac{9a^3\lambda}{2m}\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=\frac{2\lambda}{am}\left\{\frac{a}{u}-\frac{1}{2u^2}\right\}+\frac{7a\lambda}{2m} \quad \text{...(iii)}$$

Also we know that at an apse $du/d\theta=0$, so from (iii) we find that the apsidal distances are given by

$$\frac{9a^3\lambda}{2m}[u^2]=\frac{2\lambda}{am}\left[\frac{a}{u}-\frac{1}{2u^2}\right]+\frac{7a\lambda}{2am} \quad \text{(Note)}$$

or $$\frac{9a^3u^2}{2}=\frac{2}{u}-\frac{1}{au^2}+\frac{7a}{2}=\frac{4au-2+7a^2u^2}{9au^2}$$

or $9a^4u^4=4au-2+7a^2u^2$ or $9a^4u^4-7a^2u^2-4au+2=0$

or $2r^4-4ar^3-7a^2r^2+9a^4=0$, putting $u=1/r$ and simplifying. (Note)

or $2r^4-2ar^3-2ar^3+2a^2r^2-9a^2r^2+9a^4=0$

or $2r^3(r-a)-2ar^2(r-a)-9a^2(r^2-a^2)=0$

or $(r-a)[2r^3-2ar^2-9a^2(r+a)]=0$ (Note)

or $(r-a)[2r^3-6ar^2+4ar^2-12a^2r+3a^2r-9a^3]=0$

or $(r-a)[2r^2(r-3a)+4ar(r-3a)+3a^2(r-3a)]=0$

or $(r-a)(r-3a)(2r^2+4ar+3a^2)=0$,

which gives $r=a$ and $r=3a$.

But $r=a$ is the given apsidal distance, hence $r=3a$ gives the required apsidal distance. Hence proved.

Ex. 22 Show that only law for a central attraction for which the velocity in a circle at a distance is equal to the velocity acquired in falling from infinity to the distance is that of inverse cube.

Solution. Let F be the central attraction and $F=f'(r)$ ...(i)

For the particle falling from infinity under this law when it is at a distance r from the centre of force, the equation of motion is

$$v\frac{dv}{dr}=-f'(r) \quad \text{(Note)}$$

$\therefore$ Integrating $v^2=-2f(r)+C,$ .. (ii)
where C is constant of integration.

Again if v be the velocity of the particle moving in a circle at a distance r, then in the inward normal direction, the equation of motion is $\frac{v^2}{\rho}=f'(r)$, where $\rho=r$ for the circle

i.e. $v^2=rf'(r)$ (iii)

Now according to the problem the velocities given by (ii) and (iii) are equal, so we have $rf'(r)=-2f(r)+C$

or $r^2f'(r)+2rf(r)=Cr$, multiplying each term by r. (Note)

Integrating we get $r^2f(r)=\frac{1}{2}Cr^2+C_1$

or $f(r)=\frac{1}{2}C+(C_1/r^2)$

$\therefore$ $f'(r)=-\frac{2C_1}{r^3}$; differentiating both sides.

$\therefore$ From (i) we have $F=f'(r)=-\frac{2C_1}{r^3}$ or $F\propto 1/r^3$.

Hence proved.

Ex. 23. A particle acted on by a repulsive central force $\mu r/(r^2-9c^2)^2$ is projected from an apse at a distance c with velocity $\sqrt{(\mu/8c^2)}$. Find the equation of its path and show that the time to the cusp is $(4/3)\pi c^2\sqrt{(2/\mu)}$.

Solution. The differential equation (pedal form) of the path is

$$\frac{h^2}{p^3}\frac{dp}{dr}=F=-\frac{\mu r}{(r^2-9c^2)^2}$$, the force being repulsive.

or $$\frac{h^2}{p^3}dp=-\frac{\mu r}{(r^2-9c^2)^2}dr.$$

Integrating we get $-\frac{h^2}{2p^2}=\frac{\mu}{2(r^2-9c^2)}+C,$ where C is constant.

or $v^2=\frac{h^2}{p^2}=-\frac{\mu}{(r^2-9c^2)}+C_1$, where $C_1=-2C$, $pv=h$.

Initially at an apse $p=r=c$ (given) and $v=\sqrt{(\mu/8c^2)}$

$\therefore$ $\frac{\mu}{8c^2}=\frac{h^2}{c^2}=-\frac{\mu}{(c^2-9c^2)}+C_1$

$\therefore$ $h^2=(1/8)\mu$ and $C_1=0$

$\therefore$ $\frac{h^2}{p^2}=-\frac{\mu}{r^2-9c^2}+C_1$ reduces to $\frac{\mu}{8p^2}=\frac{\mu}{9c^2-r^2}$.

or $\quad 8p^2=9c^2-r^2$, which is the required pedal equation of the path and its cusp is obtained by putting $p=0$ in this equation.

∴ At the cusp we get $9c^2-r^2=0$ or $r=3c$. (Note)

∴ We are to find the time taken by the particle in moving from $r=c$ to $r=3c$.

Now we know $pv=h$

or $$\frac{h}{p}=v=\frac{ds}{dt}=\frac{ds}{dr}\cdot\frac{dr}{dt}=\frac{1}{\cos\phi}\frac{dr}{dt}, \quad \because \cos\phi=\frac{dr}{ds}$$

or $$h\,dt=\frac{p}{\cos\phi}\,dr=\frac{p}{\sqrt{(1-\sin^2\phi)}}\,dr \qquad \text{(Note)}$$

or $$h\,dt=\frac{p\,dr}{\sqrt{[1-(p/r)^2]}}, \quad \because p=r\sin\phi$$

$$=\frac{rp\,dr}{\sqrt{(r^2-p^2)}}, \text{ where } 8p^2=9c^2-r^2 \text{ is the curve}$$

$$=\frac{r\sqrt{(\frac{1}{8})}\sqrt{(9c^2-r^2)}}{\sqrt{\{r^2-\frac{1}{8}(9c^2-r^2)\}}}\,dr$$

or $$h\,dt=\frac{\sqrt{(9c^2-r^2)}\,dr}{\sqrt{(8r^2-9c^2+r^2)}}=\frac{r\sqrt{[8c^2-(r^2-c^2)]}\,dr}{3\sqrt{(r^2-c^2)}}$$

∴ If t_1 be the required time from $r=c$ to $r=3c$, then we get

$$\int_{t=0}^{t_1} h\,dt=\int_{r=c}^{3c}\frac{r\sqrt{[8c^2-(r^2-c^2)]}\,dr}{3\sqrt{(r^2-c^2)}}$$

$$=\int_{\theta=0}^{\pi/2}\frac{\sqrt{(8c^2-8c^2\sin^2\theta)}\,8c^2\sin\theta\cos\theta\,d\theta}{3\sqrt{(8c^2\sin^2\theta)}},$$

putting $r^2-c^2=8c^2\sin^2\theta$

∴ $r\,dr=8c^2\sin\theta\cos\theta\,d\theta$

$$=\int_0^{\pi/2}\frac{8c^2\sin\theta\cos^2\theta\,d\theta}{3\sin\theta}=\frac{8}{3}c^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

or $\quad ht_1=\frac{8}{3}c^2\cdot\frac{1}{2}\cdot\frac{1}{2}\pi$

or $\quad ht_1=\frac{2}{3}\pi c^2$ or $\sqrt{(\frac{1}{8}\mu)}\,t_1=\frac{2}{3}\pi c^2$, $\quad \because h^2=\frac{1}{8}\mu$ (proved)

or $\quad t_1=\frac{2}{3}\sqrt{(8/\mu)}\,\pi c^2=\frac{4}{3}\sqrt{(2\mu)}\,\pi c^2$. Hence proved.

***Ex. 24. A particle moves with central acceleration $(\mu u^2+\lambda u^3)$ and the velocity of projection at distance R is V; show that the particle will ultimately go off to infinity if $V^2>\dfrac{2\mu}{R}+\dfrac{\lambda}{R^2}$**

Solution. The differential equation of the path is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2} \qquad \text{...(i)}$$

Here $F=\mu u^2+\lambda u^3$.

$\therefore$ From (i), we get

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu u^2+\lambda u^3}{h^2u^2} \quad \text{or} \quad h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\mu+\lambda u.$$

Multiplying both sides by $2du/d\theta$ and integrating, we have

$$v^2=h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=2\left\{\mu u+\lambda\frac{u^2}{2}\right\}+c_1, \quad \text{...(ii)}$$

where c_1 is constant of integration.

Initially $r=R$ *i.e.* $u=1/R$ and $v=V$.

$\therefore$ From (ii), we get $V^2=\left\{\frac{\mu}{R}+\frac{\lambda}{2R^2}\right\}+c_1$

or $$c_1=V^2-\frac{2\mu}{R}-\frac{\lambda}{R^2}. \quad \text{...(iii)}$$

Also from (ii), we have $h^2[(du/d\theta)^2+u^2]=2\mu u+\lambda u^2+c_1$

or $$h^2(du/d\theta)^2=2\mu u+\lambda u^2-h^2u^2+c_1=u^2(\lambda-h^2)+2\mu u+c_1$$

$$=(\lambda-h^2)\left\{u^2+\frac{2\mu u+c_1}{(\lambda-h^2)}\right\}$$

or $$h\frac{du}{d\theta}=\sqrt{(\lambda-h^2)}\cdot\sqrt{\left\{u^2+\frac{2\mu}{(\lambda-h^2)}u+\frac{c_1}{(\lambda-h^2)}\right\}}$$

or $$\frac{du}{\sqrt{\left\{u^2+\frac{2\mu}{(\lambda-h^2)}u+\frac{c_1}{(\lambda-h^2)}\right\}}}=\frac{\sqrt{(\lambda-h^2)}}{h}d\theta$$

or $$\frac{du}{\sqrt{\left\{\left(u+\frac{\mu}{(\lambda-h^2)}\right)^2+\left(\frac{c_1}{(\lambda-h^2)}-\frac{\mu^2}{(\lambda-h^2)^2}\right)\right\}}}=\frac{\sqrt{(\lambda-h^2)}}{h}d\theta.$$

Integrating, we get

$$\log\left\{\left(u+\frac{\mu}{\lambda-h^2}\right)+\sqrt{\left\{\left(u+\frac{\mu}{\lambda-h^2}\right)^2+\frac{c_1}{\lambda-h^2}-\frac{\mu^2}{(\lambda-h^2)^2}\right\}}\right\}$$

$$=\frac{\sqrt{(\lambda-h^2)}}{h}\theta+c_2. \quad \text{...(iv)}$$

Now at infinity we have $r=\infty$ *i.e.* $u=0$, so the particle will be at infinity if

$$\log\left\{\frac{\mu}{\lambda-h^2}+\sqrt{\left(\frac{c_1}{\lambda-h^2}\right)}\right\}=\frac{\sqrt{(\lambda-h^2)}}{h}\theta+c_2.$$

This gives real values of θ only if c_1 is positive,

i.e. $$V^2-\frac{2\mu}{R}-\frac{\lambda}{R^2}>0, \text{ from (iii)}$$

i.e. $$V^2>\left(\frac{2\mu}{R}+\frac{\lambda}{R^2}\right).$$ Hence proved.

Exercises on § 8

Ex. 1. A particle moving under a constant force from a centre is projected in a direction perpendicular to the radius vector with velocity acquired in falling to the point of projection from the centre, show that its path is $(a/r)^2=\cos^2(\frac{1}{2}\theta)$ and that the particle will ultimately move in a straight line through the origin in the same way as if the path had always been this line.

If the velocity of projection be double that in the previous case, show that the path is

$$\tfrac{1}{2}\theta=\tan^{-1}\sqrt{[(r-a)/a]}-(1/\sqrt{3})\tan^{-1}\sqrt{[(r-a)/3a]}.$$

Ex. 2. A particle moves with central acceleration $\mu\left(r+\frac{2a^3}{r^2}\right)$ being projected from an apse at a distance a with twice the velocity for a circle at the distance; find the other apsidal distance and show that the equation to the path is

$\frac{1}{2}\theta=\tan^{-1}(t\sqrt{3})-(1/\sqrt{5})\tan^{-1}(\sqrt{\frac{5}{3}}t)$, where $t^2=(r-a)/(3a-r)$.

Ex. 3. A particle acted on by a central force μ/r^3 is projected with velocity $\sqrt{\mu}/a$ act an angle $\pi/4$ with its initial distance a from the centre of force; prove that the orbit is $r=ae^{-\theta}$.

(Agra 89)

Ex. 4. A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being nmg. It is projected from an apse at a distance a with velocity $\sqrt{(2pgh)}$; show that the other apsidal distance is given by the equation $nr^2(r-a)-2pha(r+a)=0$.

Ex. 5. A particle is projected with velocity $\sqrt{(2\mu/3c^3)}$ from a point P in a field of attractive force μ/r^4 to a point O distance c from P, where r denotes the distance from O.

If the direction of projection makes an angle 45° with PO, prove that the orbit is cardioid and the particle will arrive at O after a time $\{(3/4)\pi-2\}\sqrt{(3c^5/\mu)}$.

Ex. 6. If the law of force is $\mu(5u^3+8c^2u^5)$ and the particle is projected from an apse at a distance c with velocity $3\sqrt{\mu}/c$, prove that the orbit is $r=c\cos(2\theta/3)$.

Ex. 7. If a particle is projected from an apse at a distance a with the velocity of projection $\sqrt{\mu}/(a^2\sqrt{2})$, under the action of a central force μu^5, find the orbit. *(Garhwal 91; Meerut 92)*

Ex. 8. A particle moves under a central attractive force varying inversely as the fifth power of the distance from the centre of force. It is projected from an apse at a distance 'a' with

velocity equal to $\sqrt{5}$ times of that which would be acquired in falling from infinity, show that the other apsidal distance is $a/$..

Ex. 9. Prove that the time required to describe an arc of a parabola under the force μ/r^2 to the focus, starting from the axis, is $\sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right]$,

where a is the apsidal distance and θ is measured from the axis.

Ex. 10. A particle moves under a central attractive force μ/r^5. If it is projected from an apse at a distance 'a' with a velocity $(2/a)\sqrt{(\mu/3)}$, show that the orbit is $r\cos(\theta/2)=a$. Find the time from the apse to $r=2a/\sqrt{3}$. (*Kumaun* '2)

Ex. 11. If the law of force be $2\mu\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)$ and the particle be projected from an apse at a distance a with velocity $\sqrt{\mu}/a$, show that it will be at a distance r from the centre after a time

$$\frac{1}{2\sqrt{\mu}}\left\{r\sqrt{(r^2-a^2)}+a^2\cosh^{-1}\frac{r}{a}\right\}$$

Ex. 12. A particle moves under central force for attraction μ/r^2 per unit mass and if it describes an ellipse prove that the square of the periodic time is proportional to cube of semi-major axis.

Ex. 13. A particle moves under a central force of attraction μ/r^2 per unit mass, show that the periodic time is given by $2\pi a^{3/2}/\sqrt{\mu}$, where 'a' is the semi-major axis of the ellipse.

Ex. 14. A particle moves in an orbit under a central acceleration along the radius vector r; obtain the equation of energy and angular momentum.

Ex. 15. Complete the following sentences :—

(a) If a particle is moving under a central force, the radius vector describes......

(b) The angular velocity of a projectile about the focus of its path varies......

Ex. 16. Show that a particle can describe a rectangular hyberbola under a force from a fixed centre varying as distance and show that the time the radius vector to the particle from the centre takes in sweeping out an angle θ from the vertex is given by $\tan\theta=\tanh(t\sqrt{\mu})$, where μ is the acceleration at unit distance. (*Rohilkhand* 87)

Ex. 17. A particle moving under a central force from the centre is projected in a direction perpendicular to the radius

vector with the velocity acquired in falling to the point of projection from the centre. Show that its path is

$$\left(\frac{a}{r}\right)^2=\cos^2\left(\frac{3\theta}{2}\right).$$ (Garhwal 90; Gorakhpur 92)

Ex. 18. If the central acceleration is μ/r^5, prove that the velocities at the two apsidal distances satisfy the relation $V_1^2+V_2^2=2h^4/\mu$, where h is related by the relation $vp=h$.

(Agra 88 ; Kumaun 90; Purvanchal 91).

Ex. 19. A particle moves with a central acceleration $\mu/(\text{distance})^2$, it is projected with velocity V at a distance R from the centre of force. Show that the path is a rectangular hyperbola if the angle of projection is $\sin^{-1}\frac{\mu}{VR\sqrt{\{V^2-(2\mu/R)\}}}$.

(Gorakhpur 90)

And the periodic time is $\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r}-\frac{V^2}{\mu}\right]^{-3/2}$. (Purvanchal 92)

Planetary Motion

§ 1. Newton's Law of Attraction.

According to Newton, the mutual attraction between two particles of masses m_1 and m_2 and at a distance r apart is $\gamma \frac{m_1 m_2}{r^2}$, where γ is called the constant of gravitation.

The motion of all planets in the solar system are governed by this law. Therefore the motion of all planets (including earth) about the sun or the motion of moon (the satellite of the earth) about the earth is governed by this law.

§ 2. Motion under the inverse square law.

To show that the path of a particle which is moving in such a manner that its acceleration is always directed to a fixed point and equal to $\mu/(\text{distance})^2$ is a conic section and to discuss the three cases that may arise.

Since the acceleration is always directed to a fixed point, so this is a case of central orbit and here $F=\mu/r^2$, given.

∴ The differential equation of the path in the pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr}=F=\frac{\mu}{r^2}$$

...(See Chapter on Central Orbits)

or

$$\frac{h^2}{p^3}\,dp=\frac{\mu}{r^2}\,dr.$$

Integrating we have

$$-\frac{h^2}{2p^2}=-\frac{\mu}{r}+c \quad \text{or} \quad \frac{h^2}{p^2}=\frac{2\mu}{r}+c_1, \qquad \text{...(i)}$$

where c_1 is constant of integration.

Also we know from Chapter on Central Orbits that $h=pv$

∴ From (i), we have $v^2=\frac{h^2}{p^2}=\frac{2\mu}{r}+c_1$.

Now we know that the pedal equations referred to focus as pole of ellipse, parabola and that branch of hyperbola which is nearer the centre of force are

$$\frac{b^2}{p^2}=\frac{2a}{r}-1\ ;\ p^2=ar \quad \text{and} \quad \frac{b^2}{p^2}=\frac{2a}{r}+1 \qquad \text{(Note)}$$

respectively, where $2a$, $2b$ are the lengths of semi-axes in the case of ellipse, $4a$ is the length of latus rectum in the case of parabola

and $2a$, $2b$ are the lengths of transverse and conjugate axes in the case of hyperbola.

Now three cases arise :—

Case I. Elliptic Path.

Comparing the equation (ii) with the pedal equation of ellipse viz. $\frac{b^2}{p^2}=\frac{2a}{r}-1$, we get $\frac{h^2}{b^2}=\frac{2\mu}{2a}=\frac{c_1}{-1}$.

$\therefore\ h^2=\frac{\mu b^2}{a}$ and $c_1=-\frac{\mu}{a}$.

$\therefore$ From (ii) for the elliptic path, we have

$$v^2=\frac{h^2}{p^2}=\frac{2\mu}{r}-\frac{\mu}{a}=\mu\left[\frac{2}{r}-\frac{1}{a}\right]$$

Also from $v^2=\mu\ [(2/r)-(1/a)]$ it is evident that

$$v^2<(2\mu/r), \text{ for the elliptic path.}$$

Case II. Parabolic Path.

Comparing the equation (ii) with the pedal equation of the parabola viz. $p^2=ar$ *i.e.* $\frac{1}{p^2}=\frac{1/a}{r}$, we get $\frac{h^2}{1}=\frac{2\mu}{1/a}$ and $c_1=0$

or $h^2=2a\mu,\ c_1=0$.

$\therefore$ From (ii) for the parabolic path, we have

$$v^2=\frac{h^2}{p^2}=\frac{2a\mu}{ar}, \quad \because\ h^2=2a\mu,\ p^2=ar$$

or $v^2=(2\mu/r)$, for the parabolic path.

Case III. Hyperbolic Path.

Comparing the equation (ii) with the pedal equation of the hyperbola viz. $\frac{b^2}{p^2}=\frac{2a}{r}+1$, we get $\frac{h^2}{b^2}=\frac{2\mu}{2a}=\frac{c_1}{1}$

$\therefore\ h^2=\mu b^2/a$ and $c_1=\mu/a$.

$\therefore$ From (ii) for the hyperbolic path, we have

$$v^2=\frac{h^2}{p^2}=\frac{2\mu}{r}+\frac{\mu}{a}=\mu\left[\frac{2}{r}+\frac{1}{a}\right].$$

Also from $v^2=\mu\left[\frac{2}{r}+\frac{1}{a}\right]$ it is evident that $v^2>\frac{2\mu}{r}$ for the hyperbolic path.

From above three cases we conclude that the equation

$$v^2=(2\mu/r)+c_1,$$

always represents a conic section referred to the focus as pole (Here pole is also the centre of force) and it represents an ellipse, parabola or hyperbola according as c_1 is positive, zero or negative.

Hence we have

(i) if $v^2=\mu\,[(2/r)-(1/a)]$, then the path is elliptic,

(ii) if $v^2=2\mu/r$, then the path is parabolic,

and (iii) if $v^2=\mu\,[(2/r)+(1/a)]$, then the path is hyperbolic.

Also we conclude that

(i) if $v^2 < (2\mu/r)$, then the path is elliptic,

(ii) if $v^2=(2\mu/r)$, then the path is parabolic,

and (iii) if $v^2 > (2\mu/r)$, then the path is hyperbolic.

Note. The velocity at any point is independent of the direction of the velocity.

Again we have proved above that

(i) $h^2=\mu\,(b^2/a)=\mu l$, for the elliptic path,

(ii) $h^2=2a\mu=\mu l$, for the parabolic path,

and (iii) $h^2=\mu\,(b^2/a)=\mu l$, for the hyperbolic path.

Here l is the semi-latus rectum in all the cases.

Hence in all the three cases we have

$h^2=\mu l$ or $h=\sqrt{(\mu l)}$, where l is the semi-latus rectum.

This determines the size of the orbit.

Periodic Time of an elliptic orbit.

If T be the time to describe an elliptic path, then we have

$$T=\frac{\text{area of the ellipse}}{\text{rate of description of sectorial area}}$$

$$\therefore\quad T=\frac{\pi ab}{\frac{1}{2}h} \qquad \text{...See Chapter on Central Orbits}$$

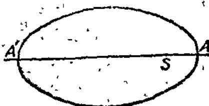
$$=\frac{2\pi\,ab}{h}=\frac{2\pi\,ab}{\sqrt{(\mu l)}} \qquad \because\ h=\sqrt{(\mu l)}$$

$$=\frac{2\pi\,ab}{\sqrt{[\mu\,(b^2/a)]}} \qquad \because\ l=b^2/a$$

i.e. $T=(2\pi a^{3/2}/\sqrt{\mu})$. *(Gorakhpur III 90)*

Perihelion and aphelion of an elliptic orbit.

The point on an elliptic orbit nearest to the focus taken as the centre of force (it is also known as the occupied focus) is called the perihelion and the farthest point on the ellipse from this focus is called aphelion. In the adjoining figure if S be the occupied focus, then A is perihelion and A' is aphelion.



(Fig. 1)

Perihelion and aphelion naturally coincide with the apses of the orbit and if $SA=r_1$ and $SA'=r_2$, then from the polar equation

of the ellipse $l/r=1+e\cos\theta$, referred to S as pole we get

$$r_1=\frac{l}{1+e}=\frac{b^2/a}{1+e}=\frac{a^2(1-e^2)}{a(1+e)}, \quad \because b^2=a^2(1-e^2)$$

or $r_1=a(1-e)$

and $$r_2=\frac{l}{1-e}=\frac{b^2/a}{1-e}=\frac{a^2(1-e^2)}{a(1-e)}, \quad \because b^2=a^2(1-e^2)$$

or $r_2=a(1+e)$.

§ 3. Circular orbit.

If r is constant *i.e.* u is constant then the path of the particle moving under an acceleration F (say) towards the centre of force is a circle and the differential equation of the path viz.

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2} \text{ reduces to } u=\frac{F}{h^2u^2}$$

i.e. $F=h^2u^3$. ...(i)

Also we know if a be the radius of the circle described by the particle; then $u=1/a$ and from $h=pv$, we get

$h=av$, since $p=a$ in the case of circular path.

$\therefore$ From (i), we get $F=a^2v^2u^3$, where $u=1/a$

i.e. $F=v^2/a$ or $v^2=aF$.

§ 4. Velocity from infinity.

If a particle describing a central orbit be imagined to start from rest at infinity under the acceleration equal to $\mu/(\text{distance})^2$ directed towards a fixed point, then from $v\dfrac{dv}{dr}=-\dfrac{\mu}{r^2}$,

we have $$\int_0^V v\,dv=-\mu\int_{r=\infty}^{R}\frac{1}{r^2}\,dr,$$

where V is the velocity acquired by the particle in falling from infinity to a distance R from the centre

i.e., $$\tfrac{1}{2}V^2=\mu\left[\frac{1}{r}\right]_\infty^R=\frac{\mu}{R} \quad \text{or} \quad V^2=2\mu/R \qquad ...(i)$$

Also from § 3 above we know that if V_1 be the velocity for the description of a circle of radius R, then

$V_1^2=RF=R\cdot\mu/R^2, \quad \because F=\mu/R^2$

or $V_1^2=\mu/R=\frac{1}{2}(2\mu/R)=\frac{1}{2}V^2$, from (i)

or $V_1=(1/\sqrt{2})\,V$

i.e. velocity for the description of a circle of radius R $=(1/\sqrt{2})$(velocity from infinity to the distance R from the centre).

****§ 5. Kepler's Laws of Planetary Motion.**

(*Gorakhpur III 89; Purvanchal 91, 89*)

The astronomer Kepler discovered the following three laws of planetary motion. (These laws according to present calculations are true only to a first approximation) :—

1. *Each planet describes an ellipse with the sun at its focus.*

2. *The rate of description of the area by the radius from the sun to the planet is constant.*

3. *The squares of the periodic times of different planets are proportional to the cubes of the major axes of their orbits.*

Deductions from Kepler's Laws.

From the first law we find that the orbit is an ellipse (*i.e.* a plane curve) and then the law of force must be inverse square distance *i.e.* the acceleration of each planet varies as the square of its distance from the sun..

By the second law we find that the real velocity is a constant and that means the force on each planet is directed towards the focus *i.e.* the sun. Hence from first and second laws we conclude that a planet of mass m at a distance r from the sun is attracted towards the sun with a force $\frac{\mu m}{r^2}$, where μ is a constant not depending upon the mass of the planet.

Also we know that the periodic time under inverse square law of a central orbit $=\frac{2\pi a^{3/2}}{\sqrt{\mu}}=T$. ...See § 2 Page 1 of this chapter

or $\quad T^2=4\pi^2a^3/\mu=\frac{1}{2}\pi^2(2a)^3/\mu$

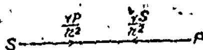
i.e. square of the periodic time varies as the cube of the major axis.

Hence from the third law we find that μ, the absolute acceleration or the acceleration at unit distance from the sun, is the same for all planets. If m and M be the respective masses of the planet and the sun, F be the force of attraction between them, then we have $F=\gamma\frac{mM}{r^2}$, where r is the distance between the planet and the sun and γ the constant of gravitation.

This law is [illegible] mass of the s[illegible] so the attra[illegible] in.

§ 6. Necessary modification of Kepler's Third Law.

Let P and S be the masses of any planet and the sun respectively. Let γ be constant of gravitation.



Then the mutual attraction (Fig. 2)

between them at any instant $=\gamma\frac{S.P}{r^2}$, where r is the distance between the sun and the planet at that instant.

The planet's acceleration is $\frac{\gamma S.P}{r^2}\times\frac{1}{P}=\frac{\gamma S}{r^2}$, towards the sun and the sun's acceleration is $\frac{\gamma S.P}{r^2}\times\frac{1}{S}=\frac{\gamma P}{r^2}$, towards the planet.

Also the acceleration of the planet relative to the sun

$$=\frac{\gamma S}{r^2}+\frac{\gamma P}{r^2}=\frac{\gamma (S+P)}{r^2}=\frac{\mu}{r^2}, \text{ where } \mu=\gamma (S+P).$$

Now we know that the time period of a complete revolution is given by $T=\frac{4\pi a^{3/2}}{\sqrt{\mu}}$ so the exact time period $T=\frac{2\pi a^{3/2}}{\sqrt{[\gamma (S+P)]}}$ ...(i)

This is the modification required in the third law of Kepler.

Let T_1 be the periodic time of another planet of mass P_1 and let $2a_1$ be major axis of its orbit then we have

$$T_1=\frac{2\pi a_1^{3/2}}{\sqrt{[\gamma (S+P_1)]}} \quad \text{... (ii)}$$

∴ From (i) and (ii) we have $\frac{T^2}{T_1^2}=\frac{S+P_1}{S+P}\cdot\frac{a^3}{a_1^3}$

or $$\frac{S+P}{S+P_1}\cdot\frac{T^2}{T_1^2}=\frac{a^3}{a_1^3} \quad \text{...(iii)}$$

If d be the mean distance (*i.e.* distance between the centres) of the planet (of mass P) from the sun (of mass S), then we have

time period $T=\frac{2\pi d^{3/2}}{\sqrt{[\gamma (S+P)]}}$ ...(iv)

Also if d_1 be the mean distance of the satellite (of mass p) of the planet (of mass P) from the planet, then we have *time period*

t (of the satellite)$=\frac{2\pi d_1^{3/2}}{\sqrt{[\gamma (P+p)]}}$ ...(v)

∴ From (iv) and (v), we get

$$\frac{T^2}{t^2}=\frac{(P+p)\, d^3}{(S+P)\cdot d_1^3} \text{ or } \frac{(S+P)}{(P+p)}\frac{T^2}{t^2}=\frac{d^3}{d_1^3}. \quad \text{.. (vi)}$$

Solved Examples on § 1 to § 6.

Ex. 1. A particle describes an ellipse under a force $\mu/(\text{distance})^2$ towards the focus. If it was projected with velocity V from a point distance r from the centre of force, show that its periodic time is $\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r}-\frac{V^2}{\mu}\right]^{-3/2}$.

Solution. Since the particle describes an ellipse and its velocity was V at a distance r from the centre of force, so we have

$$V^2=\mu\,[(2/r)-(1/a)]$$

or $$\frac{V^2}{\mu}=\frac{2}{r}-\frac{1}{a} \text{ or } \frac{1}{a}=\frac{2}{r}-\frac{V^2}{\mu} \quad \text{...(i)}$$

Also periodic time $=\frac{2\pi a^{3/2}}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{\mu}}\left[\frac{2}{r}-\frac{V^2}{\mu}\right]^{-3/2}$, from (i)

Hence proved.

Ex. 2. A particle is projected from the earth's surface with velocity v. Show that if the dimunition of gravity is taken into account, but the resistance of the air neglected, the path is an ellipse, of major axis $2ga^2/(2ga-v^2)$, where a is the earth's radius.

Solution. The path being an ellipse, we have

$$v^2=\mu\,[(2/a)-(1/a_1)], \qquad \text{...(i)}$$

where $2a_1$ is the major axis of the ellipse described by the particle projected from the earth's surface.

Also for any particle on the surface of the earth, we know the acceleration g due to gravity is given by

$$g=\mu/a^2 \quad \text{or} \quad \mu=a^2g \qquad \text{...(ii)}$$

∴ From (i) and (ii), we get

$$v^2=a^2g\left(\frac{2}{a}-\frac{1}{a_1}\right)=2ag-\frac{a^2g}{a_1}$$

or $\quad a^2g=a_1(2ag-v^2)$ or $2a_1=2a^2g/(2ag-v^2)$. Hence proved.

Ex. 3. A particle moves with a central acceleration $\mu/(\text{distance})^2$; it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is

$$\sin^{-1}[\mu/\{VR\sqrt{(V^2-2\mu/R)}\}].$$

Solution. In the case of hyperbola we know the velocity at any point at a distance r from the centre of force is given by

$$v^2=\mu\,[(2/r)-(1/a)]$$

∴ In this case $V^2=\mu\left[\frac{2}{R}+\frac{1}{a}\right]$ or $V^2-\frac{2\mu}{R}=\frac{\mu}{a}$ ...(i)

Let α be the required angle of projection, then from '$p=r\sin\phi$' we get $p_0=R\sin\alpha$, where p_0 is the initial value of p.

∴ From "$h=pv$" we get initially

$$h=p_0\,V=R\sin\alpha\,V \qquad \text{...(ii)}$$

Also we know $h=\sqrt{(\mu\times l)}=\sqrt{[\mu\cdot(b^2/a)]}$, $\because\ l=b^2/a$

or $\qquad =\sqrt{[\mu\cdot(a^2/a)]}$, $\because\ b=a$ for the rectangular hyperbola

$$h=\sqrt{(\mu a)}.$$

∴ From (ii), we get $\sqrt{(\mu a)}=R\sin\alpha\cdot V$

or $\quad \mu a=R^2\sin^2\alpha\cdot V^2$ or $\frac{\mu^2}{[V^2-(2\mu/R)]}=R^2\sin^2\alpha\cdot V^2$, from (i)

or $\quad \sin^2\alpha=\frac{\mu^2}{R^2V^2\,[V^2-(2\mu/R)]}$ or $\alpha=\sin^{-1}\left[\frac{\mu}{VR\sqrt{[V^2-(2\mu/R)]}}\right]$

Hence proved.

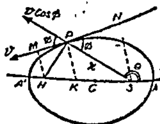
****Ex. 4** A planet is describing an ellipse about the sun as focus, show that its velocity away from the sun is greatest when the

radius vector to the planet is at right angles to the major axis of the path and that it then is $2\pi ae/T\sqrt{[(1-e^2)]}$, where $2a$ is the major axis, e the eccentricity and T the periodic time.

(*Purvanchal* 91, 90)

Solution. If v be the velocity of the planet at any point $P(r, \theta)$ of its path, which is elliptic with sun at the focus S, then its resolved part along SP and away from S is

$v \cos \phi$. (Note)



(Fig. 3)

From S and H draw SN and HM perpendiculars to the tangent at P, then from ΔSPN and ΔPHM we have $\sin \phi=\frac{SN}{SP}$ and $\sin \phi=\frac{HM}{HP}$.

Also if $SP=r$, then $HP=2a-r$ as we know for the ellipse

$SP+PH=2a$ or $HP=2a-SP=2a-r$.

$\therefore \quad \sin^2 \phi=\frac{SN}{SP}\cdot\frac{HM}{HP}=\frac{b^2}{r(2a-r)}$, $\because$ $SN\,HM=b^2$, property of ellipse

or $\quad 1-\cos^2 \phi=\frac{b^2}{r(2a-r)}$ or $\cos^2 \phi=1-\frac{b^2}{r(2a-r)}$ ...(i)

Also the path of the planet ... point is given by $v^2=\mu$ ...

$\therefore$ Velocity at any po...

$$=v \cos \phi=\mu \sqrt{\left(\frac{2a-r}{ar}\right)}\cdot\sqrt{\left[1-\frac{b^2}{r(2a-r)}\right]}, \text{ from (i) and (ii)}$$

$$=\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{\left(\frac{2ar-r^2-b^2}{r^2}\right)}=\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{\left[\frac{2a}{r}-1-\frac{b^2}{r^2}\right]}$$

$$=\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{\left[\frac{a^2}{b^2}-1-\left(\frac{b}{r}-\frac{a}{b}\right)^2\right]}$$

$$=\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{\left[\left(\frac{a^2-b^2}{b^2}\right)-\left(\frac{b}{r}-\frac{a}{b}\right)^2\right]} \quad \text{(Note)} \quad ...(iii)$$

Also its value is greatest when $\frac{b}{r}-\frac{a}{b}=0$ or $r=\frac{b^2}{a}$ i.e. l, the semi-latus rectum or the point is at the end of the latus rectum or the radius vector to the point is at right angles to the major axis.

And the greatest velocity away from the sun

$$=\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{\left(\frac{a^2-b^2}{b^2}\right)}, \text{ from (iii)}$$

$$=\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{\left[\frac{a^2-a^2(1-e^2)}{a^2(1-e^2)}\right]}, \quad \because \; b^2=a^2(1-e^2)$$

$$=\sqrt{(\mu/a)}\sqrt{[a^2/(1-e^2)]}. \quad ...(iv)$$

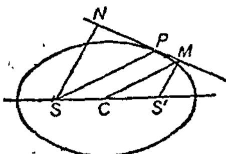
Also the periodic time T is given by

$$T=2\pi a^{3/2}/\sqrt{\mu} \quad \text{or} \quad \sqrt{(\mu/a)}=2\pi a/T \quad ...(v)$$

∴ From (iv) and (v), we have the greatest velocity away from the sun$=\dfrac{2\pi a}{T}\cdot\dfrac{e}{\sqrt{(1-e^2)}}$ Hence proved.

*Ex. 5. **Show that the velocity of a particle moving in an ellipse about a centre in the focus is compounded of two constant velocities, (μ/h) perpendicular to the radius and $(\mu e/h)$ perpendicular to the major axis.** *(Purvanchal 91, 90)*

Solution. Let S and S' be the two foci of the ellipse. Let SN and $S'M$ be perpendiculars from S and S' to the tangent at any point P of the ellipse. Let S be the centre of the force and v be the velocity of the particle at P, then we know that



(Fig. 4)

$h=pv$ or $v=(h/p)$

or $v=(h/SN)=(h/SN\cdot S'M)\cdot S'M$

or $v=(h/b^2)\ S'M,$ ...(i)

∵ $SN\ S'M=b^2$ (Property of ellipse)

∴ The velocity v of the particle at P (which is along the tangent to the ellipse at P) is proportional to $S'M$ and also perpendicular to $S'M$.

Also from $\Delta\ CS'M$, we get $\vec{S'C}+\vec{CM}=\vec{S'M}$

or $$\frac{h}{b^2}\vec{S'C}+\frac{h}{b^2}\vec{CM}=\frac{h}{b^2}\vec{S'M}. \quad ...(ii)$$

∴ From (i) and (ii) we find that velocity v is equivalent to two velocities, one $\dfrac{h}{b^2}\ S'C$ perpendicular to $S'C$ and the other $\dfrac{h}{b^2}\times CM$ perpendicular to CM.

Also as M and N lie on the auxiliary circle of the ellipse so $CM=a$ (property of ellipse). Also $CS'=ae$.

Again we know $h=\sqrt{(\mu l)}=\sqrt{(\mu\cdot b^2/a)}$, ∵ $l=b^2/a$

∴ $h^2=\mu\ (b^2/a)$ or $a/b^2=\mu/h^2$.

Hence velocity v is equivalent to two velocities :—

One $\dfrac{h}{b^2}\cdot S'C=\dfrac{h}{b^2}\ ae=\dfrac{he\mu}{h^2}=\dfrac{\mu e}{h}$ perpendicular to $S'C$ *i.e.* perpendicular to the major axis and the other $\dfrac{h}{b^2}\ CM=\dfrac{h}{b^2}\ a=\dfrac{h\mu}{h^2}=\dfrac{\mu}{h}$ perpendicular to CM *i.e.* perpendicular to radius CP.

*Ex. 6. *If the velocity of the earth at any point of its orbit, assumed circular, were increased by about one half, prove that it would describe a parabola about the sun as focus. Show also that if a body were projected from the earth with a velocity exceeding 11.2 kms per second, it will not return to earth.*

Solution. We know for the circular path the velocity at any point is given by $v^2=\mu[(2/a)-(1/a)]=\mu/a$. ...(i) (Note)

And for the parabolic path velocity v_1 at that point is given by $v_1^2=2\mu/a$.

$\therefore$ From (i) and (ii) we have $v_1^2=2v^2$ or $v_1=v\sqrt{2}$

i.e. $v_1=v+(\sqrt{2}-1)v=v+\frac{1}{2}v$, nearly.

$\therefore$ If the velocity of the earth be increased by about one half it would describe a parabola. Hence proved.

For the second part, if R be the radius of the earth then we know that the acceleration due to gravity on the surface of the earth given by $g=\frac{\text{“}\mu\text{”}}{r^2}=\frac{\mu}{R^2}$ or $\mu=R^2g$

Also for the parabolic path the velocity on the surface of the earth

$=\sqrt{(2\mu/R)}=\sqrt{(2R^2g/R)}=\sqrt{(2Rg)}$

$=\sqrt{[2\times(6400\times1000)\times9.8]}$ m/sec.

$\because$ $R=6400$ kilometres $=6400\times1000$ metres

$=11200$ m/sec. $=11.2$ km/sec.

Also velocity of the earth is 29.3 kms/sec. nearly and if changed to (29.3) $\sqrt{2}$ kms/sec, then as proved above it would describe a parabolic path. Hence if a body were projected from the surface of the earth in the direction of the earth's velocity with a velocity (29.3) $\sqrt{2}$ kms/sec. *i.e.* 41.4 kms/sec. nearly *i.e.* about 12.1 kms/sec. more than the velocity of the earth, it will describe a parabola with sun as its focus. Also the parabola being an open curve the body will not return to the earth.

Ex. 7. *If a planet were suddenly stopped in its orbit supposed circular, show that it would fall into the sun in a time which is $\sqrt{2}/8$ times the period of the planet's revolution.*

Solution. Let S be the centre and a the radius of the circular path. Let the planet be sud... at a point P on the circular path. Then as the ve... has been sud... denly reduced to zero so t... straight line PS (Here S is th... in the ... tion $\mu/(\text{distance})^2$.

$\therefore$ The equati... he ... acceler... ing along PS to ... it is mov-

Integratin...

At P, $r=a$, $v=0$ $\quad\therefore\quad c=-(2\mu/a)$.

So we have $\quad v^2=2\mu\left(\dfrac{1}{r}-\dfrac{1}{a}\right)=2\mu\left(\dfrac{a-r}{ar}\right)$

or $$v=\frac{dr}{dt}=-\sqrt{(2\mu)}\sqrt{\left(\frac{a-r}{ar}\right)},$$

negative sign is due to the fact that r decreases as t increases

or $$\sqrt{\left(\frac{r}{a-r}\right)}\,dr=-\sqrt{\left(\frac{2\mu}{a}\right)}\,dt$$

If T_1 be the time taken by the planet in reaching S from P, then we have $\quad -\sqrt{\left(\dfrac{2\mu}{a}\right)}\displaystyle\int_{t=0}^{T_1}dt=\int_{r=a}^{0}\sqrt{\left(\dfrac{r}{a-r}\right)}\,dr$

or $$-\sqrt{\left(\frac{2\mu}{a}\right)}\,T_1=-\int_{\theta=0}^{\pi/2}\sqrt{\left(\frac{a\cos^2\theta}{a\sin^2\theta}\right)}.2a\cos\theta\sin\theta\,d\theta,$$

putting $r=a\cos^2\theta$

or $$\sqrt{\left(\frac{2\mu}{a}\right)}\,T_1=2a\int_0^{\pi/2}\cos^2\theta\,d\theta=2a\cdot\tfrac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi a}{2}$$

or $$T_1=\pi a^{3/2}/[2\sqrt{(2\mu)}]. \qquad \text{...(i)}$$

Also if T be the periodic time of the planet's revolution, then we have $\quad T_1=2\pi a^{3/2}/\sqrt{\mu}$ $\quad$ (See § 2 Page 1 of this chapter)

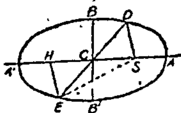
Therefore $\dfrac{T_1}{T}=\dfrac{\pi a^{3/2}}{2\sqrt{(2\mu)}}\div\dfrac{2\pi a^{3/2}}{\sqrt{\mu}}=\dfrac{1}{4\sqrt{2}}=\dfrac{\sqrt{2}}{8}$

or $\quad T_1=\dfrac{\sqrt{2}}{8}\,T.$

Hence proved.

Ex. 8. A particle describes an ellipse as a central orbit about the focus. Prove that the velocity at the end of the minor axis is geometric mean between the velocities at the ends of any diameter.

Solution. Let BB' be the minor-axis of the ellipse with S and H as focii.



(Fig. 5)

Then $SB^2=SC^2+CB^2$

$=(ae)^2+b^2$

$=a^2e^2+a^2(1-e^2)=a^2$

or $\quad SB=a.$

$\therefore$ If V be the velocity of the particle describing the ellipse at B, then

$V^2=\mu\left[\dfrac{2}{r}-\dfrac{1}{a}\right]$

or $\quad V^2=\mu\,[(2/a)-(1/a)],\quad\because\ r=a$ at B

or $\quad V^2=\mu/a. \qquad$ ...(i)

Let V_1 and V_2 be the velocities of the particle at D and E, the ends of a diameter DE of the ellipse.

Join SD and HE, then $\triangle CDS \equiv \triangle CEH$

In particular $DS = HE$

$\therefore \quad SD+SE=HE+SE, \quad \because DS=HE$

$=2a$, the sum of the focal distances of E,

i.e. $\quad DS+SE=2a$ ...(ii)

Also $V_1^2=\mu\left[\frac{2}{SD}-\frac{1}{a}\right]$ and $V_2^2=\mu\left[\frac{2}{SE}-\frac{1}{a}\right]$

$$\therefore \quad V_1^2V_2^2=\mu^2\left[\frac{2}{SD}-\frac{1}{a}\right]\left[\frac{2}{SE}-\frac{1}{a}\right]$$

$$=\mu^2\left[\frac{4}{SD.SE}-\frac{2}{a}\left(\frac{1}{SD}+\frac{1}{SE}\right)+\frac{1}{a^2}\right]$$

$$=\mu^2\left[\frac{4}{SD.SE}-\frac{2}{a}\left(\frac{SE+DS}{SD.SE}\right)+\frac{1}{a^2}\right]$$

$$=\mu^2\left[\frac{4}{DS.SE}-\frac{2}{a}\left(\frac{2a}{DS.SE}\right)+\frac{1}{a^2}\right], \text{ from (ii)}$$

$=\mu^2/a^2=V^4$, from (i) or $V^2=V_1V_2$

i.e. V is the geometric mean of V_1 and V_2. Hence proved.

Ex. 9. Show that an unresisted particle falling to the earth's surface from a great distance would acquire a velocity $\sqrt{(2ga)}$, where a is the radius of the earth. *(Purvanchal 89)*

Solution. As the particle is falling unresisted to the earth's surface from a great distance so the path of the particle is a parabola about the earth's centre as focus. (Note)

Also we know that the ve[illegible] a parabolic p[illegible]).

$\therefore$ Velocity of the p[illegible] it meets the [illegible] earth $=\sqrt{(2\mu/a)}$, $\because r=a$ [illegible] of the eart[illegible]

This shows that the subsequent path of the particle is parabola, since for the parabolic path $(\text{vel.})^2=2\mu/r$ and at B we have proved in Ex. 8 Page 11 that $r=a$. Hence proved.

Ex. 11. Prove that if the velocity of a particle when it is at a distance r from the focus is v in a direction making an angle ϕ with the radius vector, then

$$e^2\mu^2=(v^2r-\mu)^2\sin^2\phi+\mu^2\cos^2\phi.$$

Solution. We know $h=pv$ and $p=r\sin\phi$

$\therefore$ $$h=rv\sin\phi. \quad ...(i)$$

Also $$h=\sqrt{(\mu\times l)}=\sqrt{\left(\mu\times\frac{b^2}{a}\right)}=\sqrt{\left(\mu\times\frac{a^2(1-e^2)}{a}\right)}$$

or $h^2=\mu a(1-e^2)$ or $r^2v^2\sin^2\phi=\mu a(1-e^2)$, from (i)

or $$a=(v^2r^2\sin^2\phi)/[\mu(1-e^2)] \quad ...(ii)$$

Also we know that the velocity v of a particle at a distance r from the focus is given by

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]=\mu\left[\frac{2}{r}-\frac{\mu(1-e)}{v^2r^2\sin^2\phi}\right], \text{ from (ii)}$$

$$=\frac{2\mu}{r}-\frac{\mu^2(1-e^2)}{v^2r^2\sin^2\phi}$$

or $$\left(v^2-\frac{2\mu}{r}\right)v^2r^2\sin^2\phi=-\mu^2(1-e^2)$$

or $$(v^2r-2\mu)v^2r\sin^2\phi=\mu^2e^2-\mu^2$$

or $$\mu^2e^2=(v^2r-2\mu)v^2r\sin^2\phi+\mu^2$$

$$=(v^4r^2-2\cdot\mu v^2r)\sin^2\phi+\mu^2$$

$$=(v^4r^2-2\mu v^2r+\mu^2)\sin^2\phi-\mu^2\sin^2\phi+\mu^2 \quad \text{(Note)}$$

$$=(v^2r-\mu)^2\sin^2\phi-\mu^2(1-\sin^2\phi)$$

or $$\mu^2e^2=(v^2r-\mu)^2\sin^2\phi+\mu^2\cos^2\phi.$$ Hence proved.

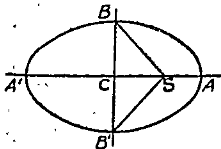
Ex. 12. Prove that the time taken by a earth to travel over half its orbit, remote from the sun, separated by the minor axis is two days more than half the year. The eccentricity of the orbit is 1/60.

Solution. The elliptic path of the earth round the sun is described in one year, so the rate of description of area

$$=\frac{\text{area of the ellipse}}{\text{one year}}$$

$$=\frac{\pi ab}{\text{one year}} \quad ...(i)$$

Now the half of the orbit remote from the sun is arc $BA'B'$ (See adjoining figure 6).



(Fig. 6)

Let V_1 and V_2 be the velocities of the particle at D and E, the ends of a diameter DE of the ellipse.

Join SD and HE, then $\triangle CDS \equiv \triangle CEH$

In particular $DS = HE$

$\therefore \quad SD+SE=HE+SE, \quad \because \; DS=HE$

$=2a$, the sum of the focal distances of E,

i.e. $\quad DS+SE=2a$ ...(ii)

Also $V_1^2=\mu\left[\frac{2}{SD}-\frac{1}{a}\right]$ and $V_2^2=\mu\left[\frac{2}{SE}-\frac{1}{a}\right]$

$$\therefore \quad V_1^2V_2^2=\mu^2\left[\frac{2}{SD}-\frac{1}{a}\right]\left[\frac{2}{SE}-\frac{1}{a}\right]$$

$$=\mu^2\left[\frac{4}{SD.SE}-\frac{2}{a}\left(\frac{1}{SD}+\frac{1}{SE}\right)+\frac{1}{a^2}\right]$$

$$=\mu^2\left[\frac{4}{SD.SE}-\frac{2}{a}\left(\frac{SE+DS}{SD.SE}\right)+\frac{1}{a^2}\right]$$

$$=\mu^2\left[\frac{4}{DS.SE}-\frac{2}{a}\left(\frac{2a}{DS.SE}\right)+\frac{1}{a^2}\right], \text{ from (ii)}$$

$=\mu^2/a^2=V^4$, from (i) or $V^2=V_1V_2$

i.e. $\quad V$ is the geometric mean of V_1 and V_2. Hence proved.

Ex. 9. Show that an unresisted particle falling to the earth's surface from a great distance would acquire a velocity $\sqrt{(2ga)}$, where a is the radius of the earth. *(Purvanchal 89)*

Solution. As the particle is falling unresisted to the earth's surface from a great distance so the path of the particle is a parabola about the earth's centre as focus. (Note)

Also we know that the velocity for a parabolic path $=\sqrt{(2\mu/r)}$.

$\therefore$ Velocity of the particle when it meets the surface of the earth $=\sqrt{(2\mu/a)}$, $\because \; r=a$ on the surface of the earth

$=\sqrt{(2a^2g/a)}$, $\because \; g=\mu/a^2$ at the surface of the earth.

$=\sqrt{(2ag)}$. Hence proved.

Ex 10. A particle describes an ellipse under a force to the focus S. When the particle is at one extremity of the minor axis, its kinetic energy is doubled without any change in the direction of motion. Prove that the particle proceeds to describe a parabola.

Solution. As in Example 8 Page 11 we can prove that the velocity V of the particle at B, one extremity of the minor axis, is given by $\quad V^2=\mu/a$.

Let V_1 be the velocity of the particle when its kinetic energy is doubled, then $\frac{1}{2}mV_1^2=2\left(\frac{1}{2}mV^2\right)$, where m is the mass of the particle.

or $\quad V_1^2=2V^2=2\mu/a$, from (i).

This shows that the subsequent path of the particle is parabola, since for the parabolic path $(\text{vel.})^2=2\mu/r$ and at B we have proved in Ex. 8 Page 11 that $r=a$. Hence proved.

Ex. 11. Prove that if the velocity of a particle when it is at a distance r from the focus is v in a direction making an angle ϕ with the radius vector, then

$$e^2\mu^2=(v^2r-\mu)^2 \sin^2 \phi+\mu^2 \cos^2 \phi.$$

Solution. We know $h=pv$ and $p=r \sin \phi$

$\therefore$ $h=rv \sin \phi.$ ...(i)

Also $h=\sqrt{(\mu\times l)}=\sqrt{\left(\mu\times\frac{b^2}{a}\right)}=\sqrt{\left(\mu\times\frac{a^2(1-e^2)}{a}\right)}$

or $h^2=\mu a(1-e^2)$ or $r^2v^2 \sin^2\phi=\mu a(1-e^2)$, from (i)

or $a=(v^2r^2 \sin^2\phi)/[\mu\ (1-e^2)]$ ...(ii)

Also we know that the velocity v of a particle at a distance r from the focus is given by

$$v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]=\mu\left[\frac{2}{r}-\frac{\mu(1-e)}{v^2r^2 \sin^2\phi}\right], \text{ from (ii)}$$

$$=\frac{2\mu}{r}-\frac{\mu^2(1-e^2)}{v^2r^2 \sin^2\phi}$$

or $\left(v^2-\frac{2\mu}{r}\right)v^2r^2 \sin^2\phi=-\mu^2(1-e^2)$

or $(v^2r-2\mu)\,v^2r \sin^2\phi=\mu^2e^2-\mu^2$

or $\mu^2e^2=(v^2r-2\mu)\,v^2r \sin^2\phi+\mu^2$

$=(v^4r^2-2\cdot\mu\, v^2r) \sin^2\phi+\mu^2$

$=(v^4r^2-2\mu\, v^2r+\mu^2) \sin^2\phi-\mu^2 \sin^2\phi+\mu^2$ (Note)

$=(v^2r-\mu)^2 \sin^2\phi-\mu^2(1-\sin^2\phi)$

or $\mu^2e^2=(v^2r-\mu)^2 \sin^2\phi+\mu^2 \cos^2\phi.$ Hence proved.

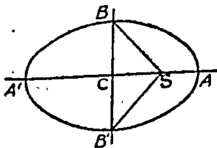
Ex. 12. Prove that the time taken by a earth to travel over half its orbit, remote from the sun, separated by the minor axis is two days more than half the year. The eccentricity of the orbit is 1/60.

Solution. The elliptic path of the earth round the sun is described in one year, so the rate of description of area

$=\frac{\text{area of the ellipse}}{\text{one year}}$

$=\frac{\pi ab}{\text{one year}}$ ...(i)

Now the half of the orbit remote from the sun is arc $BA'B'$ (See adjoining figure 6).



(Fig. 6)

Ex. 16. A comet is ... about the sun at ... ity suddenly be- ... Show that the ... $\sqrt{(1-2n^2+2n^4)}$

... is

$$\frac{l}{1-n^2},$$

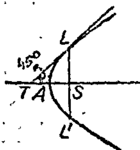
the latus rectum of the parabolic path.

Let v be the velocity of the point L, one end of the latus ... its path being parabolic $v^2=(2\mu/l)$. ...(i)

... altered velocity, then it is n ... or $V=nv$

... v, $\because n<1$

... from (i), we get $V^2<\frac{2\mu}{l}$ ...(ii) (Fig. 7)



... (ii) we conclude that the subsequent path is ... the eccentricity and $2a$ be the major axis of the ...

... have $V^2=\mu\left[\frac{2}{l}-\frac{1}{a}\right]$, ...(iii)

... path is ellipse.

... and (iii) we have $\frac{V^2}{v^2}=\frac{\mu\,[(2/l)-(1/a)]}{2\mu/l}=1-\frac{l}{2a}$

$n^2=1-\frac{l}{2a}$, $\because V=nv$

$\frac{l}{..}=1-n^2$ or $2a=\frac{l}{1-n^2}$. Hence proved.

... at L to the parabola can be proved to be in- ... of 45° to the axis of the parabola. [This can be ... value of dy/dx for the parabola $y^2=4ax$ at ... the figure it is evident that $\angle TLS=45°$.

... of perpendicular from the focus S to the ... $45°=l\sin 45°$.

$pv=\sqrt{(\mu\times l)}$, when l is the semi-latus rectum.

... L on the subsequent ellipse, we have

... $\sqrt{(\mu\times l')}$, where l' is the semi-latus rectum of the ellipse

$=\mu\,(b^2/a)$, $\because l'=b^2/a$ for ellipse

... $-e^2)$, $\because b^2=a^2(1-e^2)$

$HP^2\,\omega = HM \cdot v = h$, where ω is the angular velocity about H. (Note)

or $\quad HP^2 \cdot \omega = HM . \dfrac{h}{SN}$, $\because \; v = \dfrac{h}{p} = \dfrac{h}{SN}$ (see figure 3 Page 8)

or $$\omega = \frac{h\,HM}{SN \cdot HP^2} = \frac{h\,HM}{SN \cdot HP \cdot HP} \quad \text{...(ii)}$$

Also from similar triangles SPN and PHM we have

$$\frac{SN}{SP} = \frac{HM}{HP}$$

$\therefore$ From (ii), we have $\cdot\; \omega = \dfrac{h}{SN} \cdot \dfrac{SN}{SP} \cdot \dfrac{1}{HP}$ (Note)

or $$\omega = \frac{h}{SP \cdot HP} = \frac{h}{r\,(2a-r)}, \text{ where } r = SP. \quad \text{...(iii)}$$

Now let the eccentric angle of the point P be ϕ_1, then P is $(a \cos \phi_1, b \sin \phi_1)$ and S is $(ae, 0)$.

$\therefore \; r^2 = SP^2 = (ae - a \cos \phi_1)^2 + (0 - b \sin \phi_1)^2$

$= a^2e^2 + a^2 \cos^2 \phi_1 - 2a^2 e \cos \phi_1 + b^2 \sin^2 \phi_1$

$= a^2e^2 + a^2 \cos^2 \phi_1 - 2a^2 e \cos \phi_1 + (a^2 - a^2e^2) \sin^2 \phi_1$

$= a^2e^2 (1 - \sin^2 \phi_1) + a^2 (\cos^2 \phi_1 + \sin^2 \phi_1) - 2a^2 e \cos \phi_1$

$= a^2e^2 \cos^2 \phi_1 + a^2 - 2a \cdot ae \cos \phi_1 = (a - ae \cos \phi_1)^2$

or $\quad r = SP = a - ae \cos \phi_1$.

$\therefore$ From (iii) we get $\omega = \dfrac{h}{(a - ae \cos \phi_1)(a + ae \cos \phi_1)}$ (Note)

$$= \frac{h}{a^2 - a^2e^2 \cos^2 \phi_1} = \frac{h}{a^2 - (a^2 - b^2) \cos^2 \phi_1}$$

or $$\omega = h/(a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1) \quad \text{...(iv)}$$

Also the equation of the normal PK to the ellipse at the point $P\,(a \cos \phi_1, b \sin \phi_1)$ is $ax \sec \phi_1 - by \operatorname{cosec} \phi_1 = a^2 - b^2$.

If this normal meets the x-axis (*i.e.* the major axis) at K, then the coordinates of K are $\left(\dfrac{a^2 - b^2}{a} \cos \phi_1, 0\right)$.

$\therefore \; (\text{Normal})^2 = (PK)^2$

$$= \left(a \cos \phi_1 - \frac{a^2 - b^2}{a} \cos \phi_1\right)^2 + (b \sin \phi_1 - 0)^2$$

$$= \left(\frac{b^2}{a^2} \cos \phi_1\right)^2 + b^2 \sin^2 \phi_1$$

$$= (b^2/a^2)\,(b^2 \cos^2 \phi_1 + a^2 \sin^2 \phi_1) \quad \text{...(v)}$$

$\therefore$ From (iv) and (v), we get

$$\omega = \frac{hb^2}{a^2 \cdot PK^2} \text{ or } \omega \propto \frac{1}{PK^2}.$$

Hence proved.

Ex. 16. A comet is moving in a parabola about the sun at focus when at the end of its latus rectum its velocity suddenly becomes altered in the ratio of $n:1$, where $n<1$. Show that the comet will describe an ellipse whose eccentricity is $\sqrt{(1-2n^2+2n^4)}$ and whose major axis is

$$\frac{l}{1-n^2},$$

where $2l$ was the latus rectum of the parabolic path.

Solution. Let v be the velocity of the comet at the point L, one end of the latus rectum. Originally its path being parabolic we have

$$v^2=(2\mu/l). \quad \ldots(i)$$

Let V be the altered velocity, then it is given that $\frac{V}{v}=\frac{n}{1}$ or $V=nv$

or $V<v$, $\because$ $n<1$

$\therefore$ $V^2<v^2$ or from (i), we get $V^2<\frac{2\mu}{l}$ ...(ii) (Fig. 7)

Hence from (ii) we conclude that the subsequent path is ellipse. Let e be the eccentricity and $2a$ be the major axis of the ellipse.

$\therefore$ At L we have $V^2=\mu\left[\frac{2}{l}-\frac{1}{a}\right]$, ...(iii)

since the subsequent path is ellipse.

$\therefore$ From (i) and (iii) we have $\frac{V^2}{v^2}=\frac{\mu\{(2/l)-(1/a)\}}{2\mu/l}=1-\frac{l}{2a}$

or, $n^2=1-\frac{l}{2a}$, $\because$ $V=nv$

or $\frac{l}{2a}=1-n^2$ or $2a=\frac{l}{1-n^2}$. Hence proved.

Also the tangent at L to the parabola can be proved to be inclined at an angle of 45° to the axis of the parabola. [This can be done by calculating the value of dy/dx for the parabola $y^2=4ax$ at $L(a, 2a)$]. $\therefore$ From the figure it is evident that $\angle TLS=45°$.

$\therefore$ The length of perpendicular from the focus S to the tangent at $P=SL\sin 45°=l\sin 45°$.

Also we know $pv=\sqrt{(\mu\times l)}$, when l is the semi-latus rectum.

$\therefore$ At the point L on the subsequent ellipse, we have

$(l\sin 45°)\,V=\sqrt{(\mu\times l')}$, where l' is the semi-latus rectum of the ellipse

or $(l\sin 45°)^2.V^2=\mu l'=\mu\,(b^2/a)$, $\because$ $l'=b^2/a$ for ellipse

or $V^2 l^2\,(1/2)=\mu a\,(1-e^2)$, $\because$ $b^2=a^2\,(1-e^2)$

$HP^2\,\omega = HM \cdot v = h$, where ω is the angular velocity about H. (Note)

or $HP^2 \cdot \omega = HM . \dfrac{h}{SN}$, $\because \quad v = \dfrac{h}{p} = \dfrac{h}{SN}$ (see figure 3 Page 8)

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Also from similar triangles SPN and PHM we have

$$\frac{SN}{SP} = \frac{HM}{HP}$$

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or $$\omega = \frac{h}{SP \cdot HP} = \frac{h}{r\,(2a-r)}, \text{ where } r = SP. \quad ...(iii)$$

Now let the eccentric angle of the point P be ϕ_1, then P is $(a \cos \phi_1, b \sin \phi_1)$ and S is $(ae, 0)$.

$$\begin{aligned}\therefore \quad r^2 = SP^2 &= (ae - a\cos\phi_1)^2 + (0 - b\sin\phi_1)^2 \\ &= a^2e^2 + a^2\cos^2\phi_1 - 2a^2 e\cos\phi_1 + b^2\sin^2\phi_1 \\ &= a^2e^2 + a^2\cos^2\phi_1 - 2a^2 e\cos\phi_1 + (a^2 - a^2e^2)\sin^2\phi_1 \\ &= a^2e^2\,(1 - \sin^2\phi_1) + a^2(\cos^2\phi_1 + \sin^2\phi_1) - 2a^2e\cos\phi_1 \\ &= a^2e^2\cos^2\phi_1 + a^2 - 2a \cdot ae\cos\phi_1 = (a - ae\cos\phi_1)^2\end{aligned}$$

or $r = SP = a - ae\cos\phi_1$.

$\therefore$ From (iii) we get $\omega = \dfrac{h}{(a - ae\cos\phi_1)(a + ae\cos\phi_1)}$ (Note)

$$= \frac{h}{a^2 - a^2e^2\cos^2\phi_1} = \frac{h}{a^2 - (a^2 - b^2)\cos^2\phi_1}$$

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If this normal meets the x-axis (*i.e.* the major axis) at K, then the coordinates of K are $\left(\dfrac{a^2 - b^2}{a}\cos\phi_1, 0\right)$.

$$\begin{aligned}\therefore \quad (\text{Normal})^2 = (PK)^2 \\ &= \left(a\cos\phi_1 - \frac{a^2 - b^2}{a}\cos\phi_1\right)^2 + (b\sin\phi_1 - 0)^2 \\ &= \left(\frac{b^2}{a^2}\cos\phi_1\right)^2 + b^2\sin^2\phi_1 \\ &= (b^2/a^2)\,(b^2\cos^2\phi_1 + a^2\sin^2\phi_1) \quad ...(v)\end{aligned}$$

$\therefore$ From (iv) and (v), we get

$$\omega = \frac{hb^2}{a^2 \cdot PK^2} \text{ or } \omega \propto \frac{1}{PK^2}.$$

Hence proved.

Ex. 16. A comet is moving in a parabola about the sun at focus when at the end of its latus rectum its velocity suddenly becomes altered in the ratio of $n:1$, where $n < 1$. Show that the comet will describe an ellipse whose eccentricity is $\sqrt{(1-2n^2+2n^4)}$ and whose major axis is

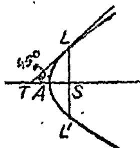
$$\frac{l}{1-n^2},$$

where $2l$ was the latus rectum of the parabolic path.

Solution. Let v be the velocity of the comet at the point L, one end of the latus rectum. Originally its path being parabolic we have $v^2=(2\mu/l)$. ...(i)

Let V be the altered velocity, then it is given that $\frac{V}{v}=\frac{n}{1}$ or $V=nv$

or $V < v$, $\because n < 1$



$\therefore V^2<v^2$ or from (i), we get $V^2 < \frac{2\mu}{l}$ ...(ii) (Fig. 7)

Hence from (ii) we conclude that the subsequent path is ellipse. Let e be the eccentricity and $2a$ be the major axis of the ellipse.

$\therefore$ At L we have $V^2=\mu\left[\frac{2}{l}-\frac{1}{a}\right]$, ...(iii)

since the subsequent path is ellipse.

$\therefore$ From (i) and (iii) we have $\frac{V^2}{v^2}=\frac{\mu\{(2/l)-(1/a)\}}{2\mu/l}=1-\frac{l}{2a}$

or, $n^2=1-\frac{l}{2a}$, $\because V=nv$

or $\frac{l}{2a}=1-n^2$ or $2a=\frac{l}{1-n^2}$. Hence proved.

Also the tangent at L to the parabola can be proved to be inclined at an angle of 45° to the axis of the parabola. [This can be done by calculating the value of dy/dx for the parabola $y^2=4ax$ at $L(a, 2a)$]. $\therefore$ From the figure it is evident that $\angle TLS=45°$.

$\therefore$ The length of perpendicular from the focus S to the tangent at $P=SL \sin 45°=l \sin 45°$.

Also we know $pv=\sqrt{(\mu \times l)}$, when l is the semi-latus rectum.

$\therefore$ At the point L on the subsequent ellipse, we have

$(l \sin 45°)\, V=\sqrt{(\mu \times l')}$, where l' is the semi-latus rectum of the ellipse

or $(l \sin 45°)^2 . V^2=\mu l'=\mu\,(b^2/a)$, $\because l'=b^2/a$ for ellipse

or $V^2 l^2 (1/2)=\mu a\,(1-e^2)$, $\because b^2=a^2(1-e^2)$

or $\frac{1}{2}n^4v^2l^2=\mu a(1-e^2)$, $\because$ $V=nv$

or $\frac{1}{2}n^4 \cdot (2\mu/l) \cdot l^2=\mu a(1-e^2)$, from (i)

or $n^4 l=a(1-e^2)$ or $n^4 l=\frac{1}{2}\frac{l(1-e^2)}{1-n^2}$, $\because$ $2a=\frac{l}{1-n^2}$

or $2n^4(1-n^2)=(1-e^2)$ or $e^2=1-2n^4+2n^6$

or $e=\sqrt{(1-2n^4+2n^6)}$. Hence proved.

Ex. 17. *A particle is describing a parabola under a force to the focus. It meets and coalesces with another particle of n times mass which was at rest before the impact. Show that the composite body will describe an ellipse whose eccentricity is given by*

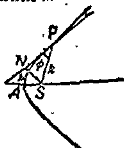
$$1-e^2=\frac{4n(n+2)}{(n+1)^4}\cos^2\tfrac{1}{2}\theta,$$

where θ is measured from the apse in the parabola.

Solution. Before impact let v be the velocity of the particle of mass m (say) at the point P on the parabolic path, where $SP=r$ and S is the focus of the parabola.

$$\therefore \quad v^2=\frac{2\mu}{r} \qquad \text{...(i)}$$

After impact let V be the velocity of the particle at P.



(Fig. 8)

Also we know that the total momentum after impact = total momentum before impact, so we have

$(m+nm)V=mv$ or $V=v/(1+n)$

or $V^2=v^2/(1+n)^2$ or $V^2 < v^2$

or $V^2 < \frac{2\mu}{r}$, from (i)

$\therefore$ The combined body after impact will move in an ellipse. Let e be the eccentricity and $2a'$ be the major axis of this elliptic path. Then for the subsequent elliptic path, we have $V^2=\mu[(2/r)-(1/a')]$

or $\frac{v^2}{(1+n)^2}=\mu\left[\frac{2}{r}-\frac{1}{a'}\right]$, $\because$ $V^2=v^2/(1+n)^2$

or $\frac{1}{(1+n)^2}\cdot\frac{2\mu}{r}=\mu\left[\frac{2}{r}-\frac{1}{a'}\right]$, from (i)

or $\frac{1}{a'}=\frac{2}{r}\left[1-\frac{1}{(1+n)^2}\right]=\frac{2}{r}\left[\frac{n^2+2n}{(1+n)^2}\right]$

or $a'=r(1+n)^2/\{2n(n+2)\}$ ...(ii)

Also from "$pv=\sqrt{(\mu l)}$", we get

or $pV=\sqrt{[\mu a'(1-e^2)]}$, $\because$ $l=b'^2/a'=a'(1-e^2)$

or $p^2V^2=\mu a'(1-e^2)$

or $$ar\cdot\frac{v^2}{(1+n)^2}=\frac{\mu r(1+n)^2}{2n(n+2)}(1-e^2),$$ from (ii)

and the pedal equation of parabola is $p^2=ar$

or $$ar\cdot\frac{2\mu}{r}\frac{1}{(1+n)^2}=\frac{\mu r(1+n)^2(1-e^2)}{2n(n+2)},$$ from (i)

or $$4an(n+2)=r(1+n)^4(1-e^2)$$

or $$1-e^2=4an(n+2)/[r(1+n)^4]$$

Also the polar equation of parabola referred to the focus S as pole is $(2a/r)=1+\cos\theta$, $\because$ $l=2a$

or $$r=\frac{2a}{1+\cos\theta}=\frac{2a}{2\cos^2\frac{1}{2}\theta}=\frac{a}{\cos^2\frac{1}{2}\theta}\quad\text{or}\quad\frac{a}{r}=\cos^2\tfrac{1}{2}\theta$$

$\therefore$ From (iii), we get $$1-e^2=\frac{4n(n+2)}{(1+n)^4}\cdot\cos^2\tfrac{1}{2}\theta.$$

Hence proved.

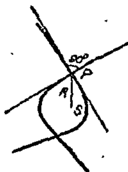
***Ex. 18.** *Two particles of masses m_1 and m_2 moving in coplanar parabolas round the sun, collide at right angles and coalesce, when their common distance from the sun is R, show that the subsequent path of the combined particles is an ellipse of major axis*

$$(m_1+m_2)^2R/(2m_1m_2).$$

Solution. Let v_1 and v_2 be the velocities of the particles of masses m_1 and m_2 respectively before collision at the common point P whose distance from the sun S is R (given).

As the particles are describing parabolas with S as focus, so at P, we have

$$v_1^2=(2\mu/R)=v_2^2 \quad ...(i)$$



(Fig 9)

Let V be the velocity of the common body, of mass (m_1+m_2) into which the particles coalesce after collision, in the direction making an angle θ, say with that of v_1.

(See Figure 10 below).

Then from the principle of conservation of momentum, in the direction of v_1 and perpendicular to it, we have

$$m_1v_1=(m_1+m_2)V\cos\theta \quad ...(ii)$$

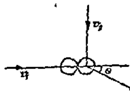
$$m_2v_2=(m_1+m_2)V\sin\theta \quad ...(iii)$$

Squaring and adding (ii) and (iii), we get

$$(m_1+m_2)^2V^2=m_1^2v_1^2+m_2^2v_2^2$$

or $$V^2=\frac{m_1^2v_1^2+m_2^2v_2^2}{(m_1+m_2)^2}=\frac{(m_1^2+m_2^2)v_1^2}{(m_1+m_2)^2}$$

$\because$ $v_2^2=v_1^2$, from (i)



(Fig. 10)

or $$V^2=\frac{(m_1^2+m_2^2)v_1^2}{(m_1+m_2)^2} \quad ...(iv)$$

$\therefore$ $V^2<v_1^2<(\mu/R)$, hence the path traced out by the

combined body after collision is an ellipse and for this elliptic path at P, we have $v^2=\mu\left(\frac{2}{R}-\frac{1}{a}\right)$,

where $2a$ is the required length of major axis.

or $\frac{(m_1^2+m_2^2)v_1^2}{(m_1+m_2)^2}=\mu\left(\frac{2}{R}-\frac{1}{a}\right)$, from (iv)

or $\frac{(m_1^2+m_2^2)2\mu}{(m_1+m_2)^2 R}=\mu\left(\frac{2}{R}-\frac{1}{a}\right)$, from (i)

or $$\frac{1}{a}=\frac{2}{R}\left[1-\frac{m_1^2+m_2^2}{(m_1+m_2)^2}\right]=\frac{2}{R}\left[\frac{2m_1 m_2}{(m_1+m_2)^2}\right]$$

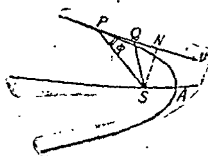
or $$2a=\frac{(m_1+m_2)^2 R}{m_1 m_2}$$

Hence proved.

Ex. 19. A particle is describing a parabolic orbit (latus rectum $4a$) about a centre of force (μ) in the focus and on its arriving at a distance r from the focus moving towards the vertex the centre of the force ceases to act for a certain time T. When the force begins again to operate, prove that the new orbit will be an ellipse, parabola or hyperbola, according, as

$$T <=> 2r\sqrt{\left[\frac{r-a}{2\mu}\right]}.$$

Solution. Let the centre of force cease to act when the particle is at P and the particle is moving towards the vertex A. Let v be the velocity of the particle at P before the centre of force ceases to act, then the path being parabolic we have



(Fig. 11)

$v^2=(2\mu/r)$. ...(i)

Let the particle be at Q after time T as given in the problem.

Then $PQ=vT$. ...(ii)

Also the velocity of the particle at Q is v.

$\therefore$ The particle will be moving in an ellipse, parabola or hyperbola according as $v^2 <=> 2\mu/SQ$

or $\frac{2\mu}{r} <=> \frac{2\mu}{SQ}$, from (i)

or $SQ <=> r$ or $SQ^2 <=> r^2$ ...(iii)

From S draw SN perpendicular to the tangent at P. Then in ΔSNP we have $\sin^2\phi=\left(\frac{SN}{SP}\right)^2=\frac{AS.SP}{SP^2}$, $\because SN^2=AS.SP$.

or $$\sin^2 \phi = \frac{AS}{SP} = \frac{a}{r} \quad \text{or} \quad \sin \phi = \sqrt{\left(\frac{a}{r}\right)} \qquad \text{...(iv)}$$

Also from ΔSQP we have $\cos \phi = \dfrac{SP^2 + PQ^2 - SQ^2}{2SP \cdot PQ}$

or $$2SP.PQ \cos \phi = SP^2 + PQ^2 - SQ^2$$

or $$2r\, vT \sqrt{(1-\sin^2 \phi)} = r^2 + v^2T^2 - SQ^2, \text{ from (ii)}$$

or $$SQ^2 = r^2 + v^2T^2 - 2r \cdot vT \sqrt{\left[1 - \frac{a}{r}\right]}.$$

$\therefore$ From (iii) the particle will describe an ellipse, parabola or hyperbola according as

$$r^2 + v^2T^2 - 2r\, vT \sqrt{\left[1 - \frac{a}{r}\right]} <=> r^2$$

or $$v^2T^2 <=> 2r\, vT \sqrt{\left[1 - \frac{a}{r}\right]}$$

or $$vT <=> 2\sqrt{(r^2 - ar)}$$

or $$\sqrt{\left[\frac{2\mu}{r}\right]} T <=> 2\sqrt{(r^2 - ar)}, \text{ from (i)}$$

or $$T <=> 2r \sqrt{\left[\frac{r-a}{2\mu}\right]}.$$

Hence proved.

Ex. 20. A body is describing an ellipse of eccentricity e under the action of a force tending to a focus and when at the nearer apse the centre of force is transferred to the other focus. Prove that the eccentricity of the new orbit is $e\,(3+e)/(1-e)$.

Solution. Refer figure 5, Page 11.

Let v be the velocity of the body at A, the nearest apse when the centre of force is S. Then we have

$$v^2 = \mu \left[\frac{2}{SA} - \frac{1}{a}\right] = \mu \left[\frac{2}{(a-ae)} - \frac{1}{a}\right], \because SA = CA - CS = a - ae$$

or $$v^2 = \mu\,(1+e)/(a-ae)$$

Now the centre of force is transferred to the other focus H, without any sudden change in the velocity of the body at A, so we have $v^2 = \mu \left[\dfrac{2}{HA} - \dfrac{1}{a'}\right]$, where $2a'$ is the major axis of the new orbit

or $$v^2 = \mu \left[\frac{2}{a+ae} - \frac{1}{a'}\right] \text{ or } \frac{\mu\,(1+e)}{a-ae} = \mu \left[\frac{2}{a+ae} - \frac{1}{a'}\right], \text{ from (i)}$$

or $$\frac{1+e}{a\,(1-e)} = \frac{2}{a\,(1+e)} - \frac{1}{a'} \qquad \text{...(ii)}$$

Also for the new orbit, H is the focus and A the nearer apse, so we have $AH = a' - a'e'$, where e' is the eccentricity of the new orbit or $a + ae = a' - a'e'$, $\because$ for the old orbit $AH = a + ae$

or $a\,(1+e) = a'\,(1-e')$ or $a' = a\,(1+e)/(1-e')$.

$\therefore$ From (ii), we get $\dfrac{1+e}{a(1-e)}=\dfrac{2}{a(1+e)}-\dfrac{(1-e')}{a(1+e)}$

or $$\frac{(1+e)^2}{(1-e)}=2-(1-e')=1+e'$$

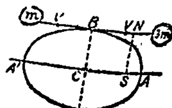
or $$e'=\frac{(1+e)^2}{(1-e)}-1=\frac{e^2+3e}{(1-e)}=\frac{e(3+e)}{(1-e)}$$

Hence proved.

Ex. 21. Two inelastic particles of masses $3m$ and m are describing the same ellipse of eccentricity e in opposite directions under a force to the focus. They collide and coalesce at an extremity of the minor axis. Prove that eccentricity of the new orbit is $\frac{1}{4}\sqrt{(9+7e^2)}$.

Solution. Let the particles collide at B, one extremity of the minor axis when both are moving with the same velocity V.

Then the path being elliptic, we have $V^2=\mu\,[(2/SB)-(1/a)]$. ...(i)



(Fig. 12)

Now from the figure it is evident that $SB^2=SC^2+CB^2=(ae)^2+b^2$

$$=a^2e^2+a^2(1-e^2)$$

or $SB^2=a^2$ or $SB=a$.

$\therefore$ From (i) we have

$$V^2=\mu\,[(2/a)-(1/a)]=(\mu/a). \quad ...(ii)$$

Let v_1 be the velocity of the combined body into which the particles coalesce after impact. Then from principle of conservation of momentum viz.

the total momentum before impact = total momentum after impact, we have $3mV+m(-V)=(3m+m)v_1$

or $2V=4v_1$ or $v_1=\frac{1}{2}V$ or $v_1^2=\frac{1}{4}V^2=(\mu/4a)$, from (ii)

so $v_1^2=(\mu/4a)<(2\mu/a)$. (Note)

After collision at B the combined body will move in an elliptic path and for the new path, we get

$$v_1^2=\mu\,[(2/SB)-(1/a')],$$

where $2a'$ is the major axis of the new elliptic path.

or $$\frac{\mu}{4a}=\mu\left[\frac{2}{a}-\frac{1}{a'}\right], \quad \because SB=a \text{ and } v_1^2=\frac{\mu}{4a}$$

or $$\frac{1}{a'}=\frac{2}{a}-\frac{1}{4a}=\frac{7}{4a} \text{ or } a'=\frac{4}{7}a. \quad ...(iii)$$

From S draw SN perpendicular to the tangent at B, then $SN=CB=b$.

Also we know "$vp=h=\sqrt{(\mu\times l)}$".

$\therefore$ At B, we have $v_1\cdot SN=\sqrt{\left[\mu\times\dfrac{(b')^2}{a'}\right]}, \quad \because l=\dfrac{b^2}{a}$

or $v_1^2 b^2 = \frac{\mu a'^2 (1-e'^2)}{a'}$, $\because$ $SN = b$ and $(b')^2 = a'^2 (1-e'^2)$

or $\frac{\mu}{4a} a^2 (1-e^2) = \mu a' (1-e'^2) = \frac{4a\mu}{7} (1-e'^2)$, from (iii)

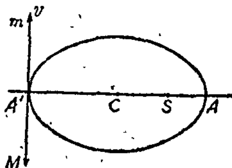
or $(1-e^2) = (16/7)(1-e'^2)$ or $e'^2 = 1-(7/16)(1-e^2) = (1/16)(9+7e^2)$

or $e' = \frac{1}{4}\sqrt{(9+7e^2)}$. Hence proved.

Ex. 22. A planet of mass M and periodic time T, when at its greatest distance from the sun; comes into collision with a meteor of mass m moving in the same orbit in the opposite direction with velocity v; if m/M be small, show that the major axis of the planet's path is reduced by $\frac{4m}{M} \cdot \frac{vT}{\pi} \sqrt{\left[\frac{1-e}{1+e}\right]}$.

Solution. Before coming into collision v is the velocity of the planet and the meteor at A', the point which is at the greatest distance from the sun at S.

(As the orbits of the planet and the meteor are the same so their velocities are also equal to A').



(Fig. 13)

$\therefore$ $v^2 = \mu \left[\frac{2}{r} - \frac{1}{a}\right]$ at A'

where $r = SC + CA' = ae + a$

or $$v^2 = \mu \left[\frac{2}{a+ae} - \frac{1}{a}\right] = \frac{\mu}{a}\left[\frac{1-e}{1+e}\right] \quad \text{.. (i)}$$

Let v_2 be the velocity of the combined body after collision. Then as the momentum after impact = the momentum before impact, so we have $(m+M) v_2 = Mv - mv$

or $$v_2 = \left(\frac{M-m}{m+M}\right) v = \left(1 - \frac{m}{M}\right)\left(1 + \frac{m}{M}\right)^{-1} v$$

$$= \left(1 - \frac{m}{M}\right)\left(1 - \frac{m}{M}\right) v, \text{ as } \frac{m}{M} \text{ is small}$$

$$= \left(1 - \frac{m}{M} - \frac{m}{M} + \frac{m^2}{M^2}\right) v = \left(1 - \frac{2m}{M}\right) v, \text{ neglecting } \frac{m^2}{M^2}$$

i.e. $$v_2 = [1-(2m/M)]\, v. \quad \text{...(ii)}$$

From this result we find that $v_2 < v$ or $v_2^2 < v^2$

or $$v_2^2 < \mu\left(\frac{2}{a+ae} - \frac{1}{a}\right), \text{ from (i)}$$

or $$v_2^2 < \frac{2\mu}{a+ae} \text{ i.e. } v_2^2 < \frac{2\mu}{SA'}$$

$\therefore$ After collision, the path of the combined body is elliptic.

Let $2a'$ be the major axis of this ellipse. So for this new elliptic path at A', we have $v_2^2=\mu\left[\frac{2}{a+ae}-\frac{1}{a'}\right]$

or $\left(1-\frac{2m}{M}\right)^2 v^2=\mu\left[\frac{2}{a+ae}-\frac{1}{a'}\right]$, from (ii)

or $\left(1-\frac{4m}{M}\right) v^2=\mu\left[\frac{2}{a+ae}-\frac{1}{a'}\right]$, neglecting $\frac{m^2}{M^2}$ on the left

or $\left(1-\frac{4m}{M}\right)\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)=\mu\left[\frac{2}{a+ae}-\frac{1}{a'}\right]$, from (i)

or $\frac{1}{a'}=\frac{2}{a+ae}-\left(1-\frac{4m}{M}\right)\cdot\frac{1}{a}\cdot\left(\frac{1-e}{1+e}\right)$

or $\frac{1}{a'}=\frac{1}{a}\left[\frac{2}{1+e}-\left(1-\frac{4m}{M}\right)\left(\frac{1-e}{1+e}\right)\right]$

$=\frac{1}{a}\left[\frac{2}{1+e}-\frac{1-e}{1+e}+\frac{4m}{M}\left(\frac{1-e}{1+e}\right)\right]$

$=\frac{1}{a}\left[1+\frac{4m}{M}\left(\frac{1-e}{1+e}\right)\right]$, after simplifications

or $a'=a\left[1+\frac{4m}{M}\left(\frac{1-e}{1+e}\right)\right]^{-1}=a\left[1-\frac{4m}{M}\left(\frac{1-e}{1+e}\right)\right]$ nearly

or $2a'-2a=-\frac{4m}{M}\left(\frac{1-e}{1+e}\right)\cdot 2a$

$\therefore$ Major axis of the subsequent elliptic path is reduced by $4\,(m/M)\,[(1-e)/(1+e)]\cdot 2a$ ...(iii)

Also the periodic time of the planet $=T=2\pi a^{3/2}/\sqrt{\mu}$.

$\therefore$ $vT=\left\{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)\right\}\cdot\frac{2\pi a^{3/2}}{\sqrt{\mu}}$, from (i)

or $vT=2\pi a\sqrt{\left(\frac{1-e}{1+e}\right)}$ or $2a=\frac{vT}{\pi}\cdot\sqrt{\left(\frac{1+e}{1-e}\right)}$.

$\therefore$ From (iii) the major axis of the subsequent path is reduced by $\frac{4m}{M}\left(\frac{1-e}{1+e}\right)\cdot\frac{vT}{\pi}\sqrt{\left(\frac{1+e}{1-e}\right)}=\frac{4MvT}{M\pi}\sqrt{\left(\frac{1-e}{1+e}\right)}$

Ex. 23. A comet describing a parabola about the sun when nearest to it suddenly breaks up without gain or loss of kinetic energy into two equal portions one of which describes a circle, prove that the other will describe a hyperbola of eccentricity 2.

Solution. The comet is nearest to the sun when it is at the vertex of the parabolic path and if V be its velocity at this point, then $V^2=(2\mu/a)$, ...(i)

since the distance of the vertex from the focus $=a$, where $4a$ is the latus rectum of the parabola.

Let v_1 and v_2 be the velocities of the two equal portions into which the comet breaks up and let v_1 be the velocity of the portion describing the circular path.

Then at the vertex, we have

$$v_1^2/a=\mu/a^2 \qquad \text{(Note)}$$

or $$v_1^2=\mu/a. \qquad \text{...(ii)}$$

Also there being no loss or gain of kinetic energy, we have

$\frac{1}{2}mv_1^2+\frac{1}{2}mv_2^2=\frac{1}{2}(m+m)V^2$, m is the mass of each portion

or $$m(\mu/a)+mv_2^2=2m(2\mu/a)$$

or $$v_2^2=2\mu/a \qquad \text{...(iii)}$$

Since $v_2^2=\dfrac{3\mu}{a}>\dfrac{2\mu}{a}$, so this portion whose velocity is v_2 (after the comet breaks into two parts) will describe a hyperbola.

$\therefore$ If $2a_1$ be the length of the transverse-axis of the hyperbola, then $v_2^2=\mu\left[\dfrac{2}{SA}+\dfrac{1}{a_1}\right]$, where S is the focus and A the vertex of the parabola,

or $$3\mu/a=\mu\,[(2/a)+(1/a_1)], \text{ from (iii)}$$

or $$1/a=1/a_1 \text{ or } a_1=a \qquad \text{(iv)}$$

Also we know "$pv=h=\sqrt{(\mu\times l)}$"

$\therefore$ At the vertex A, we have $a\,v_2=\sqrt{\left(\mu\times\dfrac{b_1^2}{a_1}\right)}$

or $a^2v_2^2=\mu\dfrac{a_1^2(e_1^2-1)}{a_1}$, where e_1 is the required eccentricity

or $a^2(3\mu/a)=\mu a_1(e_1^2-1)$ or $3a=a(e_1^2-1)$, from (iv)

or $4=e_1^2$ or $e_1=2$. Hence proved.

§ 7. Time of description of a central orbit.

****(a) Parabolic orbit, starting from the vertex.**

The polar equation of a parabola referred to its focus S as pole and axis as initial line is

$$\frac{l}{r}=1+\cos\theta \text{ or } \frac{l}{r}=2\cos^2\frac{\theta}{2}$$

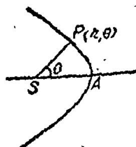
or $$r=\tfrac{1}{2}l\sec^2\tfrac{1}{2}\theta=a\sec^2\tfrac{1}{2}\theta \qquad \text{...(i)}$$

since l=semi-latus rectum$=2a$ for parabola.

Also we know $r^2\dot{\theta}=h$

or $$h\,dt=r^2\,d\theta. \qquad \text{...(ii)}$$

$\therefore$ If t be the time taken in moving from the vertex A to any point $P(r,\theta)$ on the parabolic path then from (ii), we have



(Fig. 14)

$$\int_0^t h\,dt=\int_0^\theta r^2\,d\theta$$

or $$ht=\int_0^\theta a^2\sec^4\frac{\theta}{2}\,d\theta, \text{ from (i)}$$

$$=a^2\int_0^\theta \sec^2\frac{\theta}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta$$

$$=2a^2\,[\tan\tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta],$$

for integration put $\tan\frac{1}{2}\theta=z$.

or $\sqrt{(\mu\times 2a)}\;t=2a^2\,[\tan\frac{1}{2}\theta+\frac{1}{3}\tan^3\frac{1}{2}\theta]$, $\because h=\sqrt{(\mu l)}$ and $l=2a$

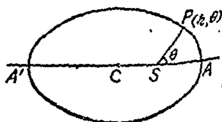
or $$t=\sqrt{(2a^3/\mu)}\,[\tan\tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta]. \quad ...(A)$$

(b) Elliptic orbit, starting from the nearer extremity of the major axis.

The polar equation of an ellipse referred to its focus S as pole and SA as initial line is

$$(l/r)=1+e\cos\theta, \quad ...(i)$$

where $e<1$.



(Fig. 15)

Also we know $r^2\dot\theta=h$

or $$h\,dt=r^2\,d\theta \quad ...(ii)$$

$\therefore$ If t be the time taken in moving from the nearer extremity A of the major axis to any point $P(r,\theta)$ on the elliptic path, then from (ii), we have $\int_0^t h\,dt=\int_0^\theta r^2\,d\theta$.

or $$ht=l^2\int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}, \text{ from (i)} \quad ...(iii)$$

Now $$\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right)=\frac{(1+e\cos\theta).\cos\theta-\sin\theta\,(e\sin\theta)}{(1+e\cos\theta)^2}$$

$$=\frac{e+\cos\theta}{(1+e\cos\theta)^2}=\frac{e^2+e\cos\theta}{e\,(1+e\cos\theta)^2}$$

$$=\frac{(1+e\cos\theta)-(1-e^2)}{e\,(1+e\cos\theta)^2}=\frac{1}{e\,(1+e\cos\theta)}-\frac{1-e^2}{e(1+e\cos\theta)^2}$$

or $$\frac{(1-e^2)}{e\,(1+e\cos\theta)^2}=\frac{1}{e\,(1+e\cos\theta)}-\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right)$$

Integrating both sides with respect to θ we get

$$\left(\frac{1-e^2}{e}\right)\int\frac{d\theta}{(1+e\cos\theta)^2}=\frac{1}{e}\int\frac{d\theta}{1+e\cos\theta}-\left(\frac{\sin\theta}{1+e\cos\theta}\right)$$

or $$\int\frac{d\theta}{(1+e\cos\theta)^2}=\frac{1}{1-e^2}\int\frac{d\theta}{1+e\cos\theta}-\left(\frac{e}{1-e^2}\right)\frac{\sin\theta}{1+e\cos\theta} \quad ...(iv)$$

Now $\int \frac{d\theta}{1+e\cos\theta}=\int \frac{d\theta}{(\cos^2 \frac{1}{2}\theta+\sin^2 \frac{1}{2}\theta)+e(\cos^2 \frac{1}{2}\theta-\sin^2 \frac{1}{2}\theta)}$

$$=\int \frac{\sec^2 \frac{1}{2}\theta\, d\theta}{(1+e)+(1-e)\tan^2 \frac{1}{2}\theta}=\frac{2}{(1-e)}\int \frac{du}{\left(\frac{1+e}{1-e}\right)+u^2},$$

where $u=\tan \frac{1}{2}\theta$

$$=\frac{2}{1-e}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\,u\right\}$$

$$=\frac{2}{\sqrt{(1-e^2)}}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan \tfrac{1}{2}\theta\right\}.$$

∴ From (iv) we have

$$\int \frac{d\theta}{(1+e\cos\theta)^2}=\frac{2}{(1-e^2)^{3/2}}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-\left(\frac{e}{1-e^2}\right)\cdot\frac{\sin\theta}{1+e\cos\theta}$$

∴ From (iii), we have

$$ht=l^2\left[\frac{2}{(1-e^2)^{3/2}}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-\left(\frac{e}{1-e^2}\right)\cdot\frac{\sin\theta}{1+e\cos\theta}\right] \quad ...(v)$$

Also $h=\sqrt{(\mu\times l)}=\sqrt{[\mu\times(b^2/a)]}=\sqrt{[\mu\times a(1-e^2)]}$,

and as proved above $l=b^2/a=a(1-e^2)$. ∵ $b^2=a^2(1-e^2)$

∴ From (v), we get

$$\sqrt{[\mu a(1-e^2)]}\,t=a^2(1-e^2)^2\left[\frac{2}{(1-e^2)^{3/2}}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-\left(\frac{e}{1-e^2}\right)\frac{\sin\theta}{1+e\cos\theta}\right]$$

$$t=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-e\sqrt{(1-e^2)}\frac{\sin\theta}{1+e\cos\theta}\right] \quad ...(B)$$

(b) **Hyperbolic orbit.**

The polar equation of a hyperbola referred to its focus as pole is

$$l/r=1+e\cos\theta, \text{ where } e>1. \quad ...(i)$$

Also we know $r^2\dot{\theta}=h$ or $h\,dt=r^2\,d\theta$. ...(ii)

∴ As in part (b) above if t be the time taken in moving from $\theta=0$ to $\theta=\theta$, then from (ii), we get

$$\int_0^t h\,dt=\int_0^\theta r^2\,d\theta$$

$$ht=l^2\int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}, \text{ from (i)} \quad ...(iii)$$

Now as in part (b) above, we can prove that

$$\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right)=\frac{e+\cos\theta}{(1+e\cos\theta)^2},\text{ after simplifying}$$

$$=\frac{e^2+e\cos\theta}{e(1+e\cos\theta)^2}=\frac{(1+e\cos\theta)+(e^2-1)}{e(1+e\cos\theta)^2},\quad \because e>1$$

$$=\frac{1}{e(1+e\cos\theta)}+\left(\frac{e^2-1}{e}\right)\cdot\frac{1}{(1+e\cos\theta)^2}$$

or $$\frac{1}{(1+e\cos\theta)^2}=\left(\frac{e}{e^2-1}\right)\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right)-\frac{1}{(e^2-1)}\cdot\frac{1}{1+e\cos\theta}$$

Integrating both sides with respect to θ, we get

$$\int\frac{d\theta}{(1+e\cos\theta)^2}=\left(\frac{e}{e^2-1}\right)\cdot\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{(e^2-1)}\int\frac{d\theta}{1+e\cos\theta}\quad \text{...(iv)}$$

$$\text{Now}\int\frac{d\theta}{1+e\cos\theta}=\int\frac{d\theta}{(\cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta)+e(\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta)}$$

$$=\int\frac{\sec^2\frac{1}{2}\theta\,d\theta}{(1+e)-(e-1)\tan^2\frac{1}{2}\theta}=\frac{1}{(e-1)}\int\frac{\sec^2\frac{1}{2}\theta\,d\theta}{\left(\frac{1+e}{e-1}\right)-\tan^2\frac{1}{2}\theta}$$

$$=\frac{1}{(e-1)}\cdot\frac{2}{2\sqrt{\left(\frac{1+e}{e-1}\right)}}\log\left[\frac{\sqrt{\left(\frac{1+e}{e-1}\right)}+\tan\frac{1}{2}\theta}{\sqrt{\left(\frac{1+e}{e-1}\right)}-\tan\frac{1}{2}\theta}\right],\text{ as } e>1$$

$$=\frac{1}{\sqrt{(e^2-1)}}\log\left[\frac{\sqrt{(1+e)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(1+e)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right]$$

$\therefore$ From (iii), we have

$$\int\frac{d\theta}{(1+e\cos\theta)^2}=\left(\frac{e}{e^2-1}\right)\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{(e^2-1)^{3/2}}\log\left[\frac{\sqrt{(1+e)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(1+e)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right]\quad \text{...(v)}$$

Also $h=\sqrt{(\mu\times l)}$ and $l=\frac{b^2}{a}=\frac{a^2(e^2-1)}{e}=a(e^2-1)$

$\therefore$ From (iii) and (v), we get

$$\sqrt{[\mu a(e^2-1)]}\,t=a^2(e^2-1)^2\left[\left(\frac{e}{e^2-1}\right)\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{(e^2-1)^{3/2}}\log\left\{\frac{\sqrt{(1+e)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(1+e)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}\right]$$

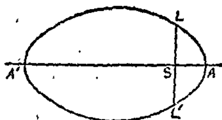
or $t=\frac{a^{3/2}}{\sqrt{\mu}}\left[e\sqrt{(e^2-1)}\frac{\sin\theta}{1+e\cos\theta}-\log\left\{\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}\right]$

Solved Examples on § 7.

Ex. 1. If the period of planet be 365 days and the eccentricity be 1/60 then show that the times of describing two halves of the orbit bounded by the latus rectum through the centre of force are

$$\frac{365}{2}\left[1\pm\frac{1}{15\pi}\right] \text{ nearly.}$$

Solution. LSL' is the latus rectum of the ellipse through its one focus S. The vectorial angle of L referred to the focus S as pole and SA are initial line is $\frac{1}{2}\pi$.



(Fig. 16)

∴ Time of description of the arc $L'AL$=2 (time of description of the arc AL)

$$=\frac{2a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\right\}\tan\frac{\theta}{2}-e\sqrt{(1-e^2)}\frac{\sin\theta}{1+e\cos\theta}\right],$$
where $\theta=\frac{1}{2}\pi$ ...see § 7 (B) Page 27

$$=\frac{2a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\right\}-e\sqrt{(1-e^2)}\right],\text{ where } e=\frac{1}{60}$$

$$=\frac{2a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-(1/60)}{1+(1/60)}\right)}\right\}-\frac{1}{60}\sqrt{\left\{1-\frac{1}{(60)^2}\right\}}\right]$$

$$=\frac{2a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\left(1-\frac{1}{60}\right)^{1/2}\left(1+\frac{1}{60}\right)^{-1/2}\right\}-\frac{1}{60}\left\{1-\frac{1}{(60)^2}\right\}^{1/2}\right]$$

$$=\frac{2a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\left(1-\frac{1}{120}\right)\left(1-\frac{1}{120}\right)\right\}-\frac{1}{60}\left\{1-\frac{1}{2(60)^2}\right\}\right),$$

neglecting higher powers of $[1/(60)^2$

$=(2a^{3/2}/\sqrt{\mu})\,[2\tan^{-1}\{1-(1/60)\}-(1/60)\,(1)],$

neglecting higher powers of $(1/60)^2$.

Now as $\tan^{-1}1=\frac{1}{4}\pi$, so $\tan^{-1}[1-(1/60)]=\frac{1}{4}\pi-z$, where z is small,

or $1-\frac{1}{60}=\tan\left(\frac{1}{4}\pi-z\right)=\frac{1-\tan z}{1+\tan z}$

or $1-(1/60)=(1-\tan z)(1+\tan z)^{-1}=(1-\tan z)(1-\tan z+\ldots)$

$=1-2\tan z$, neglecting higher powers of $\tan z$ which is small

or $\quad 2\tan z=1/60$ or $\tan z=z+\frac{1}{3}z^3+\ldots=1/120$

or $\quad z=1/120$, neglecting higher power of z.

$\therefore\ \tan^{-1}[1-(1/60)]=\frac{1}{4}\pi-z=\frac{1}{4}\pi-(1/120)$.

$\therefore$ From (i) the time of description of the arc $L'AL$

$$=\frac{2a^{3/2}}{\sqrt{\mu}}\left[2\left(\tfrac{1}{4}\pi-\frac{1}{120}\right)-\frac{1}{60}\right]=\frac{2a^{3/2}}{\sqrt{\mu}}\left[\frac{\pi}{2}-\frac{1}{30}\right]$$

$$=[\pi a^{3/2}/\sqrt{\mu}]\,[1-(1/15\pi)]$$

$$=\frac{363}{2}\left[1-\frac{1}{15\pi}\right],\ \because\ \text{time period}=\frac{2\pi a^{3/2}}{\sqrt{\mu}}=365 \text{ days (given)}$$

$\therefore$ Time of description of the remaining arc $LA'L$ (see fig)

$$=365-\frac{365}{2}\left(1-\frac{1}{15\pi}\right)=\frac{365}{2}\left(1+\frac{1}{15\pi}\right)$$

Hence the required times of description are

$$\frac{365}{2}\left(1\pm\frac{1}{15\pi}\right).$$

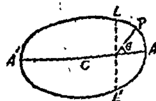
Ex. 2. Prove that the time taken to describe two portions into which an ellipse is divided by the latus rectum through the centre of force are in a ratio $\{\cos^{-1}e-e\sqrt{(1-e^2)}\}$:

$$\pi-\cos^{-1}e+e\sqrt{(1-e^2)}\}.$$

Solution. We know that the time to describe the whole elliptic path

$$=2\pi a^{3/2}/\sqrt{\mu} \qquad \ldots(i)$$

Also we know, from § 7 (b) Page 26 that the time of describing an arc of the elliptic path is given by



(Fig 17)

$$t=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}-e\sqrt{(1-e^2)}\,\frac{\sin\theta}{1-e\cos\theta}\right] \qquad \ldots(ii)$$

Also from the figure it is evident that LL' be latus rectum through the centre of force S, then at L we have $\theta=\frac{1}{2}\pi$. Therefore if T be the time of describing the arc AL, then from (ii), we get

$$T=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left(\frac{1-e}{1+e}\right)\tan\frac{\pi}{4}-e\sqrt{(1-e^2)}\frac{\sin\frac{1}{2}\pi}{1+e\cos\frac{1}{2}\pi}\right]$$

$$=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\sqrt{\left(\frac{1-\cos\lambda}{1+\cos\lambda}\right)}-e\sqrt{(1-e^2)}\right],\ \text{where } e=\cos\lambda$$

$$=(a^{3/2}/\sqrt{\mu})\,[2\tan^{-1}(\tan\tfrac{1}{2}\lambda)-e\sqrt{(1-e^2)}],\ \text{where } \lambda=\cos^{-1}e.$$

or $T=(a^{3/2}/\sqrt{\mu})\,[\cos^{-1} e-e\sqrt{(1-e^2)}]$

$\therefore$ Time of description of the arc LAL'

$=2T=(2a^{3/2}/\sqrt{\mu})\,[\cos^{-1} e-e\sqrt{(1-e^2)}]$ ...(iv)

$\therefore$ The time of describing the arc $LA'L'$

$=\dfrac{2\pi a^{3/2}}{\sqrt{\mu}}-\dfrac{2a^{3/2}}{\sqrt{\mu}}\left[\cos^{-1} e-e\sqrt{(1-e^2)}\right]$, from (i)

$=(2a^{3/2}/\sqrt{\mu})\,[\pi-\cos^{-1} e+e\sqrt{(1-e^2)}]$. ...(v)

$\therefore$ From (iv) and (v) the required ratio is

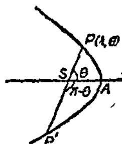
$\{\cos^{-1} e-e\sqrt{(1-e^2)}\}:\{\pi-\cos^{-1} e+e\sqrt{(1-e^2)}\}$

Hence proved.

Ex. 3. Show that if the time of describing an arc bounded by a focal chord of a periodic orbit under Newtonian law be t, then

$$t \propto (\text{focal chord})^{3/2}.$$

Solution. PSP' is a focal chord inclined at an angle θ to the initial line SA, where the focus S is taken as pole. Since θ is the vectorial angle of P, so the vectorial angle of P' is $-(\pi-\theta)$, as P' is below the initial line. (See figure).



(Fig. 18)

Also we know from § 7 (a) that the time of description of the arc of a parabolic orbit is

$\sqrt{(2a^3/\mu)}\,[\tan \tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3 \tfrac{1}{2}\theta]$

$\therefore$ In this case, the time of description of arc PAP' is given by

$$t=\sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\tan \tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3 \tfrac{1}{2}\theta\right]_{-(\pi-\theta)}^{\theta}$$

$=\sqrt{(2a^3/\mu)}\,[\{\tan \tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3 \tfrac{1}{2}\theta\}$
$\qquad -\{\tan(-\tfrac{1}{2}\pi+\tfrac{1}{2}\theta)+\tfrac{1}{3}\tan^3(-\tfrac{1}{2}\pi+\tfrac{1}{2}\theta)\}]$

$=\sqrt{(2a^3/\mu)}\,[(\tan \tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3 \tfrac{1}{2}\theta)+\{\cot \tfrac{1}{2}\theta+\tfrac{1}{3}\cot^3 \tfrac{1}{2}\theta\}]$

$=\sqrt{(2a^3/\mu)}\,[(\tan \tfrac{1}{2}\theta+\cot \tfrac{1}{2}\theta)+\tfrac{1}{3}(\tan^3 \tfrac{1}{2}\theta+\cot^3 \tfrac{1}{2}\theta)]$

$=\sqrt{(2a^3/\mu)}\,(\tan \tfrac{1}{2}\theta+\cot \tfrac{1}{2}\theta)\,[1+\tfrac{1}{3}(\tan^2 \tfrac{1}{2}\theta$
$\qquad -\tan \tfrac{1}{2}\theta.\cot \tfrac{1}{2}\theta+\cot^2 \tfrac{1}{2}\theta)]$,

$\because a^3+b^3=(a+b)(a^2-ab+b^2)$

$=\sqrt{(2a^3/\mu)}\,(\tan \tfrac{1}{2}\theta+\cot \tfrac{1}{2}\theta)\,[\tfrac{1}{3}(\tan^2 \tfrac{1}{2}\theta+\cot^2 \tfrac{1}{2}\theta+2)]$

$=\tfrac{1}{3}\sqrt{(2a^3/\mu)}\,(\tan \tfrac{1}{2}\theta+\cot \tfrac{1}{2}\theta)\,(\tan \tfrac{1}{2}\theta+\cot \tfrac{1}{2}\theta)^2$

$=\tfrac{1}{3}\sqrt{(2a^3/\mu)}\,(\tan \tfrac{1}{2}\theta+\cot \tfrac{1}{2}\theta)^3$

$=\tfrac{1}{3}\sqrt{\left(\dfrac{2a^3}{\mu}\right)}\left[\dfrac{\sin \tfrac{1}{2}\theta}{\cos \tfrac{1}{2}\theta}+\dfrac{\cos \tfrac{1}{2}\theta}{\sin \tfrac{1}{2}\theta}\right]^3$

$=\tfrac{1}{3}\sqrt{\left(\dfrac{2a^3}{\mu}\right)}\left[\dfrac{1}{\sin \tfrac{1}{2}\theta\cos \tfrac{1}{2}\theta}\right]^3=\tfrac{1}{3}\sqrt{\left(\dfrac{2a^3}{\mu}\right)}\left(\dfrac{2}{\sin\theta}\right)^3.$

or $t=(8/3)\sqrt{(2a^3/\mu)}\operatorname{cosec}^3\theta$

Also chord $PSP'=SP+SP'$, where $l/r=1+\cos\theta$ is the equation of the parabola

$$=\frac{l}{1+\cos\theta}+\frac{l}{1+\cos(-\pi+\theta)},\quad \because\ r=l/(1+\cos\theta)$$

$$=\frac{l}{1+\cos\theta}+\frac{l}{1-\cos\theta}=\frac{2l}{1-\cos^2\theta}=2l\operatorname{cosec}^2\theta$$

$=2\,(2a)\operatorname{cosec}^2\theta,\quad\because\ l=2a$ for parabola

$\therefore$ chord $PSP'=4a\operatorname{cosec}^2\theta$ ...(ii)

Now from (i), $t=\frac{8}{3}\sqrt{(2a^3/\mu)}\operatorname{cosec}^3\theta=\frac{1}{3}\sqrt{(2/\mu)}(8a^{3/2}\operatorname{cosec}^3\theta)$

$=\frac{1}{3}\sqrt{(2/\mu)}\,(4a\operatorname{cosec}^2\theta)^{3/2}$

$=\frac{1}{3}\sqrt{(2/\mu)}\,(\text{chord } PSP')^{3/2}$

i.e. $t\propto(\text{focal chord})^{3/2}$. Hence proved.

Ex. 4. Prove that in a parabolic orbit the time taken to move from the vertex to a point distant r from the focus is

$(1/3\sqrt{\mu})\,(r+l)\sqrt{(2r-l)}$, where $2l$ is the latus rectum.

Solution. From § 7 (a) Page 25 we know that the time of describing an arc of a parabolic orbit is given by

$t=\sqrt{(2a^3/\mu)}\,(\tan\frac{1}{2}\theta+\frac{1}{3}\tan^3\frac{1}{2}\theta)$ ...(i)

Also the polar equation of a parabola referred to its focus as pole is $l/r=1+\cos\theta$

or $\frac{l}{r}=2\cos^2\frac{1}{2}\theta$ or $\sec^2\frac{1}{2}\theta=\frac{2r}{l}$ or $\tan^2\frac{1}{2}\theta=\frac{2r}{l}-1$

or $\tan\frac{1}{2}\theta=\sqrt{[(2r-l)/l]}$

$\therefore$ From (i) the required time

$$=\sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\left(\frac{2r-l}{l}\right)^{1/2}+\frac{1}{3}\left(\frac{2r-l}{l}\right)^{3/2}\right]$$

$$=\sqrt{\left(\frac{2a^3}{\mu}\right)}\sqrt{\left(\frac{2r-l}{l}\right)}\left[1+\frac{2r-l}{3l}\right]$$

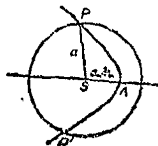
$$=\sqrt{\left\{\frac{2\,(\frac{1}{2}l)^3}{\mu}\right\}}\sqrt{\left(\frac{2r-l}{l}\right)}\cdot\frac{2\,(r+l)}{3l},\quad\because\ a=\tfrac{1}{2}l$$

$=(1/3\sqrt{\mu})\,(r+l)\sqrt{(2r-l)}$. Hence proved.

Ex. 5. The perihelion distance of a comet describing a parabolic orbit is $1/n$ of the radius on the earth's path supposed circular, show that the time that the comet will remain within the earth's orbit is $\frac{2}{3\pi}\cdot\frac{n+2}{n}\sqrt{\left(\frac{n-1}{2n}\right)}$ of a year.

Also prove that the longest time that the comet can remain within the earth's orbit is $(2/3\pi)$ of an year.

Solution. (Perihelion and Aphelion. The point on an elliptic orbit nearest to the occupied focus is known as the *perihelion* or *pericentre* and the point farthest from this focus is known as *aphelion* or *apocentre*).



(Fig. 19)

Let S be the sun and a be the radius of earth's circular path. It is given that the perihelion distance of the comet describing the parabolic orbit is $\frac{1}{n}$ of the radius of earth's path so we have

$$AS=a/n.$$

Also the polar equation of the parabolic path referred to its focus S as pole is $l/r=1+\cos\theta$, where $l=2(a/n)$

i.e. $$\frac{2a/n}{r}=1+\cos\theta. \qquad ...(i)$$

Let P be the point of intersection of the planet's parabolic orbit and earth's circular orbit. Then at P we have $r=a$ and therefore from (i) the vertical angle of P is given by

$$\frac{2a/n}{a}=1+\cos\theta \quad \text{or} \quad \frac{2}{n}=2\cos^2\frac{\theta}{2} \quad \text{or} \quad \sec^2\frac{\theta}{2}=n$$

or $$\tan\tfrac{1}{2}\theta=\sqrt{(n-1)}$$

Now we know that the time of description of an arc of a parabolic path starting from the vertex is

$\sqrt{(2a^3/\mu)}\,[\tan\frac{1}{2}\theta+\frac{1}{3}\tan^3\frac{1}{2}\theta]$, where a is the apsidal distance.

Here in this formula if a/n be substituted for a and $\sqrt{(n-1)}$ for $\tan\frac{1}{2}\theta$ we get the time of description of the arc AP of the parabolic orbit.

$\therefore$ Time of description of the arc $P'AP$

$$=2\times(\text{time of description of the arc } AP)$$

$$=2\sqrt{\left[\frac{2(a/n)^3}{\mu}\right]}\left[\sqrt{(n-1)}+\tfrac{1}{3}\{\sqrt{(n-1)}\}^3\right]$$

$$=2\sqrt{\left(\frac{2a^3}{\mu n^3}\right)}\sqrt{(n-1)}\left[1+\tfrac{1}{3}(n-1)\right]$$

$$=\tfrac{2}{3}\sqrt{\left(\frac{2a^3}{\mu n^3}\right)}\sqrt{(n-1)}\,(n+2).$$

$$=\frac{2}{3}\cdot\frac{1}{n}\sqrt{\left(\frac{4a^3}{2\mu n}\right)}\sqrt{(n-1)(n+2)}$$

$$=\frac{2}{3\pi}\cdot\frac{n+2}{n}\cdot\frac{2\pi a^{3/2}}{\sqrt{\mu}}\sqrt{\left(\frac{n-1}{2n}\right)};$$

$$=\frac{2}{3\pi}\cdot\frac{n+2}{n}\sqrt{\left(\frac{n-1}{2n}\right)} \text{ of an year,} \quad ...(ii)$$

since $(2\pi a^{3/2}/\sqrt{\mu})$=Periodic time of earth=one year.

∴ The time during which the comet will remain within the earth's orbit=time of description of the arc PAP'

$$=\frac{2}{3\pi}\cdot\frac{n+2}{n}\cdot\sqrt{\left(\frac{n-1}{2n}\right)} \text{ of an year.} \quad ...(ii)$$ Hence proved.

This time is maximum if $\frac{n+2}{n}\sqrt{\left(\frac{n-1}{2n}\right)}$ is maximum

i.e. if $\left(\frac{n+2}{n}\right)^2\left(\frac{n-1}{2n}\right)$ is maximum

Let $z=\left(\frac{n+2}{n}\right)^2\left(\frac{n-1}{2n}\right)=\frac{(n^2+4n+4)(n-1)}{2n^3}$

or $z=\frac{n^3-3n^2-4}{2n^3}=\frac{1}{2}+\frac{3}{2n}-\frac{2}{n^3}$

$$\therefore \frac{dz}{dn}=-\frac{3}{2n^2}+\frac{6}{n^4};\ \frac{d^2z}{dn^2}=\frac{3}{n^3}-\frac{24}{n^5}.$$

If z is maximum, then $\frac{dz}{dn}=0$ which gives $\frac{-3}{2n^2}+\frac{6}{n^4}=0$

i.e. $n^2=4$ or $n=\pm 2$.

For $n=2$, we have $\frac{d^2z}{dn^2}=\frac{3}{8}-\frac{24}{32}$=negative.

∴ z is maximum when $n=2$.

Hence putting $n=2$ in the above period given by (ii) we have the longest time that the comet can remain within the earth's orbit

$$=\frac{2}{3\pi}\cdot\frac{2+2}{2}\cdot\sqrt{\left(\frac{2-1}{2\times 2}\right)} \text{ of an year}$$

$$=\frac{4}{3\pi}\sqrt{(\tfrac{1}{4})} \text{ of an year}=\frac{2}{3\pi} \text{ of an year.}$$ Hence proved.

*Ex. 6. The greatest and least velocities of a certain planet in its orbit round the sun are 30 km/sec. and 29·2 km/sec. respectively. *Find the eccentricity of the orbit.*

Solution. We know the velocity at any point on an elliptic path is given by $v^2=\mu\,[(2/r)-(1/a)]$ ...(i)

This velocity is greatest when r is least and least when r is greatest as is evident from (i). Also the greatest and least values of r are $SA'=a\,(1+e)$ and $SA=a\,(1-e)$ respectively, where S is the occupied focus and A, A' are the extremities of major axis nearer and farther from S.

$\therefore$ From (i) we have

$$(30)^2=\mu\left[\frac{2}{a\,(1-e)}-\frac{1}{a}\right] \text{ and } (29\cdot2)^2=\mu\left[\frac{2}{a\,(1+e)}-\frac{1}{a}\right] \quad \text{(Note)}$$

Dividing we get

$$\frac{(30)^2}{(29\cdot2)^2}=\frac{\left(\dfrac{2}{1-e}-1\right)}{\left(\dfrac{2}{1+e}-1\right)}=\frac{\dfrac{2-(1-e)}{(1-e)}}{\dfrac{2-(1+e)}{(1+e)}}=\frac{(1+e)^2}{(1-e)^2}$$

or $$\frac{30}{29\cdot2}=\frac{1+e}{1-e} \quad\text{or}\quad 30-30e=29\cdot2+29\cdot2e$$

or $$0\cdot8=(59\cdot2)\,e \quad\text{or}\quad e=(8/592)=(1/74) \qquad \textbf{Ans.}$$

§ 8. Stability of circular orbits.

The differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2}+u=\frac{F(u)}{h^2u^2} \qquad ...(i)$$

Also we know $h=vp$ ...(ii)

Now for a circular orbit we know u=constant and so $\dfrac{d^2u}{d\theta^2}=0$ and so from (i) we have $u=\dfrac{F(u)}{h^2u^2}$ ...(iii)

or $$h^2=\frac{F(u)}{u^3} \qquad ...(iv)$$

Let the orbit be a circle of radius a, then $u=1/a$ and let $v=v_0$ for this circular orbit, then from (ii) we have

$$v_0=\text{``}h/p\text{''}=h/a, \quad \because\ p=a \text{ for the circular orbit.}$$

$\therefore$ From (iv), $$h^2=\frac{F(1/a)}{(1/a)^3}=a^3F(1/a) \qquad ...(v)$$

$$\therefore\ v_0=\frac{a^{3/2}\sqrt{[F(1/a)]}}{a}=\sqrt{[aF(1/a)]}$$

Hence the condition for stability is

$$\lambda>0 \quad i.e. \quad \frac{F'(1/a)}{aF(1/a)}<3, \text{ from (vii)}$$

Particular case. If the law of force be μu^n i.e. $F(u)=\mu u^n$

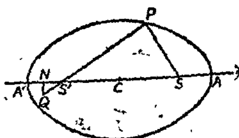
Then $\frac{F'(1/a)}{aF(1/a)}=\frac{n\mu(1/a)^{n-1}}{a\mu(1/a)^n}=n$

$\therefore$ From above the condition of stability is $n<3$.

§ 9. Disturbed Elliptic Motion (*Purvanchal* 90, 89)

Let a particle describe an elliptic orbit under a force to the focus S and let S' be the other focus.

Let the velocity v of the particle be changed to $v+\delta v$ when it reaches at a point P on its path, the direction of the velocity remaining unchanged.



(Fig. 20)

If $2a$ and $2a'$ be the major axes before and after this disturbance, then we have

$$v^2=\mu\left[\frac{2}{SP}-\frac{1}{a}\right] \quad ...(i)$$

$$(v+\delta v)^2=\mu\left[\frac{2}{SP}-\frac{1}{a'}\right] \quad ...(ii)$$

Differentiating (i) we get $2v\delta v=\frac{\mu}{a^2}\delta a$, as $SP=$ constant

$$\delta a=\frac{2va^2\,\delta v}{\mu} \quad ...(iii)$$

i.e. the increase in semi-major axis $=\frac{2va^2\,\delta v}{\mu}$.

Also we know that the time period T is given by

$$T=\frac{2\pi a^{3/2}}{\sqrt{\mu}} \quad \text{or} \quad \log T=\log\left(\frac{2\pi}{\sqrt{\mu}}\right)+\frac{3}{2}\log a$$

Now we suppose that the circular path is slightly disturbed in such a way that h remains unaltered and let at any instant $u=\frac{1}{a}+x$

Then putting $u=\frac{1}{a}+x$ the differential equation (i), becomes

$$\frac{d^2x}{d\theta^2}+\frac{1}{a}+x=\frac{F\left(\frac{1}{a}+x\right)}{h^2\left(\frac{1}{a}+x\right)^2}$$

$$=\frac{F\left(\frac{1}{a}+x\right)}{\left(\frac{1}{a}+x\right)^2 a^3 F\left(\frac{1}{a}\right)}, \text{ from (v)}$$

$$=\frac{F\left(\frac{1}{a}\right)+xF'\left(\frac{1}{a}\right)+\dots}{\frac{1}{a^2}\cdot a^3 F\left(\frac{1}{a}\right)}(1+ax)^{-2}$$

$$=\frac{F\left(\frac{1}{a}\right)+xF'\left(\frac{1}{a}\right)+\dots}{aF(1/a)}(1-2ax+\dots)$$

$$=\left[\frac{1}{a}+\frac{xF'(1/a)}{aF(1/a)}+\dots\right](1-2ax+\dots)$$

$$=\frac{1}{a}-2x+\frac{xF'(1/a)}{aF(1/a)}+\text{higher powers of } x$$

or $$\frac{d^2x}{d\theta^2}+\lambda x=0, \text{ to a first approximation.} \quad \dots(vi)$$

Here $$\lambda=3-\frac{F'(1/a)}{aF(1/a)} \quad \dots(vii)$$

Now the solutions of (vi) are

Case I. If $\lambda=0$, then $x=c_1\theta+c_2$.

Case II. If $\lambda>0$, then $x=\alpha_1\cos\sqrt{\lambda}\theta+\alpha_2\sin\sqrt{\lambda}\theta$

Case III. If $\lambda<0$, then $x=\beta_1\cosh\sqrt{\lambda}\theta+\beta_2\sinh\sqrt{\lambda}\theta$

Out of these only case II is periodic and so represents sma oscillation about the mean position. The other two cases are such that x increases as θ increases, hence not periodic. (Note)

Hence the condition for stability is

$$\lambda > 0 \quad i.e. \ \frac{F'(1/a)}{aF(1/a)} < 3, \text{ from (vii)}$$

Particular case. If the law of force be μu^n i.e. $F(u) = \mu u^n$

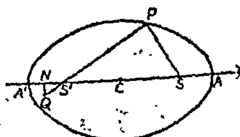
Then $\dfrac{F'(1/a)}{aF(1/a)} = \dfrac{n\mu (1/a)^{n-1}}{a\mu (1/a)^n} = n$

$\therefore$ From above the condition of stability is $n < 3$.

§ 9. Disturbed Elliptic Motion (*Purvanchal* 90, 89)

Let a particle describe an elliptic orbit under a force to the focus S and let S' be the other focus.

Let the velocity v of the particle be changed to $v + \delta v$ when it reaches at a point P on its path, the direction of the velocity remaining unchanged.



(Fig. 20)

If $2a$ and $2a'$ be the major axes before and after this disturbance, then we have

$$v^2 = \mu \left[\frac{2}{SP} - \frac{1}{a} \right] \quad \text{...(i)}$$

and

$$(v + \delta v)^2 = \mu \left[\frac{2}{SP} - \frac{1}{a'} \right] \quad \text{...(ii)}$$

Differentiating (i) we get $2v\delta v = \dfrac{\mu}{a^2} \delta a$, as SP = constant

or

$$\delta a = \frac{2va^2 \delta v}{\mu} \quad \text{...(iii)}$$

i.e. the increase in semi-major axis $= \dfrac{2va^2 \delta v}{\mu}$.

Also we know that the time period T is given by

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}} \quad \text{or} \quad \log T = \log \left(\frac{2\pi}{\sqrt{\mu}} \right) + \tfrac{3}{2} \log a$$

Differentiating we get $\frac{1}{T}\,\delta T=\frac{3}{2}\cdot\frac{1}{a}\,\delta a$

or $$\frac{\delta T}{T}=\frac{3}{2a}\cdot\frac{2va^2\,\delta v}{\mu}, \text{ from (iii)}$$

or $$\frac{\delta T}{T}=\frac{3va\,\delta v}{\mu} \qquad \text{...(iv)}$$

Again the value of 'p' will not change as the direction of motion at P remains unchanged.

∴ On differentiating the relation $h=pv$ we get

$$\delta h=p\;\delta v=\frac{h}{v}\,\delta v \qquad \text{...(v)}$$

Also we know that $h^2=\mu l$ ...See Chapter on Central Orbits

i.e. $h^2=\mu a\,(1-e^2)$, ∵ $l=b^2/a=a\,(1-e^2)$ as $b^2=a^2\,(1-e^2)$

Differentiating we get $2h\,\delta h=\mu\,[a\,(-2e\delta e)+(1-e^2)\,\delta a]$

or $$2ae\mu\delta e=\mu\,(1-e^2)\,\delta a-2h\,\delta h$$

$$=\mu\,(1-e^2)\cdot\frac{2va^2\,\delta v}{\mu}-2h\left(\frac{h}{v}\,\delta v\right), \text{ from (iii) and (v)}$$

$$=2a^2\,(1-e^2)\,v\delta v-\frac{2\mu a\,(1-e^2)}{v}\,\delta v,\ \because\ h^2=\mu a\,(1-e^2)$$

$$=2a\,(1-e^2)\left[\frac{av^2-\mu}{v}\right]\delta v$$

or $$\delta e=\left(\frac{1-e^2}{e}\right)\left(\frac{av^2-\mu}{\mu}\right)\cdot\frac{\delta v}{v}, \qquad \text{...(vi)}$$

which gives the change in eccentricity.

Again as the direction of motion of the particle remains unchanged at P so the new focus lies on PS' and let Q be the new focus (see Fig. 20 Page 37 of this chapter).

Now $S'Q=QP-S'P=(QP+SP)-(S'P+SP)$, adding and subtracting SP

$$=(2a')-(2a)=2\,(a'-a)=2\,[\delta a] \qquad \text{...(vii)}$$

Now the major axis passing through S and S' moves to the position when it is passing through S and Q *i.e.* if $\delta\psi$ be the angle through which the major axis moves, then $\delta\psi=\angle S'SQ$,

∴ $\delta\psi=\tan\,\delta\psi=\tan\,\angle S'SQ=\frac{QN}{SN}$ (see figure 20 Page 37)

$$=\frac{QS'\sin\alpha}{SS'+QS'\cos\alpha}, \text{ where } \angle QS'N=\alpha \text{ (say)}$$

or $$\delta\psi=\frac{2\,(\delta a)\sin\alpha}{2ae+2\,(\delta a)\cos\alpha},\text{ from (vii) and } SS'=2ae$$

$$=\frac{(\delta a)\sin\alpha}{ae},\text{ neglecting } (\delta a)\cos\alpha \text{ in comparison with } ae,$$

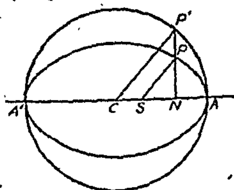
$$=\frac{2va^2\delta v}{\mu ae}\sin\alpha,\text{ from (iii)}$$

or $$\delta\psi=\left(\frac{2va}{\mu e}\sin\alpha\right)\delta v \quad \text{...(viii)}$$

The results (iii), to (viii) give the various changes due to disturbance in the elliptic path.

§ 10. Anomalies.

Let P be any point on the elliptic path described by a planet with the sun being at the focus S.



(Fig. 21)

Then the angle which the radius vector (*i.e.* SP) makes with the line from the sun towards the perihelion (*i.e.* the point A) is known as true anomaly. Here the angle is measured in the direction in which the planet moves. In the adjoining figure the anomaly is the angle ASP.

The eccentric angle ϕ of P is called the eccentric anomaly and is generally denoted by E. In the figure $\angle P'CA=E$.

Again the average angular velocity of the planet is $\frac{2\pi}{T}$, where T is its periodic time. This average angular velocity is generally denoted by n *i.e.* $n=\frac{2\pi}{T}$. Now if the time taken by the planet in reaching P from A be t then nt is defined as the mean anomaly and is generally denoted by M.

Solved Examples on § 8 to § 10.

**Ex. 1. One end of an elastic string, of unstretched length 'a' is tied to a point on the top of a smooth table, and a particle

attached to the other end can move freely on the table. If the path be nearly circular of radius 'b' show that its apsidal angle is approximately $\pi\sqrt{\{(b-a)/(4b-3a)\}}$

Solution. If a and r be the natural and extended lengths of the elastic string, then the force F acting on the particle is given by

$$F = \text{tension in the elastic string}$$

$$= \frac{\lambda(r-a)}{a}, \text{ by Hooke's Law}$$

$$= \frac{\lambda(1-au)}{au}, \ \because \ r = \frac{1}{u}$$

$\therefore$ The differential equation of the path traced out by the particle is $\dfrac{d^2u}{d\theta^2}+u=\dfrac{F}{h^2u^2}=\dfrac{\lambda(1-au)}{au\,h^2u^3}$ ...(i)

Since the path is a circle of radius b, so putting $r=b$ or $u=1/b$ and $d^2u/d\theta^2=0$ in (i) we get

$$\frac{1}{b}=\frac{\lambda(b-a)b^2}{ah^2} \quad \text{or} \quad ah^2=\lambda b^3(b-a) \qquad \text{...(ii)}$$

Now let the particle be disturbed slightly such that h remains constant and let at any instant $u=\dfrac{1}{b}+x$, where x is small *i.e.* the path remains nearly circular.

Then from (i) putting $u=\dfrac{1}{b}+x$, we get

$$\frac{d^2x}{d\theta^2}+\frac{1}{b}+x=\frac{\lambda\left[1-a\left(\frac{1}{b}+x\right)\right]}{ah^2\left(\frac{1}{b}+x\right)^3}$$

$$=\frac{\lambda b^2}{ah^2}\cdot\frac{(b-a-abx)}{(1+bx)^3} = \frac{\lambda b^2}{ah^2}(b-a-abx)(1+bx)^{-3}$$

$$=\frac{\lambda b^2}{ah^2}(b-a-abx)(1-3bx+\ldots)$$

$$=\frac{\lambda b^2}{ah^2}[b-a-abx-3b(b-a)x],$$

neglecting higher powers of x.

$$=\frac{\lambda b^2}{\lambda b^3(b-a)}[b-a+(2ab-3b^2)x], \text{ from (ii)}$$

or $$\frac{d^2x}{d\theta^2}+\frac{1}{b}+x=\frac{1}{b}+\left(\frac{2a-3b}{b-a}\right)x$$

or $$\frac{d^2x}{d\theta^2}=\left(\frac{2a-3b}{b-a}-1\right)x=\left[\frac{2a-3b-b+a}{(b-a)}\right]x$$

or $$\frac{d^2x}{d\theta^2}=\left(\frac{3a-4b}{b-a}\right)x=-\left(\frac{4b-3a}{b-a}\right)x=-\lambda x, \text{ where } \lambda=\frac{4b-3a}{b-a}$$

This equation is of S.H.M. and its solution is

$x=\alpha \cos [\theta\sqrt{\lambda}+\beta]$, where α and β are arbitrary constants.

Now $u=\dfrac{1}{b}+x=\dfrac{1}{b}+\alpha \cos [\theta\sqrt{\lambda}+\beta]$.

$$\therefore \frac{du}{d\theta}=-\alpha\sqrt{\lambda}\sin [\theta\sqrt{\lambda}+\beta].$$

At an apse we know $\dfrac{du}{d\theta}=0$, so the apses are obtained by

$\sin [\theta\sqrt{\lambda}+\beta]=0$ or $\theta\sqrt{\lambda}+\beta=n\pi$, where n is any integer. Putting $n=0, 1, 2, 3, \ldots$ in this relation, we get

$$\theta_0\sqrt{\lambda}+\beta=0$$
$$\theta_1\sqrt{\lambda}+\beta=\pi,$$
$$\theta_2\sqrt{\lambda}+\beta=2\pi, \text{ etc.}$$

From these we have

$$\theta_1-\theta_0=\frac{\pi}{\sqrt{\lambda}}=\theta_2-\theta_1=\theta_3-\theta_2=\ldots$$

$\therefore$ The difference between the successive values of θ at the apse is $\pi/\sqrt{\lambda}$.

i.e. the required apsidal angle $=\dfrac{\pi}{\sqrt{\lambda}}=\pi\sqrt{\left(\dfrac{b-a}{4b-3a}\right)}$.

Hence proved.

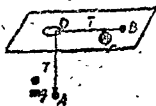
***Ex. 2.** Two masses M and m are connected by a string which passes through a hole in a smooth horizontal plane, the mass m hanging vertically. Show that M describes on the plane a curve whose differential equation is

$$\left(1+\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{mg}{M}\cdot\frac{1}{h^2u^2}.$$

Prove also that the tension in the string is

$$\frac{M}{M+m}(g+h^2u^3).$$

Solution. Let O be the hole, it is also the centre of the force and tension T (say) of the string is the central force.



(Fig. 22)

$\therefore$ The differential equation of the path traced out by the mass M on the plane is

$$\frac{d^2u}{d\theta^2}+u=\frac{(T/M)}{h^2u^2}, \because \text{ for this path } F=T/M$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{T}{Mh^2u^2} \quad ...(i)$$

Also the length of the string hanging vertically downwards

$$=OA=l-r,$$

where $OB=r$ and whole length of the string $=l$ (say).

$\therefore$ The equation of motion of the mass m is

$$m\,\frac{d^2}{dt^2}(l-r)=mg-T \quad \textbf{(Note)}$$

or $$-m\frac{d^2r}{dt^2}=mg-T. \quad ...(ii)$$

Now $r=\dfrac{1}{u}$, $\therefore$ $\dfrac{dr}{dt}=-\dfrac{1}{u^2}\dfrac{du}{dt}=-\dfrac{1}{u}\dfrac{du}{d\theta}\cdot\dfrac{d\theta}{dt}$

$$=-\frac{du}{d\theta}\left(r^2\frac{d\theta}{dt}\right)=-h\frac{du}{d\theta}, \because h=r^2\frac{d\theta}{dt}$$

or $$\frac{dr}{dt}=-h\frac{du}{d\theta}$$

$$\therefore \frac{d^2r}{dt^2}=-h\frac{d}{dt}\left(\frac{du}{d\theta}\right)=-h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$=-h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}=-h\frac{d^2u}{d\theta^2}u^2r^2\frac{d\theta}{dt}, \quad \because u^2r^2=1$$

$$=-h\frac{d^2u}{d\theta^2}u^2h, \quad \because h=r^2\frac{d\theta}{dt}.$$

or $$\frac{d^2r}{dt^2}=-h^2u^2\frac{d^2u}{d\theta^2}$$

$\therefore$ From (ii) we have $mh^2u^2\dfrac{d^2u}{d\theta^2}=mg-T$ ...(iii)

or $$mh^2u^2\frac{d^2u}{d\theta^2}=mg-Mh^2u^2\left(\frac{d^2u}{d\theta^2}+u\right), \text{ from (i)}$$

or $$(M+m)\,h^2u^2\frac{d^2u}{d\theta^2}=mg-Mh^2u^3 \quad ...(iv)$$

or $\left(1+\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{mg}{Mh^2u^2}$, dividing each term by Mh^2u^2

Again from (iii) we get $T=mg-mh^2u^2\frac{d^2u}{d\theta^2}$

or $T=mg-m\left[\frac{mg-Mh^2u^2}{M+m}\right]$, from (iv)

$$=\frac{1}{M+m}\left[mg\,(m+M)-m^2g+mMh^2u^2\right]$$

$$=\frac{mM}{m+M}\left[g+h^2u^2\right]$$

Hence proved.

Ex. 3. Assuming that the moon is acted upon by a force μ/r^2 to the earth and that the effect of the sun's disturbing force is to cause a force m^2r from the earth to the moon, show that the orbit being nearly circular, the apsidal angle is $\pi\left(1+\frac{3m^2}{2n^2}\right)$ nearly, where $2\pi/n$ is the lunar month and cubes of m neglected.

Solution. The forces acting on the moon are—(i) the force μ/r^2 or μu^2 towards the earth and (ii) the force m^2r or m^2/u from the earth.

Hence $$\text{"}F\text{"}=\mu u^2-\frac{m^2}{u} \quad \text{(Note)}$$

∴ The differential equation of the path of the moon is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}=\frac{\mu u^2-(m^2/u)}{h^2u^2}$$

or $$\frac{d^2u}{d\theta^2}+u=\frac{1}{h^2}\left(\mu-\frac{m^2}{u^3}\right) \quad \text{...(i)}$$

Since the path is a circle of radius $1/b$ (say), so putting $r=1/b$ or $u=b$ and $d^2u/d\theta^2=0$ in (i) we get

$$b=\frac{1}{h^2}\left(\mu-\frac{m^2}{b^3}\right). \quad \text{...(ii)}$$

Now let the moon be disturbed slightly such that h remains constant and let at any instant $u=b+x$, where x is small *i.e.* the path remains nearly circular.

Putting $u=b+x$ in (i) we get

$$\frac{d^2x}{d\theta^2}+b+x=\frac{1}{h^2}\left[\mu-\frac{m^2}{(b+x)^3}\right]=\frac{1}{h^2}\left[\mu-\frac{m^2}{b^3}\left(1+\frac{x}{b}\right)^{-3}\right]$$

or $\dfrac{d^2x}{d\theta^2}+b+x=\dfrac{1}{h^2}\left[\mu-\dfrac{m^2}{b^3}\left(1-\dfrac{3x}{b}+...\right)\right]$

$$=\frac{1}{h^2}\left[\mu-\frac{m^2}{b^3}+\frac{3m^2}{b^4}x\right],$$

neglecting higher powers of x

$$=\frac{1}{h^2}\left(\mu-\frac{m^2}{b^3}\right)+\frac{3m^2}{h^2b^4}x=b+\frac{3m^2}{h^2b^4}x, \text{ from (ii)}$$

or $$\frac{d^2x}{d\theta^2}=-\left(1-\frac{3m^2}{h^2b^4}\right)x$$

This equation is of S.H.M. and hence as in last example the required apsidal angle $=\pi\Big/\sqrt{\left(1-\dfrac{3m^2}{h^2b^4}\right)}$ ...(iii)

Also it is given that lunar month $=2\pi/n=T$ (say) ...(iv)

Also if v be the velocity of the moon, then

$$T=\frac{2\pi r}{v}, \text{ where } r=\frac{1}{b} \quad ...(v)$$

∴ From (iv) and (v) we get $\dfrac{2\pi r}{v}=\dfrac{2\pi}{n}$, where $r=\dfrac{1}{b}$.

or $$1/(bv)=1/n \quad \text{or} \quad bv=n \quad ...(vi)$$

Also we know that $h=pv$.

Here the path being circular $p=$ radius $=1/b$

∴ $h=\dfrac{1}{b}v$ or $h=\dfrac{n}{b^2}$, from (vi)

∴ From (iii), the required apsidal angle

$$=\pi\Big/\sqrt{\left(1-\frac{3m^2}{n^2}\right)}, \quad \because \; h^2=n^2/b^4 \text{ i.e. } h^2b^4=n^2$$

$$=\pi\left(1-\frac{3m^2}{n^2}\right)^{-1/2}=\pi\left(1+\frac{3m^2}{2n^2}\right) \text{ nearly.}$$

as m^4 is neglected. Hence proved.

Ex. 4. Two particles of masses M and m are connected by a light string; the string passes through a small hole in the table; m hangs vertically and M describes a curve on the table which is very nearly a circle whose centre is the hole; show that the apsidal angle of the orbit of M is $\pi\sqrt{\{(M+m)/3M\}}$.

Solution. Refer Fig. 22 Page 42 of this chapter.

In Ex. 2 Page 41 it has been proved that the path described by the particle of mass M on the table is given by

$$\left(1+\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{mg}{M}\cdot\frac{1}{h^2u^2} \qquad \text{...(i)}$$

If this path is a circle of radius $1/a$ *i.e.* $r=1/a$ *i.e.* $u=a$, then $\frac{d^2u}{d\theta^2}=0$ and so from (i) we have

$$a=\frac{mg}{Mh^2a^2} \quad \text{or} \quad a^3=\frac{mg}{Mh^2} \qquad \text{...(ii)}$$

$\therefore$ From (i) we get $\left(1-\frac{m}{M}\right)\frac{d^2u}{d\theta^2}+u=\frac{a^3}{u^2}$ ...(iii)

Let the particle be slightly disturbed from its circular path by putting $u=a+x$ and in such a way that h remains constant, then from (iii) we have

$$\left(1+\frac{m}{M}\right)\frac{d^2x}{d\theta^2}+(a+x)=\frac{a^3}{(a+x)^2}$$

$$=\frac{a^3}{a^2(1+x/a)^2}=a\left(1+\frac{x}{a}\right)^{-2}$$

$$=a\left[1-2\left(\frac{x}{a}\right)+\ldots\right]=a-2x, \text{ nearly}$$

or $\left(1+\frac{m}{M}\right)\frac{d^2x}{d\theta^2}=-3x$ or $\frac{d^2x}{d\theta^2}=-\frac{3M}{(m+M)}x$,

which represents a S. H. M.

$\therefore$ The required apsidal angle$=\pi\Big/\sqrt{\left(\frac{3M}{m+M}\right)}=\pi\sqrt{\left(\frac{m+M}{3M}\right)}$

Hence proved.

Ex. 5. A particle moves in an orbit under a central acceleration μ/r^2 along the radius vector. Obtain the equations of energy and angular momentum and show that if the particle is projected with velocity u at right angles to the radius at a distance c from the origin, $\left(\frac{dr}{dt}\right)^2=\left\{\frac{2\mu}{c}-u^2\left(1+\frac{c}{r}\right)\right\}\left(\frac{c}{r}-1\right)$.

Solution. We know that the radial and transverse velocities of a particle at any point (r, θ) are $\dot{r}$ and $r\dot{\theta}$ respectively.

$\therefore$ The resultant velocity at any point$=\sqrt{[(\dot{r})^2+(r\dot{\theta})^2]}$

Also the given central acceleration is μ/r^2 along the radius vector.

Now we know that change in kinetic energy$=$work done

i.e. $\frac{1}{2}[(\dot{r})^2+(r\dot{\theta})^2]=-\int\frac{\mu}{r^2}\,dr+C$ (Note)

or $\frac{1}{2}[(\dot{r})^2+(r\dot{\theta})^2]=\frac{\mu}{r}+C,$ ...(i)

which is the required energy equation.

Also for the central orbits we know $r^2\dot{\theta}=h.$ ...(ii)

which is the required equation of angular momentum.

Now if the particle is projected initially with velocity u at right angles to the radius at a distance c from the origin, then we have initially $r=c,\ \dot{r}=0$ and $r\dot{\theta}=u$ (Note)

$\therefore$ From (i) we have $\frac{1}{2}u^2=\frac{\mu}{c}+C$ ...(iii)

$\therefore$ From (i) and (iii) eliminating C we have

$$\frac{1}{2}\left[(\dot{r})^2+(r\dot{\theta})^2\right]-\frac{1}{2}u^2=\mu\left(\frac{1}{r}-\frac{1}{c}\right) \quad ...(iv)$$

And from (ii) we have $h=r.r\dot{\theta}=c.u$, initially.

$\therefore$ From (ii) we have $r^2\dot{\theta}=cu$ ...(v)

$\therefore$ From (iv) and (v) we have

$$\frac{1}{2}\left[(\dot{r})^2+\left(\frac{cu}{r}\right)^2\right]-\frac{1}{2}u^2=\mu\left(\frac{1}{r}-\frac{1}{c}\right)$$

or $$(\dot{r})^2+\frac{c^2u^2}{r^2}-u^2=2\mu\left(\frac{1}{r}-\frac{1}{c}\right)$$

or $$(\dot{r})^2=u^2\left(1-\frac{c^2}{r^2}\right)+\frac{2\mu}{c}\left(\frac{c}{r}-1\right)$$

$$=\left(\frac{c}{r}-1\right)\left[-u^2\left(1+\frac{c}{r}\right)+\frac{2\mu}{c}\right]$$

$$=\left[\frac{2\mu}{c}-u^2\left(1+\frac{c}{r}\right)\right]\left(\frac{c}{r}-1\right).$$ Hence proved.

Ex. 6. When a periodic comet is at its greatest distance from the sun, its velocity v is increased by a small quantity δv. Show that the comet's least distance from the sun is increased by

$$4\delta v\left\{\frac{a^2(1-e)}{\mu(1+e)}\right\}^{1/2}.$$

Solution. Let S be the occupied focus of the elliptic path and A, A' be the nearest and farthest points from S on the elliptic path, then $SA=a-ae$ and $SA'=a+ae$ (see figure 20 Page 37), if $2a$ is the major axis.

Also we know in an elliptic orbit velocity at any point at a distance r from the focus S is given by $v^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]$, ...(i)

where $2a$ is the major axis of the ellipse.

$\therefore$ If v be the velocity of the comet at A', then from (i) we have

$$v^2=\mu\left[\frac{2}{a+ae}-\frac{1}{a}\right]=\mu\left[\frac{2a-a-ae}{a\,(a+ae)}\right]$$

or

$$v^2=\frac{\mu}{a}\left[\frac{1-e}{1+e}\right] \qquad \text{...(ii)}$$

Also from § 9·Page 37 we know that for the disturbed orbit

$$\delta a=\frac{2va^2}{\mu}\delta v \qquad \text{...(iii)}$$

and

$$\delta e=\frac{(1-e^2)}{e}\cdot\frac{av^2-\mu}{\mu}\,\frac{\delta v}{v} \qquad \text{...(iv)}$$

Now from (ii) $\dfrac{av^2}{\mu}=\dfrac{1-e}{1+e}$ or $\dfrac{av^2}{\mu}-1=\dfrac{1-e}{1+e}-1$

or

$$\frac{av^2-\mu}{\mu}=\frac{-2e}{1+e}$$

$\therefore$ From (iv) we have $\delta e=\dfrac{(1-e^2)}{e}\cdot\left(\dfrac{-2e}{1+e}\right)\dfrac{\delta v}{v}$

or

$$\delta e=-\frac{2}{v}\,(1-e)\,\delta v \qquad \text{...(v)}$$

Also we know least distance from S is given by

$$SA=a-ae.$$

Let $SA=R$. Then $R=a-ae=a\,(1-e)$

$\therefore$ $\delta R=(1-e)\,\delta a-a\,\delta e$, differentiating both sides

or

$$\delta R=(1-e)\frac{2va^2}{\mu}\,\delta v-a\left(-\frac{2}{v}\right)(1-e)\,\delta v, \text{ from (iii), (v)}$$

$$=2a(1-e)\left[\frac{av}{\mu}+\frac{1}{v}\right]\delta v$$

$$=2a(1-e)\left[\frac{a}{\mu}\sqrt{\left\{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)\right\}}+\sqrt{\left\{\frac{a}{\mu}\left(\frac{1+e}{1-e}\right)\right\}}\right], \text{ from (ii)}$$

$$=\frac{2a^{3/2}\,(1-e)}{\sqrt{\mu}}\left[\sqrt{\left(\frac{1-e}{1+e}\right)}+\sqrt{\left(\frac{1+e}{1-e}\right)}\right]\delta v$$

$$=\frac{2a^{3/2}(1-e)}{\sqrt{\mu}}\left[\frac{(1-e)+(1+e)}{\sqrt{(1-e^2)}}\right]\delta v$$

or $\delta R=4\left[\dfrac{a^3\,(1-e)}{\mu\,(1+e)}\right]^{1/2}\delta v$, where δR is the increase in R i.e. the least distance from S.

EXERCISES ON PLANETARY MOTION

Ex. 1. If v_1 and v_2 are the linear velocities of a planet when it is respectively nearest and farthest from the sun, prove that

$$(1-e)\,v_1=(1+e)\,v_2$$

Ex. 2. If the velocity of a body in an elliptic orbit of major axis $2a$, is the same at a certain point P, whether the orbit is being described in a periodic time T about one focus S or in periodic time T' about the other focus S', prove that

$$SP=\frac{2aT'}{T+T'} \quad \text{and} \quad S'P=\frac{2aT'}{T+T'}.$$

Ex. 3. A body is moving in an ellipse about a centre of force in the focus; when it arrives at P, the direction of motion is turned through a right angle, the speed being unaltered, show that the body will describe an ellipse whose eccentricity varies as the distance of P from the centre.

Ex. 4. A particle is moving in an ellipse of eccentricity e, under the acceleration μ/r^2 to a focus, when the particle is nearest to a focus, the acceleration is suddenly replaced by an acceleration $\mu' r$ towards the centre of the ellipse. If the particle continues to move in the same ellipse, prove that $\mu=\mu'(1-e)^2a^3$.

Ex. 5. A particle describes a path which is nearly a circle about a centre of force μu^n at its centre. Find the condition that this may be a stable motion. (*Gorakhpur 91; Purvanchal 90*)

—:o:—

Impact

§ 1. **Impact.** When two bodies strike against each other, the state of striking is known as their impact. Impact is of two kinds **Direct and Oblique.**

Direct Impact. Impact between two bodies is said to be direct when the direction of motion of each of them before impact is along the common normal at their point of contact.

Oblique Impact. Impact between two bodies is said to be oblique when the direction of motion of one or both of them before impact is not along the common normal at their point of contact.

Line of Impact. The direction of this common normal at the point of contact is called the line of impact.

§ 2. **Coefficient of Restitution.**

When two bodies strike against each other, they remain in contact with each other for a very short interval of time before they separate. This short interval of time during which the impact lasts is divided into two parts of periods namely **period of compression and period of restitution**

During the period of compression the two bodies press each other and their surfaces get slightly compressed due to the elasticity of the bodies. This period of compression lasts from the moment of striking till the velocities of the bodies become the same and at that instant the compression is maximum and after this the period of restitution starts.

During the period of restitution, two bodies try to regain their original shapes and this period lasts from the moment of maximum compression till the bodies separate from each other.

Newton found by experiments that when two bodies collide their relative velocity after impact is in a constant ratio of their relative velocity before impact and is in opposite direction. This constant ratio is called **coefficient of restitution** and is denoted by the letter e.

Note. 1. e depends upon the nature of the substances which collide, it does neither depend upon the masses nor the velocities of the bodies.

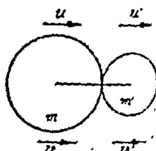
Note 2. The value of e lies between 0 and 1.

Note 3. The bodies are perfectly elastic when $e=1$ and the bodies are perfectly inelastic when $e=0$.

§ 3. **Direct Impact. *Two spheres of masses m and m′ collide* [illegible]*ectively, to determine motion* [illegible] *one and gained by the other is* [illegible] *is the coefficient of restitution*

(Gorakhpur 89)

Let v and v' be the velocities of the spheres after impact. Then by the Principle of conservation of momentum we have



(Fig. 1)

Total momenta before impact

= Total momenta after impact,

i.e. $mu+m'u' = mv+m'v'$ ...(i)

And by Newton's Experimental Law, we get $v-v' = -e(u-u')$ ...(ii)

Solving (i) and (ii) we get,

$$v = \frac{mu+m'u'-em'(u-u')}{m+m'}. \quad ...(iii)$$

$$v' = \frac{mu+m'u'+em(u-u')}{m+m'} \quad ...(iv)$$

Results (iii) and (iv) give the velocities of the spheres after impact

Now change in momentum of the 1st sphere of mass m

$$= m(v-u) = m\left[\frac{mu+m'u'-em'(u-u')}{m+m'} - u\right], \text{ from (iii)}$$

$$= m\left[\frac{m'u'-em'(u-u')-m'u}{m+m'}\right] = \frac{mm'}{m+m'}[(u-u')(-1-e)]$$

$$= -[mm'/(m+m')][(u-u')(1+e)], \quad ...(v)$$

which being negative, momentum is lost by this sphere of mass m.

Similarly change in momentum of the 2nd sphere of mass m'

$$= m'(v'-u') = m'\left[\frac{mu+m'u'+em(u-u')}{m+m'} - u'\right], \text{ from (iv)}$$

$$= m'\left[\frac{m(u-u')+em(u-u')}{m+m'}\right] = \frac{mm'}{(m+m')}(u-u')(1+e) \quad ...(vi)$$

which being +ve, momentum is gained by this sphere of mass m'.

Hence from (v) and (vi) we find that momentum lost by 1st sphere of mass m = momentum gained by 2nd sphere of mass m'.

$= [mm'/(m+m')][(u+u')(1+e)]$ Hence proved.

Cor. If the two spheres are equal and perfectly elastic then $m=m'$ and $e=1$. And from (iii) and (iv) we get

$$v = \frac{mu+mu'-m(u-u')}{m+m} = \frac{2u'}{2} = u'$$

and

$$v' = \frac{mu+mu'+m(u-u')}{m+m} = \frac{2u}{2} = u$$

i.e. the spheres interchange their velocities after impact.

§ 4. Loss of Kinetic Energy due to direct impact.

Two elastic spheres of masses m and m' moving with velocities

u and u' impinge directly. If e be the coefficient of restitution to find the loss of energy due to impact.

Let v and v' be the velocities of the spheres after impact.

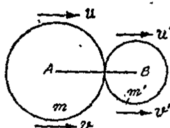
Then by the principle of coservation of momentum we have

$$mv+m'v'=mu+m'u' \quad ...(i)$$

And by Newton's Experimental Law we get

$$v-v'=e(u-u') \quad ..(ii)$$

To the sphere of (i) adding the square of (ii) multiplied by mm' we have



(Fig. 2)

(Note)

$(mv+mv')^2+mm'(v-v')^2$
$= mu+m'u')^2+mm'e^2(u-u')^2$

or $m^2v^2+m'^2v'^2+mm'v^2+mm'v'^2$
$=(mu+m'u')^2+e^2mm'(u-u')^2$

or $(m^2+mm')v^2+(m'^2+mm')v'^2$
$=(mu+m'u')^2+mm'(u-u')^2$
$-mm'(1-e^2)(u-u')^2$

or $m(m+m')v^2+m'(m+m')v'^2$
$=m^2u^2+m'^2u'^2+mm'u^2+mm'u'^2-mm'(1-e^2)(u-u')^2$
$=m(m+m')u^2+m'(m+m')u'^2-mm'(1-e^2)(u-u')^2$

or $mv^2+m'v'^2=mu^2+m'u'^2-\dfrac{mm'}{(m+m')}(1-e^2)(u-u')^2.$

or $(\tfrac{1}{2}mv^2+\tfrac{1}{2}m'v'^2)-(\tfrac{1}{2}mu^2+\tfrac{1}{2}m'u'^2)=-\dfrac{mm'}{2(m+m')}(1-e^2)(u-u')^2$

or (Total K E after impact)−(Total K. E befor impact)

$= \tfrac{1}{2}[mm'/(m+m')](1-e^2)(u-u')^2$

or Change in K.E $=-\tfrac{1}{2}[mm'/(m+m')](1-e^2)(u-u')^2$.
which being negative shows that K E. is lost due to impact.

$\therefore$ K Energy lost due to impact $=\dfrac{mm'}{2(m+m')}(1-e^2)(u-u')^2$

$$=\tfrac{1}{2}[mm'/(m+m')](1-e^2)V^2, \quad (iii)$$

where $V=u-u'$=the relative velocity before impact.

Corolary. If two inelastic spheres have direct impact then the Kinetic energy lost by the impact is that of a body whose mass is half the harmonic mean between those of the spheres, and whose velocity equals their relative velocity before impact

Proof. Putting $e=0$ in the result of the above article we have K.E. lost due to impact $=\tfrac{1}{2}[mm'/(m+m')]V^2$ Hence proved.

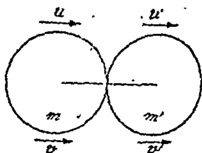
Note. If the elasticity is perfect, then $e=1$ and from (iii) there is no loss of Kinetic energy when the two smooth spheres impinge directly.

Solved Examples on § 3 – § 4 (Direct Impact)

Ex. 1 A sphere impinges directly on an equal sphere at rest,

If the coefficient of restitution be e, show that their velocities after impact are as $(1-e):(1+e)$.

Sol. Let u be the velocity of the impinging sphere before impact and v be its velocity after impact. Let v' be the velocity after impact of the other sphere which was at rest initially. Let m be the mass of each sphere. Then from the principle of conservation of momentum and Newton's Experimental Law we have



(Fig. 3)

$mv+mv'=mu+m.0$ ...(i)

and $v-v'=-e\,(u-0)$ ...(ii)

∴ from (i) we get $v+v'=u$ ...(iii)

and from (ii) we get $v-v'=-eu$ ...(iv)

Adding and subtracting (iii) and (iv) we have

$v=\frac{1}{2}(1-e)\,u$ and $v'=\frac{1}{2}u\,(1+e)$

∴ $v:v'=(1-e):(1+e)$ Hence proved.

Ex. 2. Two bodies of equal mass impinge directly. Show that if e is the coefficient of restitution, the amount of K. E. gained by [illegible]

[illegible]

Sol. Let the velocities before impact of 1st and 2nd bodies be $(3+e)\,u$ and $-(1-e)\,u$ respectively (Note the negative sign)

Let the velocities after impact of the bodies be v and v' respectively. Let m be the mass of each body. Then from the Principle of conservation of momentum we have

total momenta after impact = total momenta before impact

i.e. $mv+mv'=m\,(3+e)\,u-m\,(1-e)\,u$

or $v+v'=2\,(1+e)\,u$ ...(i)

And from Newton's experimental law we have

"$v-v'=-e\,(u-u')$"

or $v-v'=-e\,[(3+e)\,u-\{-(1-e)\,u\}]$ or $v-v'=-4eu$ ...(ii)

Solving (i) and (ii) we get

$v=u\,(1-e)$ and $v'=u\,(1+3e)$ ...(iii)

K.E. of 1st body after impact $=\frac{1}{2}mv^2=\frac{1}{2}m\,(1-e)^2\,u^2$. ...(iv)

∴ Change in K.E. of 1st body

$=$ K. E. after impact $-$ K. E. before impact

$=\frac{1}{2}m\,(1-e)^2\,u^2-\frac{1}{2}m\,(3+e)^2\,u^2$

$=\frac{1}{2}mu^2\,[1+e^2-2e-9-6e-e^2]$

$=-\frac{1}{2}mu^2\,[8+8e]=-4\,mu^2\,(1+e)$. ...(v)

negative sign shows that K. E. is lost by the 1st body.

Similarly K.E. of 2nd body before impact $=\frac{1}{2}m(1-e^2)u^2$
and K.E. of 2nd body after impact
$=\frac{1}{2}mv'^2=\frac{1}{8}m(1+3e)^2u^2$, from (iii)
$\therefore$ Change in K.E. of 2nd body $=\frac{1}{8}m(1+3e)^2u^2-\frac{1}{2}m(1-e)^2u^2$
$=\frac{1}{8}mu^2[1+9e^2+6e-1-e^2+2e]$
$=\frac{1}{8}mu^2[8e^2+8e]=e[\frac{1}{4}mu^2(1+e)]$, ...(v)
positive sign shows that K.E. is gained by 2nd body.

From (iv) and (v) we conclude that K.E. gained by 2nd body is e times the K.E. lost by the 1st body. Hence proved.

Ex 3 Two smooth spheres A and B of masses 4 and 8 kgs. move with velocities 9 and 3 metres/sec in opposite directions. If A rebounds with velocity 1 m. per second, find the velocity of B after impact, the coefficient of impact and the loss of kinetic energy.

Sol. Let v and v' be the velocities in m./second of the balls A and B after impact. Also we are given that the balls are moving in opposite directions before impact and $v=-1$ m./sec., ...(i)
since A rebounds after impact.

$\therefore$ From the principle of Conservation of momentum and Newton's experimental law we have

$4v+8v'=4[9+8(-3)]$ (Note) ...(ii)
and $v-v'=-e[9-(-3)]$ (Note) .. (iii)

$\therefore$ From (i) and (ii) we get $-4+8v'=12$ or $v'=2$ m./sec.
And from (i) and (ii) we get $-1-v'=-12e$
or $-1-2=-12e$, $\because v'=2$ m./sec. Ans.
or $12e=3$ or $e=\frac{1}{4}$.

Again K.E. of A after impact $=\frac{1}{2}(4)(-1)^2=2$ units;
K.E. of B after impact $=\frac{1}{2}(8)(2)^2=16$ units.
$\therefore$ Total K.E. after impact $=2+16=18$ units
K.E. of A before impact $=\frac{1}{2}(4)(9)^2=162$ units
K.E. of B before impact $=\frac{1}{2}(8)(-3)^2=36$ units
$\therefore$ Total K.E. before impact $=162+36=198$ units.
$\therefore$ Loss of K.E. $=198-18=180$ units.

Ex. 4. Three equal spheres are in a straight line on a table and one moves towards the other two which are at rest and not in contact. If $e=\frac{1}{4}$, find how many impact will take place and show that the ultimate speeds of the spheres are in the ratio 13 : 15 : 36.

Sol. Let m be the mass of each sphere. Let the sphere move with a velocity u towards the other two which are at rest. Then there will be an impact between first and second spheres. Let v and v' be the velocities of them after impact. For these two spheres from the principle of conservation of momentum and Newton's experimental law we have.

$mv+mv'=mu+m\cdot 0$ and $v-v'=-e(u-0)$
or $v+v'=u$ and $v-v'=-\frac{1}{4}u$, $\because e=\frac{1}{4}$ (given)
Solving $v=\frac{3}{8}u$ and $v'=\frac{5}{8}u$

Since $v' > v$, $\therefore$ second sphere moves faster than the first after impact, hence first will be left behind and second will strike third sphere at rest with a velocity $v' = \frac{3}{4}u$.

Let V and V' be their velocities after impact.

$\therefore$ for impact of second and third spheres we have

$$mV + mV' = m.\tfrac{3}{4}u + m.0$$

and $$V - V' = -e\left(\tfrac{3}{4}u - 0\right)$$

or $V + V' = \frac{3}{4}u$ and $V - V' = -\frac{3}{8}u$, $\because$ $e = \frac{1}{2}$ (given)

Solving $V = \frac{3}{16}u$ and $V' = \frac{9}{16}u$

After second impact (of 2nd and 3rd spheres) we observe that V' being greater than V, third sphere moves faster than 2nd and v (velocity of the 1st sphere after first impact) being greater than V (velocity of the second sphere after 2nd impact), the first sphere will strike the second. This will be the third impact If v_1 and v_2 be the velocities of 1st and 2nd spheres after this impact, then we have

$$m\,v_1 + m.v_2 = mv + mV = m\,\tfrac{1}{4}u + m\,\tfrac{3}{16}u$$

and $$v_1 - v_2 = -e(v - V) = -\tfrac{1}{2}\left(\tfrac{1}{4}u - \tfrac{3}{16}u\right)$$

i.e. $$v_1 + v_2 = 7u/16 \text{ and } v_1 - v_2 = -u/32$$

whence $$v_1 = \tfrac{13}{64}u \text{ and } v_2 = \tfrac{15}{64}u$$

Now $v_2 > v_1$ and $V' > v_2$; $\therefore$ there will be no further impact. Hence there are three impacts only and final velocities of the spheres are in the ratio.

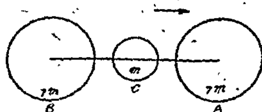
$$v_1 : v_2 : V' \text{ i.e } \frac{13u}{64} : \frac{15u}{16} : \frac{9u}{16} \text{ i.e. } 13 : 15 : 36.$$

Ex. 5. **The masses of three spheres, A, B, C are $7m$, $7m$, m; their coefficient of restitution is unity, their centres are in a straight line and C lies between A and B. Initially A and B are at rest and C is given a velocity along the line of centres towards A. Show that it strikes A twice and B once, and that final velocities of A, B, C are in the ratio 21 : 12 : 1.**

Sol. Let sphere C move with a velocity u towards A. Then first impact will be between A and C.

Impact of A and C spheres :

Let v and v' be the velocities of spheres A and C after impact. Then by the principle of conservation of momentum and Newton's experimental law, we have,



(Fig. 4)

$7mv+mv'=7m\,0+mu$ and $v-v'=-e\,(0-u)$.

or $7v+v'=u$ and $v-v'=u$, $\because\ e=1$ (given)

Solving $v=\frac{1}{4}u$ and $v'=-\frac{3}{4}u$

As v' is negative, so after impact C changes direction *i.e.* moves towards B which is at rest and thus second impact will be between B at rest and C moving with a velocity $3u/4$.

Impact of B and C spheres. Let V and V' be the velocities of spheres B and C after this impact. Then by the principle of conservation of momentum and Newton's Experimental Law we have

$$7mV+mV'=7m.0+m\,\tfrac{3}{4}u$$

and $\quad V-V'=-e\,(0-\frac{3}{4})$, $\because\ e=1$ (given)

Solving $7V=3u/16$ and $V'=-9u/16$ ...(i)

As V' is negative, so after second impact C changes direction *i.e.* moves towards A which is already moving with a velocity $u/4$. Since $9u/16>u/4$, so C will overtake A and third impact will be between C and A which are moving before third impact with velocities $9u/16$ and $u/4$ respectively.

Impact of A and C spheres for the second time. Let v_1 and v_2 be the velocities of A and C spheres after this impact.

Then from principle of conservation of momentum and Newton's Experimental Law we have

$$7m\,v_1+m\,v_2=7m\,(\tfrac{1}{4}u)+m(\tfrac{9}{16}u)$$

and $\quad v_1-v_2=-e\,(\frac{1}{4}u-\frac{9}{16}u)$

or $\quad 7v_1+v_2=\frac{37}{16}u$ and $v_1-v_2=\frac{5}{16}u$, $\because\ e=1$

Solving $v_1=\frac{21}{64}u$ and $v_2=\frac{1}{64}u$

Since $v_1>v_2$ so A sphere will move faster than C and there will be no further impact between A and C

Hence after this impact the final velocities of the spheres A, B and C are in the ratio $v_1:V:v_2$

i.e. $\frac{21}{64}u:\frac{3}{16}u:\frac{1}{64}u$ from (*i*) and (ii)

i.e. 21 : 12 : 1.

+Ex. 6. Three balls ... **the last two being at rest** ... **velocity u strikes the second which afterwards strikes the third. Find the velocity of the third ball after the impact**

****Prove that if the masses of the first and the third balls be given, the velocity of the third ball after impact is greatest if the masses are in geometric progression and that the velocity then is**

$$\frac{m_1\,(1+e)^2\,u}{(\sqrt{m_1}+\sqrt{m_3})^2}$$

Sol Let after impact of first and second balls the velocities of these be v_1 and v_2. Then from the principle of conservation of momentum and Newton's Experimental Law we have

$$m_1v_1+m_2v_2=m_1u+m_2\,0 \quad\text{and}\quad v_1-v_2=-e\,(u-0)$$

or $\quad m_1v_1+m_2v_2=u$ and $v_1-v_2=-eu$

Solving these we have $v_2 = \frac{(1+e)\,m_1 u}{(m_1+m_2)}$. ...(i)

Next impact takes place between second and third balls.

Let the velocity of third ball after impact be v_3. Before this impact second ball was moving with velocity v_2 and third was at rest. Then on the basis of (i) we have

$$v_3 = \frac{(1+e)\,m_2 v_2}{(m_2+m_3)} = \frac{(1+e)m_2}{(m_2+m_3)} \cdot \frac{(1+e)\,m_1 u}{m_1+m_2}, \text{ from (i)} \quad \text{(Note)}$$

$$= \frac{(1+e)^2 m_1 m_2 u}{(m_2+m_3)\,(m_1+m_2)},$$ which gives velocity of third ball after impact.

Rewriting $v_3 = (1+e)^2\, m_1 u. \frac{m_2}{m_2^2 + m_2(m_1+m_3) + m_1 m_3}$

or $$v_3 = (1+e)^2 . m_1 u . \frac{1}{m_2 + (m+m_3) + (m_1 m_3/m_2)} \quad ...(ii)$$

If m_1 and m_3 are given then only m_2 is variable on R.H.S. of the above result. Hence v_3 will be greatest when

$m_2 + (m_1+m_3) + [m_1 m_3/m_2]$ is least

i.e. when $[m_2 + (m_1 m_3/m_2)]$ is least as m_1+m_3 is given

i.e. when the first differential of $[m_2 + (m_1 m_3/m_2)]$ with respect to m_2 is zero

i.e. when $1-(m_1 m_3/m_2^2)=0$ *i.e.* when $m_2^2 = m_1 m_3$

i.e. when m_1, m_2 and m_3 are in G.P.

And then from (ii) we get

$$v_3 = (1+e)^2 m_1 u \frac{1}{\sqrt{(m_1 m_3)} + (m_1+m_3) + \sqrt{(m_1 m_3)}}$$

$$= \frac{(1+e)^2 m_1 u}{m_1+m_3+2\sqrt{(m_1 m_3)}} = \frac{(1+e)^2\, m_1 u}{(\sqrt{m_1}+\sqrt{m_3})^2}$$ Hence proved.

Ex. 6. Three elastic balls of masses m_1, m_2, m_3 lie in a straight line on a horizontal table and m_1 is projected towards m_2. If velocity of m_1 after striking m_2 is equal to that of m_2 after striking m_3, prove that

$$(m_1+m_2)\,(m_2+m_3)\,e = m_1 m_3\,(1+e)^2.$$

Sol. Let m_1 be projected towards m_2 with a velocity u (say). ... and m_2 their velocities be v_1 and v_2 respec... ...ntum and

...(i)

$m_1 v_1 + m_2 v_2$ ...

i.e. $m_1 v_1 + m_2 v_2 = m_1 u$ and $v_1 - v_2 = -eu$.

Solving these we get

$$v_1 = \frac{(m_1 - e m_2)\,u}{(m_1+m_2)} \text{ and } v_2 = \frac{(1+e) m_1 u}{(m_1+m_2)} \quad ...(i)$$

Now let m_2 strike m_3. Then before impact m_2 is moving with velocity v_2 whereas m_3 is at rest. Let after this impact the velocities of m_2 and m_3 be v_3 and v_4. Then on the basis of (i), we get

$$v_3=\frac{(m_2-em_3)\,v_2}{(m_2+m_3)}=\frac{(m_2-em_3)\,(1+e)\,m_1 u}{(m_2+m_3)\,(m_1+m_2)}, \text{ from (i)} \qquad \text{(Note)}$$

Now if $v_3=v_1$, then

$$\frac{(m_2-em_3)\,(1+e)\,m_1 u}{(m_2+m_3)\,(m_1+m_2)}=\frac{(m_1-em_2)\,u}{(m_1+m_2)}, \text{ from (i)}$$

or $(m_2-em_3)\,(1+e)\,m_1=(m_1-em_2)\,(m_2+m_3)$

or $[(m_2-em_3)\,(1+e)-(m_2+m_3)]\,m_1=-em_2\,(m_2+m_3)$

or $[m_2e-em_3-e^2m_3-m_3]\,m_1=-em_2\,(m_2+m_3)$

or $[-m_3\,(1+e+e^2)+m_2e]\,m_1=-em_2\,(m_2+m_3)$

or $[m_3\,(1+2e+e^2)-(m_2+m_3)\,e]\,m_1=em_2\,(m_2+m_3)$

or $m_1m_3\,(1+e)^2=e\,(m_2+m_3)\,m_2+em_1\,(m_2+m_3)$

or $m_1m_3\,(1+e)^2=e\,(m_2+m_3)\,(m_1+m_2)$ Hence proved.

Ex. 7. **A sphere impinges directly on an equal sphere which is at rest. Show that a fraction $\frac{1}{2}(1-e^2)$ of the original kinetic energy is lost during the impact.**

Solution. Let m be the mass of each sphere. Let the first sphere impinge second which is at rest with a velocity u. Let their velocities after impact be v and v' respectively. Then from the Principle of Conservation of momentum and Newton's experimental law, we have

$$mv+mv'=mu+m.0 \text{ and } v-v'=\;-e\,(u-0)$$

or

$$v+v'=u \text{ and } v-v'=-eu$$

Solving these we get

$$v=\tfrac{1}{2}(1-e)\,u \text{ and } v'=\tfrac{1}{2}(1+e)\,u \qquad \text{.. (i)}$$

Now kinetic energy before impact $=\frac{1}{2}mu^2+\frac{1}{2}m\,0=\frac{1}{2}mu^2$

and kinetic energy after impact $=\frac{1}{2}mv^2+\frac{1}{2}mv'^2$

$$=\tfrac{1}{2}m\,\tfrac{1}{4}\,(1-e)^2\,u^2+\tfrac{1}{2}m\,\tfrac{1}{4}\,(1+e)^2\,u^2, \text{ from (i)}$$

$$=\tfrac{1}{8}mu^2\,[(1-e)^2+(1+e)^2]=\tfrac{1}{4}mu^2\,(1+e^2)$$

$\therefore$ Loss of kinetic energy due to impact

= kinetic energy before impact − kinetic energy after impact

$=\frac{1}{2}mu^2-\frac{1}{4}mu^2\,(1+e^2)=\frac{1}{4}mu^2\,(2-1-e^2)=\frac{1}{4}(1-e^2)\,\frac{1}{2}mu^2$

$=\frac{1}{2}(1-e^2)$ of ...

Ex. 8. **A ... is itself reduced : ... the impact. If half of the initial K. E. is lost in collision, find e.**

Solution Let m be the mass of the ball which strikes the ball of mass m' at rest. Let u be the velocity before impact of the ball of mass m and v' that of ball of mass m' after impact.

Then from the Principle of conservation of momentum and Newton's Experimental law, we have

$$m\,0+m'v'=mv+m'.0 \text{ and } 0-v'=-e\,(u-0)$$

or

$$mu=m'v' \text{ and } v'=eu \qquad \text{...(i)}$$

Solving these we get $\qquad mu=m'\,(eu)$ or $m=m'e \qquad$ ...(ii)

K.E. before impact $=\frac{1}{2}mu^2+\frac{1}{2}m'\,(0)^2=\frac{1}{2}mu^2$

K.E. after impact $=\frac{1}{2}m'0)+\frac{1}{2}m'v'^2$

$$=\tfrac{1}{2}m'\,(eu)^2, \text{ from (i)}$$

Loss of K. E. due to impact
$= $ K.E. before impact $-$ K.E. after impact
$= \frac{1}{2}mu^2 - \frac{1}{2}m'e'u^2 + \frac{1}{2}(m'e)u^2 - \frac{1}{2}m'e^2u^2$, from (ii)
$= \frac{1}{2}m'u^2(e - e^2)$

According to the problem,

Loss of K.E. due to collision $= \frac{1}{4}$ (K.E. before impact)

i.e. $\frac{1}{2}m'u^2(e - e^2) = \frac{1}{4}(\frac{1}{2}mu^2)$
or $m'(e - e^2) = \frac{1}{2}(m'e)$, from (ii)
or $2(e - e^2) = e$ or $1 - 2e = 0$, $e \neq 0$
or $e = \frac{1}{2}$ Ans

Ex. 9. A series of n elastic balls whose masses are $1, e, e^2$ etc. are at rest, separated by intervals, with their centres on a straight line. The first is made to impinge directly on the second with velocity u. Show that finally the first $(n-1)$ balls will be moving with the same velocity $(1-e)u$ and the last with velocity u. Prove that the kinetic energy of the system is $\frac{1}{2}(1-e+e^n)u^2$.

Sol. Consider the impact of the first and second balls whose masses are 1 and e (given). The first ball strikes with a velocity u the second ball which is at rest. After impact let v_1 and v_2 be their velocities.

Then by the Principle of conservation of momentum and Newton's Experimental Law, we have

$$1 \cdot v_1 + e \cdot v_2 = 1 \cdot u + e \cdot 0 \text{ and } v_1 - v_2 = -e(u - 0)$$

i.e. $v_1 + ev_2 = u$ and $v_1 - v_2 = -eu$

Solving the equation we get $v_1 = (1-e)u$ and $v_2 = u$ ...(i)

Hence we conclude that if a ball strikes another ball at rest with a velocity u and the ratio of the mass of the ball at rest to that of the impinging ball be e, then after impact we have the velocity of the impinging ball $= (1+e)$ times its velocity before impact and the velocity of the second ball at rest $=$ the velocity of impinging ball.

Hence if $v_3, v_4, v_5, \ldots, v_{n-1}, v_n$ be the velocities of the remaining balls after impact, then we have

$$v_3 = v_4 = v_5 = v_6 = \ldots = v_{n-1} = (1-e)u \quad \ldots \text{(ii)}$$

and $v_n = u$ ...(iii)

Also from (i), $v_1 = (1-e)u$ and after impact of 2nd and 3rd balls velocity of 2nd ball $= (1-e)u = v'_2$ (say) ...(iv)

Also kinetic energy of the system after impact

$= \frac{1}{2} \cdot 1 \cdot v_1^2 + \frac{1}{2}ev_2'^2 + \frac{1}{2}e^2v_3^2 + \ldots + \frac{1}{2}e^{n-2}v_{n-1}^2 + \frac{1}{2}e^{n-1}v_n^2$ (Note)

$= \frac{1}{2}(1-e)^2u^2(1+e+e^2+\ldots+e^{n-2}) + \frac{1}{2}e^{n-1}u^2$, from (i), (ii), (iii) and (iv)

$$= \frac{1}{2}(1-e)^2u^2\left[\frac{1-e^{n-1}}{1-e}\right] + \frac{1}{2}e^{n-1}u^2$$

$= \frac{1}{2}u^2[(1-e)(1-e^{n-1}) + e^{n-1}] = \frac{1}{2}u^2(1-e+e^n)$ Hence proved.

Exercise on § 3 – § 4

Ex. 1. Two spheres of masses M, m impinge directly when

moving in opposite directions with speeds u, v respectively and the sphere of mass m is brought to rest by the collision, prove that

$$v\,(m-eM)=M\,(1+e)\,u$$

After the collision, the sphere of mass M is acted on by a constant retarding force which brings it to rest after travelling a distance a. Prove that the magnitude of this force is $Ma^2\,(u+v)^2/2$.

Ex. 2. If two balls of masses m, m' moving with velocities u, u' impinge directly prove that the condition that each loses the same amount of kinetic energy is

$$(3+e)\,(mu+m'u')+(1-e)\,(m'u+mu')=0.$$

Ex. 3. If two inelastic spheres have direct impact, show that the kinetic energy lost by the impact is that of a body whose mass is half the harmonic mean between the mass of the two impinging spheres and whose velocity is equal to their relative velocity before impact.

[Hint : See § 4 cor. Page 3 of this chapter]

Ex 4. Two elastic spheres impinge directly with equal and opposite velocities. Find the ratio of their masses so that one of them may be reduced to rest by the impact, the coefficient of restitution being e.

Ex. 5. Two perfectly inelastic bodies of masses m_1 and m_2 moving with velocities u_1 and u_2 in the same direction impinges directly. Show that the loss of kinetic energy due to impact is

$$\tfrac{1}{2}\,[m_1m_2/(m_1+m_2)]\,(u_1-u_2)^2.$$

[Hint : See cor. of § 4 Page 3 of this chapter]

Oblique Impact

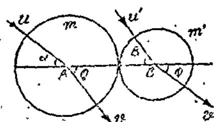
§ 5. Newton's Experimental Law on oblique impact.

When two bodies strike each other obliquely, then relative velocity after impact resolved along their common normal bears a constant ratio to their relative velocity before impact resolved in the same direction and is of opposite sign. This constant ratio is e.

** § 6. Oblique impact of two imperfectly elastic spheres

(Gorakhpur 89, 87; Purvanchal 91, 88)

Let two smooth imperfectly elastic spheres of masses m and m' strike each other obliquely with velocities u and u' making angles α and β with AB, the line joining the centres A and B of the spheres. Let after impact v and v' be their velocities making angles θ and ϕ with the line AB.



(Fig 5)

The spheres being smooth, there is no force between them

at right angles to the common normal AB. On each sphere the only force is along the line of centres *i.e.* the common normal. Hence velocities at right angles to the comman normal AB remain unaltered due to impact, so we get

$$v \sin\theta = u \sin\alpha \quad ...(i)$$

and

$$v' \sin\phi = u' \sin\beta \quad ...(ii)$$

Also by the Principle of conservation of momentum, we have

$$mv \cos\theta + m'v' \cos\phi = m\, u \cos\alpha + m'u' \cos\beta \quad ...(iii)$$

Also by Newton's Experimental Law (See § 5 above) we get

$$v \cos\theta - v' \cos\phi = -e\,(u \cos\alpha - u' \cos\beta) \quad ...(iv)$$

Equations (i), (ii), (iii) and (iv) are sufficient to find out the values of the four unknown quantities v, v', θ and ϕ

From (iii) and (iv) we get [multiplying (iv) by m' and adding to (iii)].

$$v \cos\theta = \frac{mu \cos\alpha + m'u' \cos\beta - em'(u \cos\alpha - u' \cos\beta)}{(m+m')} \quad ...(v)$$

Again from (iii) and (iv) we get [multiplying (iv) by m and subtracting from (iii)].

$$v' \cos\phi = \frac{mu \cos\alpha + m'u' \cos\beta + em(u \cos\alpha - u' \cos\beta)}{(m+m')} \quad ...(vi)$$

Squaring and adding (i) and (v) we have the value of v^2 or v and dividing (i) by (v) we have the value of $\tan\theta$ or θ.

Similarly from (ii) and (vi) we can find the value of v' and ϕ.

Cor 1. If $m = m'$ and $e = 1$, we have from (v) and (vi)

$$v \cos\theta = u' \cos\beta \text{ and } v' \cos\phi = u \cos\alpha$$

i.e. if a moving smooth sphere impinges another moving smooth sphere of equal mass obliquely, then they interchange their velocities resolved in the direction of the line of centres (Gorakhpur 92)

Cor. 2. If $u_1 = 0$ we have from (ii), $\phi = 0$.

i.e. if moving smooth sphere impinges obliquely another sphere at rest, the latter begins to move in the sense of the common normal

**§ 7. Loss of kinetic energy due to oblique impact.

(Purvanchal 91)

Proceeding exactly as in § 6 Pages 11–12 of this chapter we can obtain the results (i), (ii), (iii) and (iv).

Adding mm' times the square of (iv) to the square of (iii), we have

$$(mv \cos\theta + m'v' \cos\phi)^2 + mm'(v \cos\theta - v' \cos\phi)^2$$
$$= (mu \cos\alpha + m'u' \cos\beta)^2 + mm'\, e^2 (u \cos\alpha - u' \cos\beta)^2$$

or

$$(m^2 + mm')\, v^2 \cos^2\theta + (m'^2 + mm')\, v'^2 \cos^2\phi$$
$$= (mu \cos\alpha + m'u' \cos\beta)^2 + mm'\, e^2 (u \cos\alpha - u' \cos\beta)^2$$

or

$$(m+m')\, mv^2 \cos^2\theta + (m+m')\, m'v'^2 \cos^2\phi$$
$$= (mu \cos\alpha + m'u' \cos\beta)^2 + mm'e^2 (u \cos\alpha - u' \cos\beta)^2$$
$$+ mm'(u \cos\alpha - u' \cos\beta)^2 - mm'(u \cos\alpha - u' \cos\beta)^2 \text{ (Note)}$$

introducing the term $mm'(u \cos\alpha - u' \cos\beta)^2$ with positive and

negative signs.

or $(m+m')[mv^2\cos^2\theta+m'v'^2\cos^2\phi]$

$$=(m+m')(mu^2\cos^2\alpha+m'u'^2\cos^2\beta)$$
$$-mm'(1-e^2)(u\cos\alpha-u'\cos\beta)^2$$

or $\frac{1}{2}mv^2\cos^2\theta+\frac{1}{2}m'v'^2\cos^2\phi=\frac{1}{2}mu^2\cos^2\alpha+\frac{1}{2}m'u'^2\cos^2\beta$

$$-\frac{1}{2}\frac{mm'}{(m+m')}(1-e^2)(u\cos\alpha-u'\cos\beta)^2 \quad ...(vii)$$

Also squaring and adding (i) and (ii) we get

$\frac{1}{2}mv^2\sin^2\theta+\frac{1}{2}m'v'^2\sin^2\phi=\frac{1}{2}mu^2\sin^2\alpha+\frac{1}{2}m'u'^2\sin^2\beta$ ...(viii)

Adding (viii) and (vii) we have $\frac{1}{2}mv^2+\frac{1}{2}m'v'^2=\frac{1}{2}mu^2+\frac{1}{2}m'u'^2$

$$-\frac{1}{2}\cdot\frac{m'm}{(m+m')}(1-e^2)(u\cos\alpha-u'\cos\beta)^2$$

or $(\frac{1}{2}mv^2+\frac{1}{2}m'v'^2)-(\frac{1}{2}mu^2+\frac{1}{2}m'u'^2)$

$$=-\frac{1}{2}\cdot\frac{mm'}{(m+m')}(1-e^2)(u\cos\alpha-u'\cos\beta)^2$$

or (K.E. after impact)$-$(K.E. before impact)

$$=-\frac{1}{2}\cdot\frac{mm'(1-e^2)}{(m+m')}(u\cos\alpha-u'\cos\beta)^2,$$

which being negative, K.E. is lost due to impact.

∴ Amount of K.E. lost

$$=\frac{1}{2}\cdot\frac{mm'}{(m+m')}(1-e^2)(u\cos\alpha-u'\cos\beta)^2$$
$$=\frac{1}{2}\cdot\frac{mm'}{(m+m')}(1-e^2)V^2,$$

where V is the relative velocity of the two spheres before impact in the direction of the common normal i.e. $V=(u\cos\alpha-u'\cos\beta)$.

§ 8 Impulse of the blow. Let I be the impulse of the blow on each sphere.

The impulse of blow on the sphere of mass m=change in the momentum of mass m in the direction of the impulse.

i.e. $$-I=m(v\cos\theta-u\cos\alpha) \quad ...(ix)$$

negative sign is due to the fact that the direction of I on m is in direction BA i.e. opposite to the direction of $u\cos\alpha$ and $v\cos\theta$ which are in the direction AB. See Fig. 5 Page 11 of this chapter.

Again for the impulse on m', we have

$$I=m'(v'\cos\phi-u'\cos\beta) \quad ...(x)$$

here the direction of I and $v'\cos\phi$, $u'\cos\beta$, are in the same sense i.e. in the sense of AB.

From (ix) and (x) we have

$$(I/m)+(I/m')=(v'\cos\phi-u'\cos\beta)-(v\cos\theta-u\cos\alpha)$$

or $I[(1/m)+(1/m')]=(u\cos\alpha-u'\cos\beta)-(v\cos\theta-v'\cos\phi)$

$$=(u\cos\alpha-u'\cos\beta)+e(u\cos\alpha-u'\cos\beta),$$

from (iv) Page 11

$$=(1+e)(u\cos\alpha-u'\cos\beta)$$

or $I=\frac{mm'}{(m+m')}(1+e)(u\cos\alpha-u'\cos\beta)=\frac{mm'}{(m+m')}(1+e).V,$

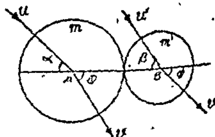
where V is the relative velocity of the sphere before impact in the direction of the common normal.

Solved Examples on § 5 § 8 (Oblique Impact).

Ex. 1. The velocity of a ball A, after its oblique impact on a ball B, has the same magnitude and direction as the velocity of B before impact. Show that the ratio of the masses of the balls is equal to the coefficient of restitution.

Sol. Let m and m' be the massses of the balls. u, u' be their velocities before impact making angles α and β respectively with the line of centres.

Let v and v' be their velocities after impact making angles θ and ϕ respectively with the line of centres. Since we are given that the velocity of A after impact has the same magnitude and direction as the velocity of B before impact, so we get $v=u'$ and $\theta=\beta$ ...(i)



(Fig 6)

Now by the Principle of conservation of momentum, we have

$m'v'\cos\phi+mv\cos\theta=m'u'\cos\beta+mu\cos\alpha$

or $m'(v'\cos\phi-u'\cos\beta)=m(u\cos\alpha-v\cos\theta)$

or $m'(v'\cos\phi-u'\cos\beta)=m(u\cos\alpha-u'\cos\beta)$, ...(ii)

since from (i) $v=u'$ and $\theta=\beta$

Also by *Newton's Experimental Law we have*

$v'\cos\phi-v\cos\theta=-e\,(u'\cos\beta-u\cos\alpha)$

or $(v'\cos\phi-u'\cos\beta)=e\,(u\cos\alpha-u'\cos\beta)$, ...(iii)

since from (i) $v=u'$ and $\theta=\beta$. Hence proved

***Ex. 2. If two equal perfectly elastic spheres impinge obliquely show that they interchange their velocities in the direction of the line of centres.** *(Gorakhpur 92)*

Sol. Refer figure 6 of last example, remembering that here $m=m'$, balls being equal.

Then as in last example, with the same notations from principle of conservation of momentum and Newton's Experimental Law we have $v'\cos\phi+v\cos\theta=u'\cos\beta+u\cos\alpha$ ...(i)

and $v'\cos\phi-v\cos\theta=-(u'\cos\beta-u\cos\alpha)$ ...(ii)

(Here $m'=m$ and $e=1$)

Adding and subtracting (i) and (ii) we get

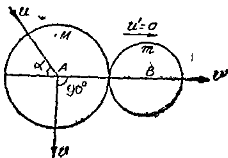
$v' \cos \phi = u \cos \alpha$; $v \cos \theta = u' \cos \beta$

i.e. in the direction of line of centres velocity after impact of the second ball = velocity before impact of the first ball.

And the velocity after impact of the first ball = velocity before impact of the second ball.

Ex. 3. A ball of mass M strikes another ball of mass *m* at rest. If they separate in mutually perpendicular directions, prove that the coefficient of elasticity is equal to the ratio of their masses.

Sol. Let u be the velocity before impact of the ball of mass M making an angle α with the line joining the centres. The ball of mass m before impact was at rest. Since during the period of impact the only force acting in the direction of the common normal, so the ball of mass m will move in the sense of the common normal after impact.



(Fig. 7)

Again as the balls separate in mutually perpendicular directions, so ball of mass M will move after impact in a direction at right angles to the common normal. Let v and v' be the velocities of the masses M and m after impact. By the principle of conservation of momentum, we get

$M v \cos 90° + m v' = M u \cos \alpha + m \cdot 0$, in the direction of common normal, or $mv' = Mu \cos \alpha$ ... (i)

Also by Newton's Experimental Law, we get

$v' - v \cos 90° = -e (0 - u \cos \alpha)$ or $v' = eu \cos \alpha$ ... (ii)

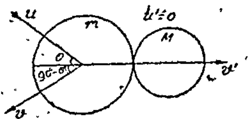
Dividing (i) by (ii) we have $m = M/e$ or $e = M/m$

or e = ratio of masses of the balls. Hence proved.

Ex. 4. A smooth sphere of mass m, travelling with velocity u impinges ...

... through a right angle if $\tan^2 \theta = (eM - m)/(M + m)$.

Sol. After impact the second ball of mass M should move along the line of common normal because the force on it during the period of impact is only along the common normal. Let v' be its velocity after impact. Also the impinging ball moves after impact at right angles to the former direction. Let v be its velocity after impact.



(Fig. 8)

Since there is no force in the direction perpendicular to the line joining the centres, so for ball of mass m, we have

$$u \sin \theta = v \sin (90° - \theta) \quad \text{or} \quad u \sin \theta = v \cos \theta \quad \ldots(i)$$

Also by principle of conservation of momentum, we have

$$Mv' + m\{-v \cos (90° - \theta)\} = M.0 + Mu \cos \theta$$

or $$Mv' - mv \sin \theta = mu \cos \theta \quad \ldots(ii)$$

And by Newton's experimental Law, we have

$$v' - \{-v \cos (90° - \theta)\} = -e\,(0 - u \cos \theta)$$

or $$v' + v \sin \theta = eu \cos \theta \quad \ldots(iii)$$

Multiplying (iii) by M and subtracting from (ii) we get

$$-mv \sin \theta - Mv \sin \theta = mu \cos \theta - M\,eu \cos \theta$$

or $v\,(m + M) \sin \theta = u\,(eM - m) \cos \theta$

or $[(u \sin \theta)/\cos \theta]\,(m + M) \sin \theta = u\,(eM - m) \cos \theta$, from (i)

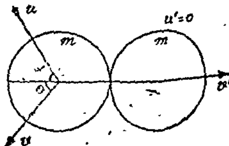
or $\tan^2 \theta = (eM - m)/(m + M)$ Hence proved.

Ex. 5. A smooth ball impinges on another equal ball at rest in a direction making an angle α with the line of centres. If the coefficient of elasticity of the two balls is 1/3, prove that the angle through which the direction of the impinging ball is deviated is

$$\tan^{-1} \{2 \tan \alpha/(1 + 3 \tan^2 \alpha)\}.$$

Sol. Let after impact the velocity of the impinging ball be v making an angle θ with the direction of the common normal.

Also after impact the second ball should move along the direction of the common normal because the force on it during the period of impact is only along the common normal. Let v' be its velocity after impact.



(Fig. 9)

In the direction perpendicular to the line joining the centres, three being no force we have $u \sin \alpha = v \sin \theta$. .. (i)

By the Principle of conservation of momentum, we get

$$mv' + m(-v \cos \theta) = m.0 + m\,u \cos \alpha,$$

where m is the mass of each ball

or $$v' - v \cos \theta = u \cos \alpha \quad \ldots(ii)$$

By Newton's Experimental Law, we have

$$v' - (-v \cos \theta) = -e\,(0 - u \cos \alpha), \quad \text{where } e = \tfrac{1}{3}$$

or $$v' + v \cos \theta = \tfrac{1}{3} u \cos \alpha \quad \ldots(iii)$$

Solving (iii) and (ii), we get $v \cos \theta = -\tfrac{1}{3} u \cos \alpha$

or $[u \sin \alpha/\sin \theta] \cos \theta = -\tfrac{1}{3} u \cos \alpha$, from (i)

or $$\tan \theta = -3 \tan \alpha \quad \ldots(iv)$$

$\therefore$ Required angle $= \theta + \alpha$ (see figure).

$$= \tan^{-1} [\tan (\theta + \alpha)] = \tan^{-1} \left[\frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha}\right]$$

$$=\tan^{-1}\left[\frac{-3\tan\alpha+\tan\alpha}{1-(-3\tan\alpha)\tan\alpha}\right], \text{ from (iv)}$$

$$=\tan^{-1}\left[\frac{-2\tan\alpha}{1+3\tan^2\alpha}\right], \text{ numerically,}$$

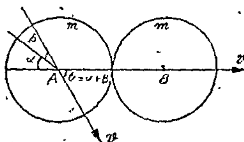
(negative sign shows this angle is obtuse).

*Ex. 6. A smooth sphere impinge obliquely on an equal sphere at rest. Before impact the first sphere was moving in a direction making an angle α with the line of centres at the moment of impact. If the direction of motion of the first sphere is turned through an angle β by the impact, show that

$$\tan\beta=[(1-e)\tan\alpha]/(1-e+2\tan^2\alpha].$$

Show also that the max deviation is $\sin^{-1}[(1+e)/(3-e)]$.

Sol. Let u be the velocity before impact of the impinging sphere making an angle α with the common normal. The second sphere being at rest before impact will move along the common normal after impact as the force on it during the period of impact is only along the common normal. Let v' be its velocity after impact.



(Fig. 10)

Let v be the velocity after impact of the impinging sphere making an angle θ with the common normal. As the direction of motion of this sphere is turned through an angle β by the impact, so

$$\theta=\alpha+\beta \qquad \text{(i) (See figure)}$$

Let m be the mass of each sphere.

Then by the Principle of Conservation of momentum, we get

$$mv\cos\theta+mv'=mu\cos\alpha+m.0 \quad \text{or} \quad v\cos\theta+v'=u\cos\alpha \qquad \text{(ii)}$$

And by the Newton's Experimental law, we have

$$(v\cos\theta-v')=-e\,(u\cos\alpha-0)$$

or

$$v\cos\theta-v'=-eu\cos\alpha \qquad \text{(iii)}$$

Also as there is no force in the direction perpendicular to the common normal so we have for the impinging sphere

$$v\sin\theta=u\sin\alpha \qquad \text{(iv)}$$

From (ii) and (iii) we can show that

$$v\cos\theta=\tfrac{1}{2}(1-e)\,u\cos\alpha \qquad \text{(v)}$$

Dividing (iv) by (v) we get $\tan\theta=(2\tan\alpha)/(1-e)$ .. (vi)

Also from (i) we get $\beta=\theta-\alpha$ or $\tan\beta=\tan(\theta-\alpha)$

or

$$\tan\beta=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}=\frac{\{2\tan\alpha/(1-e)\}-\tan\alpha}{1+2\{\tan^2\alpha/(1-e)\}}, \text{ from (vi)}$$

or

$$\tan\beta=\{(1+e)\tan\alpha\}/(1-e+2\tan^2\alpha) \qquad \text{(vii)}$$

Hence proved.

Now deviation will be max. when $\tan\beta$ is maximum

i.e. when $\{(1+e)\tan\alpha\}/(1-e+2\tan^2\alpha)$ is maximum

i.e. $\dfrac{1-e+2\tan^2\alpha}{\tan\alpha}$ is minimum, $(1+e)$ being constant.

i.e. when $(1-e)\cot\alpha+2\tan\alpha$ is minimum.

i.e. when $\dfrac{d}{d\alpha}\left[(1-e)\cot\alpha+2\tan\alpha\right]=0$ and second differential coefficient is positive.

i.e. when $-(1-e)\operatorname{cosec}^2\alpha+2\sec^2\alpha=0$

i.e. when $(1-e)=2\tan^2\alpha$ or $\tan\alpha=\sqrt{\{\tfrac{1}{2}(1-e)\}}$

And then from (vii), $\tan\beta=\dfrac{(1+e)\sqrt{\{(1-e)/2\}}}{1-e+(1-e)}=\dfrac{(1+e)\sqrt{(1-e)}}{2\sqrt{2}(1-e)}$

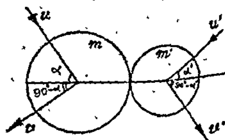
or $\tan\beta=\dfrac{(1+e)}{2\sqrt{\{2(1-e)\}}}$ or $\sin\beta=\dfrac{1+e}{3-e}$

or $\beta=\sin^{-1}[(1+e)/(3-e)]$, for max. deviation. Proved.

Ex. 7. Two balls whose masses are m and m' and moving in the directions, making angles α, α' with the line of centres, collide. If their directions of motion after the impact be perpendicular to their direction before impact; prove that

$$e=(m\sin^2\alpha'+m'\sin^2\alpha)/(m\cos^2\alpha'+m'\cos^2\alpha).$$

Sol. Let v and v' be velocities after impact of the balls of masses m and m' respectively, making angles $90°-\alpha$ and $90°-\alpha'$ with the common normal, as their directions of motion after impact are perpendicular to their directions before impact. Let u and u' be their velocities before impact.



(Fig. 11)

Since there is no force in the direction perpendicular to the common normal, so we have

$$u\sin\alpha=v\sin(90°-\alpha) \quad\text{or}\quad u\sin\alpha=v\cos\alpha \qquad ...(i)$$

and $$u'\sin\alpha'=v'\sin(90°-\alpha') \quad\text{or}\quad u'\sin\alpha'=v'\cos\alpha' \qquad ...(ii)$$

Also by the Principle of conservation of momentum, we get

$$m'v'\cos(90°-\alpha')+m\{-v\cos(90°-\alpha)\}=mu\cos\alpha+m'\{-u'\cos\alpha'\}$$

or $$m'v'\sin\alpha'-mv\sin\alpha=mu\cos\alpha-m'u'\cos\alpha' \qquad ...(iii)$$

By Newton's Experimental Law, we have

$$v'\cos(90°-\alpha')-\{-v\cos(90°-\alpha)\}=-[(-u'\cos\alpha')-(u\cos\alpha)]$$

$$v'\sin\alpha'+v\sin\alpha=e(u'\cos\alpha'+u\cos\alpha) \qquad ...(iv)$$

From (iv), $e = \frac{v' \sin \alpha' + v \sin \alpha}{u' \cos \alpha' + u \cos \alpha}$

$= \frac{\left(\frac{u' \sin \alpha'}{\cos \alpha'}\right) \sin \alpha' + \left(\frac{u \sin \alpha}{\cos \alpha}\right) \sin \alpha}{u' \cos \alpha' + u \cos \alpha}$, from (i) and (ii)

or $e = \frac{(u'/\cos \alpha') \sin^2 \alpha' + (u/\cos \alpha) \sin^2 \alpha}{u' \cos \alpha' + u \cos \alpha}$ .. (v)

From (i), $m (u \cos \alpha + v \sin \alpha) = m' (v' \sin \alpha' + u' \cos \alpha')$

or $m\left[u \cos \alpha + \left(\frac{u \sin \alpha}{\cos \alpha}\right) . \sin \alpha\right] = m'\left[\left(\frac{u' \sin \alpha'}{\cos \alpha'}\right) \sin \alpha' + u' \cos \alpha'\right]$, from (i) and (ii)

or $m [u/\cos \alpha] = m' [u'/\cos \alpha'] = k$ (say)

or $\frac{u}{\cos \alpha} = \frac{k}{m}$ and $\frac{u'}{\cos \alpha'} = \frac{k}{m'}$

Substituting in (v) we get

$$e = \frac{(k/m') \sin^2 \alpha' + (k/m) \sin^2 \alpha}{\{(k \cos \alpha')/m'\} \cos \alpha' + \{(k \cos \alpha)/m\} \cos \alpha}$$

$$= \frac{m \sin^2 \alpha' + m' \sin^2 \alpha}{m \cos^2 \alpha' + m' \cos^2 \alpha}$$ Hence proved.

Ex. 8. Two spheres of masses m and m', moving with velocities u and u' at right angles to one another, collide. Prove that if their direction of motion after impact are also at right angles, then

$$e = \frac{m^2 u \cos \alpha + m'^2 u' \sin^2 \alpha}{mm' (u \cos \alpha + u' \sin \alpha)},$$

where the direction of u makes angle α with the line of centres. *(Purvanchal 92, 89)*

Sol. The spheres of masses m and m' are moving with velocities u and u' before impact making angles α and $90° - \alpha$ with the line of centres. Let these spheres move with velocities v and v' after impact making angles β and $90° - \beta$ with the line of centres.

Since there is no force in the direction perpendicular to the common normal, so we have $u \sin \alpha = v \sin \beta$ ...(i)

and $u' \sin (90° - \alpha) = v' \sin (90° - \beta)$ or $u' \cos \alpha = v' \cos \beta$ .. (ii)

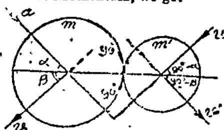
By the Principle of Conservation of Momentum, we get

$m' v' \cos (90° - \beta) + m (-v \cos \beta) = mu \cos \alpha + m' \{-u' \cos (90° - \alpha)\}$.

$m'v' \sin \beta - mv \cos \beta = mu \cos \alpha - m'u' \sin \alpha$ ...(iii)

And by Newton's Experimental Law, we get

$v' \cos (90° - \beta) - (-v \cos \beta) = -e [\{-u' \cos (90° - \alpha)\} - u \cos \alpha]$



(Fig. 12)

or $\quad v' \sin\beta + v\cos\beta = e\,[u\cos\alpha + u'\sin\alpha]$

or $\quad e = \dfrac{v'\sin\beta + v\cos\beta}{u\cos\alpha + u'\sin\alpha} = \dfrac{\left(\dfrac{u'\cos\alpha}{\cos\beta}\right)\sin\beta + \left(\dfrac{u\sin\alpha}{\sin\beta}\right)\cos\beta}{u\cos\alpha + u'\sin\alpha}$,

from (i) and (ii)

or $\quad e = \dfrac{u'\cos\alpha\tan\beta + u\sin\alpha\cot\beta}{u\cos\alpha + u'\sin\alpha}$ ...(iv)

From (iii), $m\,(u\cos\alpha + v\cos\beta) = m'\,(v'\sin\beta + u'\sin\alpha)$

or $m\left[u\cos\alpha + \left(\dfrac{u\sin\alpha}{\sin\beta}\right)\cos\beta\right] = m'\left[\left(\dfrac{u'\cos\alpha}{\cos\beta}\right)\sin\beta + u'\sin\alpha\right]$,

from (i) and (ii)

or $\quad mu\,[\cos\alpha + \sin\alpha\cot\beta] = m'u'\,[\cos\alpha\tan\beta + \sin\alpha]$

or $\quad mu\,[\sin(\alpha+\beta)/\sin\beta] = m'u'\,[\sin(\alpha+\beta)/\cos\beta]$

or $\quad \tan\beta = mu/m'u'$ and $\cot\beta = m'u'/mu$.

Substituting these values in (iv), we get

$$e = \frac{u'\cos\alpha\,(mu/m'u') + u\sin\alpha\,(m'u'/mu)}{u\cos\alpha + u'\sin\alpha}$$

$$= \frac{m^2 u\cos\alpha + m'^2 u'\sin\alpha}{mm'\,(u\cos\alpha + u'\sin\alpha)}$$

Hence *proved.*

[illegible] tic spheres *moving* [illegible] Show that after [illegible] h other.

(Gorakhpur 92)

Sol. Refer Fig. 12 Page 19 of this chapter.

Let m be the mass of each sphere and u, u' be their velocities before impact making angles α and $90° - \alpha$ with the line of centres as shown in the figure. Let v and v' be their velocities after impact making angles β and γ with the line of centres. (In the figure take γ for $90° - \beta$).

$\because$ There is no force at right angles to the common normal,

so we have $\quad u\sin\alpha = v\sin\beta$ .. (i)

and $\quad u'\sin(90° - \alpha) = v'\sin\gamma$ (Note)

i.e. $\quad u'\cos\alpha = v'\sin\gamma$ ...(ii)

By the Principle of Conservation of Momentum, we get

$mv'\cos\gamma + m\,(-v\cos\beta) = m\,\{-u'\cos(90° - \alpha)\} + mu\cos\alpha$

or $\quad v'\cos\gamma - v\cos\beta = u\cos\alpha - u'\sin\alpha$ ...(iii)

And by Newton's Experimental Law, we get

$v'\cos\gamma - (-v\cos\beta) = -1\,[\{-u'\cos(90° - \alpha)\} - u\cos\alpha]$

or $\quad v'\cos\gamma + v\cos\beta = u'\sin\alpha + u\cos\alpha$ ...(iv)

Adding and subtracting (iii) and (iv) we get

$v'\cos\gamma = u\cos\alpha$ ...(v) and $v\cos\beta = u'\sin\alpha$ ...(v)

From (ii) and (v) we have $[(u'\cos\alpha)/\sin\gamma]\cos\gamma = u\cos\alpha$,

eliminating v'.

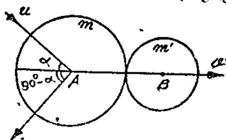
or $\cot \gamma = \frac{u}{u'} = \frac{u \sin \alpha}{u' \sin \alpha} = \frac{v \sin \beta}{v \cos \beta}$, from (i) and (vi)

or $\cot \gamma = \tan \beta$ or $\gamma = 90° - \beta$

i.e. the balls after impact move at right angles to each other.

Ex. 10. A smooth ball strikes another which is at rest. the angle betwen the line of centres of the balls and the direction of motion of the first ball is α. The ball moves off in a direction perpendicular to its original direction. Show that the masses of the balls are in the ratio $(e \cos^2 \alpha - \sin^2 \alpha) : 1$, where e is the coefficient of restitution.

Sol Let the masses of the ball at rest and the impinging ball be m' and m respectively. Let u be the velocity of the impinging ball before impact. Since the force during impact is in the sense of the common normal, so the ball of mass m' moves in the sense of the common normal after impact and let its velocity after impact be v'. Let v be the velocity after impact of the impinging ball which is moving off in a direction perpendicular to its original direction *i.e.* making an angle $90° - \alpha$ with the common normal.



(Fig. 13)

Since in the direction perpendicular to the common normal there is no force, so there is no change in velocity in that direction and we have $u \sin \alpha = v \sin (90 - \alpha)$ or $u \sin \alpha = v \cos \alpha$ (i)

Also by the Principle of Conservation of Momentum, we get

$m'v + m\{-v \cos (90° - \alpha)\} = m'0 + m.u \cos \alpha$

or $m'v' - mv \sin \alpha = mu \cos \alpha$ ...(ii)

And by Newton's Experimental Law, we get

$v' - \{-v \cos (90° - \alpha)\} = -e [0 - u \cos \alpha]$

or $v' + v \sin \alpha = eu \cos \alpha$ .. (iii)

From (ii), $m'v' = m [u \cos \alpha + v \sin \alpha]$

or $= m [u \cos \alpha + (u \sin \alpha / \cos \alpha) \sin \alpha]$, from (i)

$m'v' = mu [1/\cos \alpha]$ or $v' = mu/(m' \cos \alpha]$

Also from (i) we get $v = u \sin \alpha / \cos \alpha$

Substituting these values of v and v' in (iii) we have

$$\frac{mu}{m' \cos \alpha} + \left(\frac{u \sin \alpha}{\cos \alpha}\right) . \sin \alpha = eu \cos \alpha$$

or $$\frac{m}{m' \cos \alpha} = e \cos \alpha - \frac{\sin^2 \alpha}{\cos \alpha} = \frac{e \cos^2 \alpha - \sin^2 \alpha}{\cos \alpha}$$

or $m/m' = (e \cos^2 \alpha - \sin^2 \alpha)/1$. Hence proved.

Ex. 11. Two balls of elasticity e, moving in a parallel direction with equal momenta impinge, prove that if their directions of motions be opposite, they will move after impact in parallel directions with equal momenta.

Sol. Let m and m' be the masses of the balls and u and u' their velocities before impact each making an angle α with the common normal. Also u and u' are in opposite directions. (In the figure reverse the direction of u').

Let the velocities of these balls after impact be v and v' making angles θ and ϕ with the common normal.

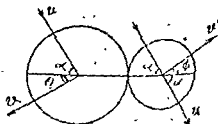
Since there is no force acting on the balls in the direction perpendicular to the common normal, so we get

$v \sin \theta = u \sin \alpha$ ...(i)

and $v' \sin \phi = u' \sin \alpha$ ...(ii)

Also the momenta before impact of the balls are given to be equal, so

$mu = m'u'$ ...(iii)



(Fig. 14)

By the Principle of Conservation of momentum we have

$m'v' \cos \phi + m(-v \cos \theta)$

$= m'(-u' \cos \alpha)$

$+ m(u \cos \alpha)$ ...(iv)

or $m'v' \cos \theta - mv \cos \theta = (mu - m'u') \cos \alpha = 0$, from (iii)

or $\dfrac{v \cos \theta}{v' \cos \phi} = \dfrac{m'}{m} = \dfrac{u}{u'}$; from (iii)

$= \dfrac{u \sin \alpha}{u' \sin \alpha}$, multiplying num. and denom. by $\sin \alpha$

$\therefore$ from (i) and (ii), $\dfrac{v \cos \theta}{v' \cos \phi} = \dfrac{v \sin \theta}{v' \sin \phi}$ or $\dfrac{\cos \theta}{\cos \phi} = \dfrac{\sin \theta}{\sin \phi}$

or $\tan \theta = \tan \phi$ or $\theta = \phi$

i.e. directions of motions of the balls after impact are parallel.

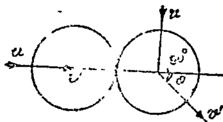
Also from (iii) and (iv) we have $m'v' \cos \phi - mv \cos \theta = 0$.

or $m'v' = mv$, since $\theta = \phi$ (proved)

i.e. balls move with equal momenta after impact.

Ex. 12. A ball impinges on another equal ball moving with the same speed in a direction perpendicular to its own, the line joining the centres of the balls at the instant of impact being perpendicular to the direction of motion of the second ball. If e be the coefficient of elasticity, show that direction of motion of second ball is turned through an angle $\tan^{-1} \{(1+e)/2\}$.

Sol. Let u be the velocity before impact of each ball. One ball is moving in the direction of the common normal and the other is moving in a direction perpendicular to it (as shown in the figure). Let v and v' be the velocities of these balls after impact the former moving in the direction of the common normal i.e. the direction in which it was originally moving (there being no change in the direction of motion of this ball as the only force acting on it during the period of impact is also in this sense) and the latter moving in a direction making an angle θ with the common normal. Let m be the mass of each ball.



(Fig. 15)

As the velocity remains unaltered in the direction perpendicular to the line joining the centres, so we have $v' \sin\theta = u$ .. (i)

Also by the Principle of Conservation of momentum, we have

$$mu + mu\cos 90° = mv + mv'\cos\theta$$

or $$v + v'\cos\theta = u \qquad ...(ii)$$

And by Newton's Experimental Law we get

$$v - v'\cos\theta = -e\,(u - u\cos 90°) = -eu \qquad ...(iii)$$

Solving (i) and (iii) we have $v' = (1+e)\,u/(2\cos\theta)$. (iv)

∴ From (i) and (iv) we have $u = [(1+e)\,u\,\sin\theta]/(2\cos\theta)$

or $\tan\theta = 2/(1+e)$

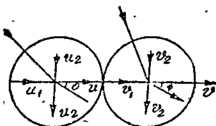
∴ Required angle $= 90° - \theta = \tan^{-1}\{\tan(90° - \theta)\}$

$$= \tan^{-1}(\cot\theta) = \tan^{-1}\{(1+e)/2\}.$$

*Ex. 13. Two equal spheres of elasticity e impinge, havi before impact velocities u_1, v_1, in the direction of the comm on normal and u_2, v_2 perpendicular to this normal. If after impact be spheres move perpendicular to each other, prove that

$$(u_1 + v_1)^2 + 4u_2 v_2 = e^2 (u_1 \quad v_1)^2$$

Sol. After impact only the velocity of the sphere in the direction of the line joining the centres will be altered. Let them be u and v after impact, as shown in the figure. The velocities in direction perpendicular to the common normal will remain u_2 and v_2 as shown in th figure. After impact the direc-



(Fig. 16)

tion of the motion of the spheres are mutually perpendicular, therefore product of their gradients $=-1$.

i.e $\tan\theta \; \tan\phi=-1$, where θ and ϕ are the angles which the directions of motion of spheres after impact make with the common normal

i.e. $$\frac{u_1}{u} \cdot \frac{v_2}{v}=-1.$$ (Note) (See figure)

or $$u_2v_2=-uv \qquad \ldots(i)$$

Also from Principle of conservation of momentum, we get

$$mu+mv=mu_1+mv_1 \quad \text{or} \quad u+v=u_1+v_1 \qquad \ldots(ii)$$

And by Newton's Experimental Law, we get

$$v-u=-e\,(v_1-u_1) \qquad \ldots(iii)$$

Squaring and subtracting (ii) and (iii) we get

$$(u_1+v_1)^2-e^2\,(v_1-u_1)^2=(u+v)^2-(v-u)^2=4uv$$

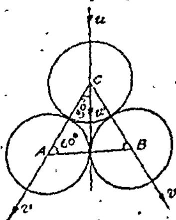
$$=-4u_2v_2, \text{ from (i)}$$

or $$(u_1+v_1)^2+4u_2v_2=e^2\,(u_1-v_1)^2$$ Hence proved.

Ex. 14. Two equal spheres of mass m are in contact on a smooth horizontal table. A third equal ball of mass m' impinges symmetrically on them and is reduced to rest. Prove that $e=2m'/3m$ and find the loss of K.E due to impact.

Sol. The mass of the spheres with centres A and B are m each and that with C is m'. Let u be the velocity before impact of the impinging sphere with centre C whose direction of motion is along the common tangent of the spheres with centres A and B lying in contact at rest on the table.

During the period of [illegible] ing sphere at right angles [illegible] fore it continues to move [illegible] velocity u' (say) and the force on each of the two equal spheres of mass m is along the normal common with the impinging sphere (*i.e.* the line CA or CB as the case may be). Therefore each equal sphere of mass m moves after impact along the common normal whose inclination to the common tangent is 30°, since ABC is an equilateral triangle whose each side $=$ twice the radius of each sphere.



(Fig. 17)

Let v be the velocity after impact of each of the two balls with centres A and B.

By the Principle of conservation of momentum, we have

$$m'u=m'u'+2mv\cos\tfrac{1}{6}\pi \quad \text{(Note)}$$

or $$m'u=m'u'+mv\sqrt{3} \qquad \ldots(i)$$

Considering the impact of the impinging ball with one of the two balls of mass m, from Newton's Experimental Law, we have

$$u' \cos \tfrac{1}{6}\pi - v = -e\,(u \cos \tfrac{1}{6}\pi - 0) \qquad \text{(Note)}$$

or

$$u'\sqrt{3} - 2v = -eu\sqrt{3} \qquad \text{(ii)}$$

But we are given that after Impact, the impinging ball is reduced to rest i.e. $u'=0$ then from (i) and (ii) we get

$$m'u = mv\sqrt{3} \quad \ldots\text{(iii)} \quad \text{and} \quad 2v = eu\sqrt{3} \quad \ldots\text{(iv)}$$

From (iii) and (iv) we have $\dfrac{m'}{e\sqrt{3}} = \dfrac{m\sqrt{3}}{2}$ or $2m' = 3em$

or $\qquad e = 2m'/3m.$ Hence proved.

Also K.E. before impact $= \frac{1}{2}\, m'u^2$

And K.E. after impact $= 2 \cdot \frac{1}{2}\, mv^2 = mv^2 = m\,(\frac{1}{2}eu\sqrt{3})^2$, from (iv)

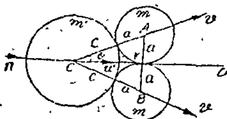
$= \frac{3}{4}me^2u^2 = \frac{3}{4} \cdot \frac{2}{3}m'eu^2$; since $2m' = 3\,em$

$= \frac{1}{2}m' \cdot eu^2.$

$\therefore$ Loss of K.E. $= \frac{1}{2}m'u^2 - \frac{1}{2}\, m'\,eu^2 = \frac{1}{2}\,(1-e)\, m'u^2.$ **Ans.**

****Ex. 15.** Two equal balls of radius a are in contact and are struck simultaneously by a ball of radius c moving in the direction of their common tangent; if all the balls be of the same material the coefficient of elasticity being e, prove that the impinging ball will be reduced to rest if $2e = c^3\,(a+c)^2/a^2\,(2a+c)$. *(Purvanchal 93)*

Sol. Let m' be the mass of the impinging ball with centre C and m be the mass of each ball at rest with centres A and B.



(Fig. 18)

Let u be the velocity before impact of the impinging ball.

Since during the period of impact there is no force on the impinging ball at right angles to its original direction of motion, therefore it continues to move in the original direction after impact with velocity u' (say) and the force on each of the equal balls is along the normal common with the impinging ball $\therefore$ Each equal ball of mass m moves after impact along that common normal whose inclination to the common tangent CD is θ (say). Let v be the velocity of each of these two balls after impact.

By the Principle of Conservation of momentum, we get

$$m'u = m'u' + 2mv \cos\theta \qquad \text{(i)}$$

Consider the impact of the impinging ball with one of the two balls of mass m, from Newton's Experimental Law we have

$u \cos \theta - v = -e(u \cos \theta - 0)$ or $u' \cos \theta - v = -eu \cos \theta$ ∴ ...(ii)

If the impinging ball be reduced to rest after impact, then $u' = 0$ and from (i) and (ii) we get $m'u = 2mv \cos \theta$ ...(iii)

and $v = eu \cos \theta$ ...(iv)

Substituting value of u from (iv) in (iii) we get

$m'u = 2m\, eu \cos \theta \cos \theta$ or $m' = 2em \cos^2 \theta$ ...(v)

Let ω be the mass per unit volume of the material of the spheres, then $\dfrac{m}{m'} = \dfrac{\frac{4}{3}\pi a^3 \omega}{\frac{4}{3}\pi c^3 \omega} = \dfrac{a^3}{c^3}$. ...(vi)

Also from $\triangle AKC$, we find that $\sin \theta = AK/AC = a/(a+c)$

$\therefore \cos^2 \theta = 1 - \sin^2 \theta = 1 - \{a^2/(a+c)^2\} = \dfrac{c(c+2a)}{(a+c)^2}$

Substituting this value in (v) we get

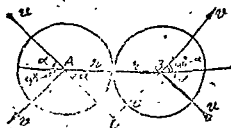
$\dfrac{m'}{m} = \dfrac{2ec(c+2a)}{(a+c)^2}$ or $\dfrac{c^3}{a^3} = \dfrac{2ec(c+2a)}{(a+c)^2}$, ∵ from (vi) $\dfrac{m}{m'} = \dfrac{a^3}{c^3}$

or $2e = c^2(a+c)^2/a^3(c+2a)$. Hence proved.

***Ex. 16. Two equal smooth spheres of radius r, move with the same speed in opposite directions in parallel lines which are at a distance c apart; prove that the direction of motion of each deviates on account of impact, through a right angle if $(1+e)c^2 = 4er^2$, e being the coefficient of restitution.** *(Purvanchal 90)*

Sol. Before impact let u be the velocity of each sphere making an angle α with the line joining the centres but their velocities are given to be in opposite directions.

Due to symmetry their velocities after impact will also be equal in magnitude say v, as they deviate through a right angle, hence after impact their directions will make angles of $(90° - \alpha)$ with the common normal



(Fig. 19)

By Newton's Experimental Law, we have

$v \cos (90° - \alpha) - [-v \cos (90° - \alpha)] = -e[(-u \cos \alpha) - (u \cos \alpha)]$ (Note)

or $v \sin \alpha = eu \cos \alpha$ ...(i)

Also as the velocities of the spheres in the direction perpendicular to the common normal remain unaltered, so we get

$u \sin \alpha = v \sin (90° - \alpha)$ or $v \cos \alpha = u \sin \alpha$ ...(ii)

Dividing (i) by (ii) we get $\tan \alpha = e \cot \alpha$ or $\tan^2 \alpha = e$ ...(iii)

Again from the figure we find that in $\triangle ABC$,

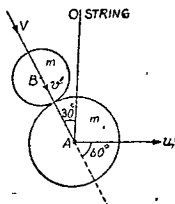
$\sin \alpha = BD/AB = c/2r$ or $\tan \alpha = c/\sqrt{(4r^2 - c^2)}$.

Substituting this value in (iii) we get $e=c^2/(4r^2-c^2)$

or $4er^2=c^2(1+e)$. Hence proved.

Ex. 17. A sphere is suspended from a fixed point by an inextensible string A second sphere of small radius and equal mass m moving downwards in a direction making an angle of 30° with the vertical impinges direct on the first spheres with speed V. If the coefficient of restitution between the spheres be $\frac{1}{2}$; prove that the initial velocity of the first sphere after the impact is 3V/5. Calculate also the impulsive force in the string at the moment of impact.

Sol. Due to the string, the sphere of mass m, suspended from the fixed point O, will be compelled to move in a direction perpendicular to the string (i.e. the horizontal direction) after impact.



(Fig. 20)

The other ball with centre B will remain moving in the same direction of the common normal BA after impact as it was moving before impact.

Let u and v' be the velocities after impact of the spheres with centres A and B respectively.

Since there is no constraint in the horizontal direction, so from Principle of conservation of momentum, in the horizontal direction, we get

$mu+mv'\cos 60°=m.0+mV\cos 60°$ or $2u+v'=V$ ...(i)

Also by Newton's Experimental Law, (in the sense of the common normal), we get $v'-u\cos 60°=-e(V-0)$

or $v'-u(\frac{1}{2})=-\frac{1}{2}(V)$ or $u-2v'=V$ (ii)

Solving (i) and (ii) we get $u=3V/5$ and $v'=-V/5$.

Also if T be the impulse of the tension in the string at the moment of impact, then T= change in momentum along the string

$=m(V\cos 30°)-m(v'\cos 30°)$ (Note)

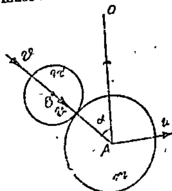
$=(m\cos 30°)(V-v')=m\frac{1}{2}\sqrt{3}(1+\frac{1}{5})V$, $\because v'=-\frac{1}{5}V$

$=\frac{3}{5}\sqrt{3}m\ V$ units of impulse. Ans.

***Ex. 18. A smooth sphere of mass m is tied to a fixed point by an inextensible string and another sphere of mass m' impinges directly on it with velocity v in a direction making an acute angle α with the thread. Show that m begins to move with velocity**

$$\frac{m'(1+e)v\sin\alpha}{m+m'\sin^2\alpha}$$

where e is the coefficient of restitution.



(Fig. 21)

Sol. Due to the string, the sphere of mass m will be compelled to move in a direction perpendicular to the string (*i.e.* the horizontal direction) after impact.

The other sphere of mass m' will remain moving in the same direction, of the common normal after impact as it was moving before impact.

Let u and v' be the velocities of the spheres of masses m and m' after impact

Since there is no constraint in the horizontal direction so from Principle of conservation of momentum we have in the horizontal direction

$$m'v \sin \alpha = mu + m'v' \sin \alpha \quad \text{.. (i)}$$

Also by Newton's Experimental Law (in the sense of common normal), $v' - u \sin \alpha = -e(v-0)$ or $v' = u \sin \alpha - ev$ .. (ii)

Substituting the value of v' from (ii) in (i), we get.

$$m'v \sin \alpha = mu + m'(u \sin \alpha - ev) \sin \alpha$$

or $$m'(1+e) v \sin \alpha = (m + m' \sin^2 \alpha) u$$

or $$u = m'(1+e) v \sin \alpha / (m + m' \sin^2 \alpha).$$ Hence proved.

Exercises on § 5 - § 8

Ex 1 A ball moving with a velocity u impinges obliquely on another ball at rest. If the two balls are smooth, perfectly elastic and equal in mass, prove that they move at right angles after impact.

Ex 2. Two equal balls are in contact on a table, a third equal ball strikes them simultaneously and remains at rest after impact. Show that the coefficient of restitution is $\frac{2}{3}$.

(Hint : Use $m = m'$ in Ex. 14 Page 24 of this chapter).

***Ex. 3** Two equal balls are lying in contact on a smooth table, and a third equal ball moving along their common tangent, strikes them simultaneously Prove that $\frac{3}{5}(1-e^2)$ of its kinetic energy is lost by the impact, e being the coefficient of restitution for each pair of balls.

Ex. 4. A sphere of mass m impinges obliquely on a sphere of mass M which is at rest. Show that if $m = eM$, the direction of motion after impact are at right angles.

**§9. Impact of a smooth sphere against a fixed plane.

(Gorakhpur 90)

Let the sphere strike the fixed plane at A with a velocity u making an angle α with the normal to the plane at A. Let v be the velocity of the sphere after impact making an angle θ with the nor-

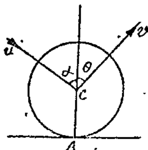
mal to the plane at A. α and θ are called angles of incidence and reflection respectively.

By Newton's Experimental Law in the sense of the common normal, we have

$$v \cos \theta - 0 = -e\,[(-u \cos \alpha) - 0]$$ (Note)

or $$v \cos \theta = eu \cos \alpha \quad \ldots(i)$$

The plane being smooth, there is no force acting on the sphere in the direction perpendicular to the common normal at A, hence the velocity of the sphere remains unchanged in the direction perpendicular to the common normal



(Fig. 22)

$$\therefore \quad v \sin \theta = u \sin \alpha \quad \text{(ii)}$$

Dividing (i) by (ii) we get $\cot \theta = e \cot \alpha$ (iii)

(Gorakhpur 90)

Squaring and adding (i) and (ii) we get

$$v^2 = u^2 (\sin^2 \alpha + e^2 \cos^2 \alpha) \quad \ldots(iv)$$

Equations (iv) and (iii) give the velocity and direction of motion of the sphere after impact.

Also from (i) we conclude that *velocity of the sphere after impact along the normal = e times its velocity before impact along the normal*

Loss of kinetic energy $= \frac{1}{2}mu^2 - \frac{1}{2}mv^2$

$$= \tfrac{1}{2}mu^2 - \tfrac{1}{2}mu^2 (\sin^2\alpha + e^2 \cos^2\alpha), \text{ from (iv)}$$
$$= \tfrac{1}{2}mu^2 (1 - \sin^2 \alpha \quad e^2 \cos^2 \alpha)$$
$$= \tfrac{1}{2}mu^2 (1 - e^2) \cos^2 \alpha$$

Impulse of the blow : The force acting on the sphere during the period of impact is in the sense of the common normal, hence the implse of the blow = change in momentum of the sphere in the direction of the common normal.

$$= m\,(v \cos \theta) - m\,(-u \cos \alpha) = m\,(v \cos \theta + u \cos \alpha)$$
$$= m\,[eu \cos \alpha + u \cos \alpha], \text{ from (i)}$$
$$= m\,(1+e)\,u \cos \alpha$$

**Cor. 1. If $\alpha = 0$ then $\theta = 0$ and $v = eu$, from (iii) and (iv) *i.e. when the impact is direct, the direction of motion of the sphere is reversed after impact, and its velocity after impact is e times its velocity just before impact*

Cor. 2. If $e = 1$ then $\theta = \alpha$ and $v = u$, from (iii) and (iv) *i.e. when a perfec... its veloci... angle of r...*

Cor. 3. If $e = 0$, then $\theta = 90°$ and $v = u \sin \alpha$, from (iii) and (iv) *i.e. when a sphere strikes against a fixed inelastic plane, the sphere slides along the plane with a velocity u sin α (parallel to the plane) and does not rebound.*

(Note)

Solved Examples on § 9.

Ex. 1. A particle falling vertically from a height impinges on a horizontal fixed plane and rebounds to a height h_1. Show that $h_1=e^2h$, where e is the coefficient of restitution between the sphere and the plane.

Sol. Let u be the velocity of the particle when it impinges (*i.e.* strikes) the horizontal plane, then $u=\sqrt{(2gh)}$ ...(i)

∵ The velocity after rebound $=e$ times the velocity before rebound (See cor. 1 of § 9 Page 29).

∴ The particle leaves the plane after impinging with velocity

$$=eu=e\sqrt{(2gh)}.$$

As the particle comes to momentary rest after rising a height h_1 (given) so from "$v^2=u^2+2fs$" we get

$$0=[e\sqrt{(2gh)}]^2+2(-g)\,h_1 \qquad \text{(Note)}.$$

or $$2gh_1=(2gh)\,e^2 \quad \text{or} \quad h_1=e^2h$$ Hence proved.

***Ex. 2.** A sphere falls from a height h above a horizontal plane and rebounds continually, show that the whole distance described by the particle is $(1+e^2)\,h/(1-e^2)$, and that the whole time before it comes to rest is $\sqrt{(2h/g)}\,[(1+e)/(1-e)]$, e being the coefficient of restitution.

Sol. Let u be the velocity of the sphere when it strikes the horizontal plane for the first time, then $u=\sqrt{(2gh)}$ ... (i)

Since the velocity after rebound $=e$ times the velocity before rebound. (See Cor. 1 § 9 Page 29)

∴ The velocities of successive rebounds are

$$eu,\ e^2u,\ e^3u,\ e^4u, ...$$

Also from "$v^2=u^2+2fs$" the distance travelled by the particle after 1st impact and before second impact $=2\,[(0-e^2u^2)/(-2g)]$, this includes the distance after 1st impact to the highest point reac- ... the

and 4th impacts etc. are

$$(2/g)\,(e^2u)^2,\ (1/g)\,(e^3u)^2,\ (1/g)\,(e^4u)^2 ...$$

∴ Total distance covered $=h+(1/g)\,(eu)^2+(1/g)\,(e^2u)^2+...$ ...ad. inf. (Note)

$=h+(1/g)\,(eu)^2\,[1+e^2+e^4+\text{...ad. inf.}]$

$=h+\dfrac{e^2u^2}{g}\left[\dfrac{1}{1-e^2}\right]$, since sum of an infinite G P. $=\dfrac{a}{1-r}$

$=h+\dfrac{e^2\,(2gh)}{g\,(1-e^2)}$, ∵ from (i) $u^2=2gh$

$=\dfrac{h\,(1-e^2)+2e^2h}{(1-e^2)}=\dfrac{(1+e^2)\,h}{(1-e^2)}$ Hence proved.

Also time between the first and second impact $=2\times$(time after 1st impact to reach the highest point attained by the sphere before it starts falling to have the second impact).

$=2\,[(0-eu)/(-g)]$, from '$v=u+ft$'

$=(2/g)\,(eu)$.

Similarly times between 2nd and 3rd impacts, 3rd and 4th impacts etc. are $(2/g)\,(e^2u)$, $(2/g)\,(e^3u)$, $(2/g)\,(e^4u)$,...

Also the time taken by the spheres from its point of start to the instant when it strikes the plane for the first time is given by

i.e. "$s=ut+\frac{1}{2}ft^2$"

$h=0+\frac{1}{2}gt^2$ or $t=\sqrt{(2h/g)}$

The whole time taken by the sphere before it ceases rebounding

$=\sqrt{(2h/g)}+(2/g)\,(eu)+(2/g)\,(e^2u)+(2/g)\,(e^3u)+$.. ad. inf.

$=\sqrt{(2h/g)}+(2/g)\,(eu)\,[1+e+e^2+$. ad. inf.]

$=\sqrt{(2h/g)}+(2/g)\,(eu)\,[1/(1-e)]$,

since sum of an infinite G.P. $=a/(1-r)$

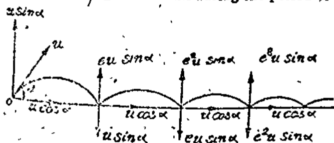
$=\sqrt{(2h/g)}+\dfrac{2e\,[\sqrt{(2gh)}]}{g\,(1-e)}$, since $u^2=2gh$ from (i)

$=\sqrt{(2h/g)}\left[1+\dfrac{2e}{1-e}\right]=\sqrt{\left(\dfrac{2h}{g}\right)\left(\dfrac{1+e}{1-e}\right)}$. Hence proved.

Ex. 3. From a point on a smooth horizontal plane a ball is projected with velocity u at an angle α to the horizon. Show that it will keep rebounding from the plane for a time $\dfrac{2u\sin\alpha}{g\,(1-e)}$ and will have a range $(u^2\sin 2\alpha)/g\,(1-e)$, e being the coefficient of elasticity.

Sol. Let the particle be projected from O. The vertical and horizontal components of initial velocity are $u\sin\alpha$ and $u\cos\alpha$ respectively.

The vertical component becomes $eu\sin\alpha$ after the first rebound and the horizontal component remains $u\cos\alpha$, since there is no horizontal force acting on the ball during the period of impact.



(Fig. 21)

Therefore the particle begins to describe trajectories on the horizontal plane.

Every time the particle rebounds after striking the horizontal plane, only the vertical velocity is altered and becomes e times of that before striking. Thus the vertical velocities for successive trajectories (right from the start) are $u \sin \alpha$, $eu \sin \alpha$, $e^2 u \sin \alpha$, ..

∴ Total time taken in describing all the trajectories (remembering that the time of flight for a horizontal range $= \left(\frac{2u \sin \alpha}{g} \right)$

$$= \frac{2u \sin \alpha}{g} + \frac{2\,(eu \sin \alpha)}{g} + \frac{2\,(e^2 u \sin \alpha)}{g} + \ldots \text{ad. inf.}$$

$$= \frac{2u \sin \alpha}{g} [1 + e + e^2 + \ldots \text{ad. inf.}] = \frac{2u \sin \alpha}{g\,(1-e)}.$$ Hence proved.

Also the horizontal component of velocity remains unaltered and equal to $u \cos \alpha$ throughout the motion, so the required range

$= (u \cos \alpha) \times$ total time taken

$$= (u \cos \alpha) . \frac{2u \sin \alpha}{g\,(1-e)} = \frac{u^2 \sin 2\alpha}{g\,(1-e)}$$ Hence proved.

Ex. 4. **A ball is projected from a point in a horizontal plane and makes one rebound, show that if the second range is equal to the greatest height, the angle of projection is $\tan^{-1} (4e)$, e being the coefficient of elasticity.**

Sol. Let u be the velocity of projection making an angle α with the horizon. Then as in last example we can prove that the initial vertical component of velocity for first and second ranges are $u \sin \alpha$ and $eu \sin \alpha$, whereas the horizontal component of velocity remains constant and equal to $u \cos \alpha$ throughout the motion.

∴ Second range $= (2/g)$ (Horizontal comp. of velocity) $\times$ (Initial vertical velocity for the range)

$= (2/g)\,(u \cos \alpha)\,(eu \sin \alpha) = (2/g)\,eu^2 \sin \alpha \cos \alpha$ ...(i)

And the greatest height attained in the first range

$= (u^2 \sin^2 \alpha)/(2g)$

According to the problem, second range = greatest height

i.e. $(2/g)\,eu^2 \sin \alpha \cos \alpha = (u^2 \sin^2 \alpha)/(2g)$, from (i) and (ii)

or $4e \cos \alpha = \sin \alpha$ or $\tan \alpha = 4e$ or $\alpha = \tan^{-1} (4e)$

Exercise on § 9

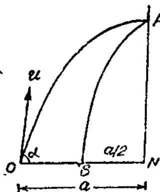
A particle falls from a height 'h' in time 't' upon a horizontal plane. Prove that it rebounds and reaches the maximum height '$e^2 h$' in time '$e^2 t$'.

**Solved Examples on Impact with the smooth vertical walls.

Ex. 1. **A particle is projected with velocity u from a point on the ground so as to strike a smooth vertical wall and after impact to strike the ground at a point midway between the point of**

projection and the wall; show that the angle of projection is $\frac{1}{2}\sin^{-1}\{(1+2e)ga/2eu^2\}$, where a is the distance of the ball from the point of projection and e is the coefficient of restitution between the particle and the wall.

Sol. Let O be the point of projection at a distance a from the wall. The velocity of projection is given as u. Let the angle of projection be α.



(Fig. 24)

The vertical and horizontal components of initial velocity are $u \sin \alpha$ and $u \cos \alpha$ respectively.

Since horizontal component of velocity from O to A, where the particle strikes the wall, remains $u \cos \alpha$ uniformly, so the time from O to A

$=a/(u \cos \alpha)$,

After striking the wall at A, the horizontal component of velocity becomes $eu \cos \alpha$.

∴ Time from A to $B = \dfrac{(\frac{1}{2}a)}{eu \cos \alpha} = \dfrac{a}{2eu \cos \alpha}$,

where B is the point midway between O and the wall.

∴ Total time from O to A and A to B

$$= \frac{a}{u \cos \alpha} + \frac{a}{2\, eu \cos \alpha} = \frac{a\,(1+2e)}{2eu \cos \alpha}$$

In this time vertical distance travelled is zero.

Also as the vertical component of velocity is not affected by the impact at A, so from "$s=ut+\frac{1}{2}ft^2$" considering motion from O to B

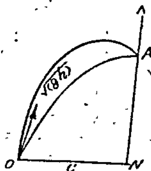
we have $\quad 0=u \sin \alpha \left[\dfrac{a\,(1+2e)}{2eu \cos \alpha}\right]-\frac{1}{2}g\left[\dfrac{a\,(1+2e)}{2eu \cos \alpha}\right]^2$

or $\quad u \sin \alpha = \dfrac{ag\,(1+2e)}{4eu \cos \alpha}$ or $2 \sin \alpha \cos \alpha = \{ag\,(1+2e)/2eu^2\}$

or $\quad \sin 2\alpha = \{ag\,(1+2e)/2eu^2\}$ or $\alpha = \frac{1}{2}\sin^{-1}[ag\,(1+2e)/2eu^2]$

Ex. 2. An imperfectly elastic ball is projected with velocity $\sqrt{(gh)}$ at an angle α with the horizon, so that it strikes a vertical wall distant c from the point of projection, and returns to the point of projection, show that the coefficient of restitution between the ball and the wall is

$c/(h \sin 2\alpha - c)$.



Sol. Let O be the point of projection at a distance c from the wall. The velocity and angle of projection are given as $\sqrt{(gh)}$ and α respectively. The horizontal and intial vertical components of velocity are $\sqrt{(gh)} \cos \alpha$ and $\sqrt{(gh)} \sin \alpha$ respectively.

Since the horizontal component of velocity from O to A, where the ball strikes the wall, remains $\sqrt{(gh)}\cos\alpha$ uniformly, so the time from O to $A = c/[\sqrt{(gh)}\cos\alpha]$

After striking the wall at A, the horizontal component of velocity becomes $e\sqrt{(gh)}\cos\alpha$.

$\therefore$ Time from C to $O = \dfrac{c}{e\sqrt{(gh)}\cos\alpha}$

$\therefore$ Total time from O to A and back to O

$$=\frac{c}{\sqrt{(gh)}\cos\alpha}+\frac{c}{e\sqrt{(gh)}\cos\alpha}=\frac{c}{\sqrt{(gh)}\cos\alpha}\left(1+\frac{1}{e}\right)$$

$=(1+e)\,c/[e\sqrt{(gh)}\cos\alpha]$

In this time the vertical distance travelled is zero. Also as the vertical component of velocity is not affected by the impact at A, so considering vertical motion from O to A and back to O from "$s=ut+\frac{1}{2}ft^2$" we have

$$0=\sqrt{(gh)}\sin\alpha\left[\frac{(1+e)c}{e\sqrt{(gh)}\cos\alpha}\right]-\tfrac{1}{2}g\left[\frac{(1+e)c}{e\sqrt{(gh)}\cos\alpha}\right]^2$$

or $2\sqrt{(gh)}\sin\alpha=g(1+e)c/[e\sqrt{(gh)}\cos\alpha]$

or $2egh\sin\alpha\cos\alpha=g(1+e)c$ or $eh\sin2\alpha=c+ec$

or $e(h\sin2\alpha-c)=c$ or $e=c/(h\sin2\alpha-c)$. Hence proved.

*Ex. 3. A ball thrown with a velocity of 48 ft/sec at an elevation of 15°, hits a smooth vertical wall 12 ft. away and returns to the point of projection. Find the coefficient of restitution.

Sol. Proceed exactly as in last example.

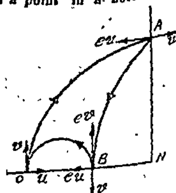
[Here initial velocity $\sqrt{(gh)}=48$ ft./sec., where $g=32$ ft./sec².

i.e. $\sqrt{(32h)}=48$ or $h=(48\times48)/32=72$ ft.

Also 'c' = 12 ft. and $\alpha=15°$

$\therefore$ Here $e=\dfrac{\text{"}c\text{"}}{h\sin2\alpha-c}=\dfrac{12}{72\sin30°-12}=\frac{1}{2}$ Ans.

Ex. 4. A particle is projected from a point in a horizontal plane so as to strike a vertical wall at right angles; and after rebounding from the wall and once from the horizontal plane returns to the point of projection, prove that the coefficient of elasticity of the particle is $\frac{1}{2}$.



(Fig. 26)

Sol. Let the point of projection O be at a distance a from wall AN. Let the horizontal and vertical components of velocity at O be u and v respectively. Since the horizontal component of velocity from O to A, where the particle strikes the wall, remains u uniformly, so the time from O to $A=(a/u)$

The vertical component of velocity will be reduced to zero at A as the particle strikes the wall at right angles at A. So A is the highest point of the trajectory from O to A.

Hence $ON=\frac{1}{2}$ (Horizontal range of the particle projected from O).

i.e. $a=\frac{1}{2}[(2/g)\,uv]=uv/g$ ...(i)

After striking the wall at A, the horizontal componet of velocity becomes eu. The particle strikes the horizontal plane at B. At B, the horizontal component of velocity is eu and the vertical component of velocity before striking is v again.

As the vertical distance moved from O to A=the vertical distance moved from A to B and the vertical velocity at A is zero, so the time of flight from A to B=time of flight from O to A $=a/u$ (Proved)

∴ NB=horizontal distance moved in time (a/u) with uniform horizon velocity $eu=(a/u)\,eu=ae$

∴ $OB=ON-NB=a-ae$ ...(ii)

After striking the horizontal plane at B, the vertical component of velocity becomes ev and the horizontal component of velocity eu remains unaffected due to this impact.

∴ BO=horizontal range of the particle when it leaves B with horizontal and vertical velocities as eu and ev.

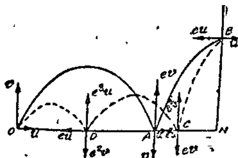
$=(2/g)\,(eu)\,(ev)=(2e^2/g)\,(uv)$

or $(a-ae)=(2/g)\,e^2\,(ag)$, from (i) and (ii)

or $2e^2+e-1=0$ or $(2e-1)(e+1)=0$ or $e=\frac{1}{2}$. Ans.

Ex. 5. A ball is projected from a point in a horizontal plane; after one rebound from the plane it strikes directly against a vertical wall; after two more rebounds with the horizontal plane ball returns to the point of projection, prove that $e=2\,(1-e^2)$, e being the elasticity of the ball.

Sol. The ball is projected from O with a velocity whose horizontal and vertical components are u and v respectively, the



(Fig. 27)

ball on its way out strikes the horizontal plane through O at A and then strikes the wall directly (*i.e.* at right angle) at B. While coming back from B, the ball strikes the horizontal plane through O at C and D and then returns to O. In the figure the path traced out by the particle on its way back from B has been shown in dotted lines.

Now OA = horizontal range of the ball when projected from O

$$=(2/g)\,(u)\,(v)=2uv/g. \qquad \text{.. (i)}$$

When the ball strikes at A, the vertical component of velocity becomes ev after impact, the horizontal component remaining the same and the ball strikes the wall directly at B.

$\therefore$ $AN=\frac{1}{2}$ (Horizontal range of the ball when it is projected from A with vertical and horiz[illegible] velocity as ev [illegible] (ii)

[illegible] ents of velocity becomes eu and just before striking at C the vertical component of velocity is ev, which remains unchanged due to this impact. Also at B, the impact is direct.

$$\therefore \quad CN=\tfrac{1}{2}\,[(2/g)\,(eu)\,(ev)]=e^2uv/g \qquad \text{...(iii)}$$

After striking at C, the vertical component of velocity becomes e^2v, horizontal component of velocity eu remaining unaltered due to impact.

$\therefore$ CD = Horizontal range of the ball with horizontal and vertical components of velocity as eu and e^2v respectively.

$$=(2/g)\,(eu)\,(e^2v)=(2/g)\;e^3uv \qquad \text{.. (iv)}$$

Also after striking at D, the vertical component of velocity becomes e^3v, horizontal component of velocity remaining unaltered due to impact at D. (correct the figure).

$$\therefore \quad \text{As before } DO=(2/g)\,(eu)\,(e^3v)=(2/g)\,e^4uv \qquad \text{.. (v)}$$

Now $OA+AN=NC+CD+DO$, see figure,

$$\text{or} \quad \frac{2uv}{g}+\frac{euv}{g}=\frac{e^2uv}{g}+\frac{2}{g}\,e^3uv+\frac{2}{g}\,e^4uv, \text{ from (i) to (v)}$$

$$\text{or} \quad 2+e=e^2+2e^3+2e^4 \quad \text{or} \quad 2e^4+2e^3+e^2-e-2=0$$

$$\text{or} \quad (2e^3+e-2)\,(e+1)=0 \quad \text{or} \quad 2e^3+e-2=0, \qquad \because e\neq-1$$

$$\text{or} \quad e=2-2e^3 \quad \text{or} \quad e=2\,(1-e^3).$$

Hence proved.

***Ex. 6. A ball is projected from a point in one of the two smooth parallel vertical walls against the other in a plane perpendicular to both after being reflected at each wall impinge again on the second at a point in the same horizontal plane as it started from, show that $be^3=a\,(1+e+e^2)$, where e is the coefficient of elasticity, b the free range on a horizontal plane and a the distance between the walls.**

Sol. The ball is projected from O, a point on the first wall, with a velocity whose horizontal and vertical components are u

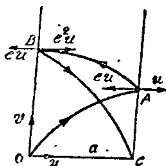
and v respectively. It then strikes the second wall at A, then first wall at B and strikes again the second wall at C such that O and C are in the same horizontal plane.

The ball strikes A with a horizontal velocity u, which after impact becomes eu. With this horizontal velocity eu, the ball strikes at B and after impact at B, the horizontal velocity of the ball is e^2u and with this horizonal velocity ball strikes the second wall at C.

The horizontal distance moved as the particle moves from O to A or A to B or B to C is the same viz a, the horizontal distance between the walls

$\therefore$ Total time taken in moving from O to C

$$=\frac{a}{u}+\frac{a}{eu}+\frac{a}{e^2u}=\frac{a}{e^2u}\,(e^2+e+1)$$



(Fig. 28)

Throughout the motion from O to C, the vertical velocity remains unaffected due to impact with walls, and vertical distance moved from O to C is zero. Hence from "$s=ut+\frac{1}{2}ft^2$" we have for vertical velocity.

$$0=v.(a/e^2u)(e^2+e+1)-\tfrac{1}{2}g\,\{(a/e^2u)\,(e^2+e+1)\}^2, \text{ from (i)}$$

or $\;2v=(ga/e^2u)\,(e^2+e+1)$ or $(2uv/g)\,e^2=a\,(e^2+e+1)$ ...(ii)

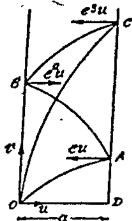
Also it is given that the free range on the horizontal plane $=b$.

i.e. $$2uv/g=b$$

$\therefore$ From (ii) we get $be^2=a\,(e^2+e+1)$ Hence proved.

***Ex. 7. A particle is projected from a given point at the foot of one of the two smooth vertical walls, so as after three impacts, it may return to the point of projection. If the last impact be direct, show that $e^3+e^2+e=1$.**

Sol. Let O be the point of projection from the 1st wall. From O the particle strikes the other wall at A, then it strikes the 1st wall at B, then strikes the 2nd wall at C and return back to O.



(Fig. 29)

Let u and v be the horizontal and vertical components of the initial velocity.

Then after striking at A, B and C the horizontal components of velocities become eu, e^2u and e^3u. Let $OD=a$

$\therefore$ Time from O to $A=a/u$

Similarly time from A to $B = a/eu$ and time from B to $C = a/e^2u$ and from C to $O = a/e^3u$.

Since the impact at C is direct *i.e.*, at right angles to the wall and the vertical component of velocity is not altered by the impacts.

$\therefore$ Time from O to C = Time from C to O,

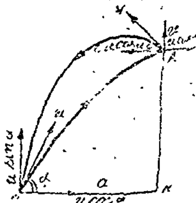
since the vertical height moved in rising from O to C is the same as in falling from C to O.

i.e. $\quad \dfrac{a}{u} + \dfrac{a}{eu} + \dfrac{a}{e^2u} = \dfrac{a}{e^3u}$ or $1 + \dfrac{1}{e} + \dfrac{1}{e^2} = \dfrac{1}{e^3}$

or $\quad e^3 + e^2 + e = 1$. Hence proved.

Ex. 8. **A particle is projected at an angle α to the horizontal to strike a smooth vertical wall and after rebounding it passes through the point of projection. If ϕ be the angle of inclination to the horizontal at which the particle rebounds from the wall, e the coefficient of restitution, show that** $\quad e(1+e)\tan\phi = (1-e)\tan\alpha$.

Sol. Let the particle be projected from O with a velocity u making an angle α with the horizontal. Let the distance of O from the wall be a. Let the particle strike the wall at A and then return to O.



(Fig. 30)

The horizontal and vertical com- [illegible] cos α

[illegible]

$= (a/u \cos\alpha)$ ...(i)

Let v be the vertical component of velocity at A, before the particle strikes the wall. Then from

"$v = u + ft$" we have

$v = u\sin\alpha - g\,(a/u\cos\alpha) = (u^2\sin\alpha\cos\alpha - ag)/(u\cos\alpha)$ ...(ii)

After impact at A, the horizontal velocity $= eu\cos\alpha$ and vertical velocity remains unaffected due to this impact and so remains v given by (ii).

$\therefore$ If ϕ be the angle which the direction of the particle after impact at A makes with the horizontal, then

$$\tan\phi = \frac{v}{eu\cos\alpha} = \frac{u^2\sin\alpha\cos\alpha - ag}{eu^2\cos^2\alpha}, \text{ from (ii)} \qquad \text{...(iii)}$$

Also time from A to $O = a/(eu\cos\alpha)$

$\therefore$ Total time taken in moving from O to A and back to O

$$= \frac{a}{u\cos\alpha} + \frac{a}{eu\cos\alpha} = \frac{a}{u\cos\alpha}\left(1 + \frac{1}{e}\right)$$

Also in this time the vertical distance moved is zero.

Hence from "$s = ut + \frac{1}{2}ft^2$" we have

$$0=\sin\alpha\left[\frac{a}{u\cos\alpha}\left(1+\frac{1}{e}\right)\right]-\tfrac{1}{2}g\left[\frac{a}{u\cos\alpha}\left(1+\frac{1}{e}\right)\right]^2$$

or $\quad 2u^2\sin\alpha\cos\alpha=ga\{(e+1)/e\}$

or $\quad ag=(2eu^2\sin\alpha\cos\alpha)/(e+1)$ .. (iv)

$\therefore$ from (iii), $\tan\phi=\dfrac{u^2\sin\alpha\cos\alpha-\{(2eu^2\sin\alpha\cos\alpha)/(e+1)\}}{eu^2\cos^2\alpha}$

$=\left[\dfrac{1-\{2e/(e+1)\}}{e}\right]\tan\alpha=\dfrac{(1-e)\tan\alpha}{e\,(e+1)}$

or $\quad e\,(e+1)\tan\phi=(1-e)\tan\alpha.$ Hence proved.

Solved Examples on Billiard Tables.

Note : In such examples we are to use cotangent of angle of reflection $=e$ times cotangent of angle of incidence.

Ex. 1. An imperfectly elastic ball, reflected at the circumference of a circle, describes the sides of an inscribed quadrilateral in order, prove that two of the angles of the quadrilateral are right angles

And find the initial direction of projection.

Sol. Let the ball be projected from A at an angle α with the radius through A. Let β, γ and δ be the angles of reflection at the points of impact B, C and D respectively.

The angles of incidence at B, C and D can easily seen to be α, β and γ respectively

Since cotangent of angle of reflection $=e\times$ cotangent of angle of incidence.

(Fig. 31)

$\therefore$ At A, $\cot\beta=e\cot\alpha$

or $\quad \tan\beta=(1/e)\tan\alpha$ .. (i)

Similarly at C, $\cot\gamma=e\cot\beta$ or $\tan\gamma=(1/e)\tan\beta$

or $\quad \tan\gamma=(1/e)^2\tan\alpha$, from (i) (ii)

Also at D, $\cot\delta=e\cot\gamma$ or $\tan\delta=(1/e)\tan\gamma$

or $\quad \tan\delta=(1/e^3)\tan\alpha$, from (ii) (iii)

Also for the quadrilateral $ABCD$,

Sum of the angles, A, B, C and $D=2\pi$

or $\quad 2\alpha+2\beta+2\gamma+2\delta=2\pi$ or $\alpha+\beta+\gamma+\delta=\pi$ (iv)

or $\quad \alpha+\beta=\pi-(\gamma+\delta)$ or $\tan(\alpha+\beta)=-\tan(\gamma+\delta)$

or $\quad \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=-\left[\dfrac{\tan\gamma+\tan\delta}{1-\tan\gamma\tan\delta}\right]$

or $\quad \dfrac{\tan\alpha+(1/e)\tan\alpha}{1-(1/e)\tan^2\alpha}=-\left[\dfrac{(1/e^2)\tan\alpha+(1/e^3)\tan\alpha}{1-(1/e^5)\tan^2\alpha}\right],$

from (i), (ii) and (iii)

or $\quad \dfrac{e+1}{e-\tan^2\alpha}=-\dfrac{e^2\,(e+1)}{e^5-\tan^2\alpha}$ or $e^5-\tan^2\alpha=-e^2\,(e-\tan^2\alpha)$

or $\quad (e^2+1)\tan^2\alpha=e^5+e^3$ or $\tan^2\alpha=e^3$, ..

which gives the initial direction of projection.

$$\text{Now } \tan(\beta+\gamma)=\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\frac{(1/e)\tan\alpha+(1/e^2)\tan\alpha}{1-(1/e^3)\tan^2\alpha},$$

from (i) and (ii)

or $$\tan(\beta+\gamma)=\frac{(1/e+1/e^2)\tan\alpha}{1-1}, \text{ from (v)}$$

or $\tan(\beta+\gamma)=\infty$ or $\beta+\gamma=\frac{1}{2}\pi$

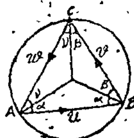
$\therefore$ from (iv) we get $\alpha+\delta=\pi-(\beta+\gamma)=\pi-\frac{1}{2}\pi=\frac{1}{2}\pi$

Hence two angles of quadrilateral $ABCD$ viz. A and C are right angles.

****Ex. 2** **A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that ball whose coefficient of restitution is e, projected along the table from a point in the rim making an angle α with the radius through the point will return to the point of projection after two impacts if $\tan^2\alpha=e^3/(1+e+e^2)$.**

Prove also that when the ball returns to the point of projection its velocity is to the original velocity as $e^{3/2}:1$. *(Gorakhpur 91)*

Sol. Let the ball be projected from A with a velocity u making and angle α with the radius through A. Let the ball strike the rim of the table at B and C and then come back to A. Let the velocities after impact at B and C be v and w respectively. Let the angles of reflection at B and C be β and γ. Then the angle of incidence at B and C are α and β respectively.



(Fig. 23)

$\therefore$ Cotangent of angle of reflection $=e\times$cotangent of angle of incidence.

$\therefore$ At B we get $\cot\beta=e\cot\alpha$

or $\tan\beta=(1/e)\tan\alpha$ ...(i)

Also at C we get $\cot\gamma=e\cot\beta$ or $\tan\gamma=(1/e)\tan\beta$

or $\tan\gamma=(1/e^2)\tan\alpha$, from (i).

In $\triangle ABC$, we have $\angle A+\angle B+\angle C=\pi$

or. $2\alpha+2\beta+2\gamma=\pi$ or $\alpha+\beta+\gamma=\frac{1}{2}\pi$ or $\beta+\gamma=\frac{1}{2}\pi-\alpha$ ...(i)

or $\tan(\beta+\gamma)=\tan(\frac{1}{2}\pi-\alpha)=\cot\alpha$ or $\dfrac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\cot\alpha$

or $$\frac{(1/e)\tan\alpha+(1/e^2)\tan\alpha}{1-(1/e)\tan\alpha\,(1/e^2)\tan\alpha}=\cot\alpha, \text{ from (i) and (ii)}$$

or $$\left(\frac{1}{e}+\frac{1}{e^2}\right)\tan\alpha=\cot\alpha\left(1-\frac{1}{e^3}\tan^2\alpha\right)=\cot\alpha-\frac{1}{e^3}\tan\alpha$$

or $$\left(\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\right)\tan\alpha=\cot\alpha=\frac{1}{\tan\alpha}$$

or $\tan\alpha=\dfrac{1}{[(1/e)+(1/e^2)+(1/e^3)]}=\dfrac{e^3}{(1+e+e^2)}$ (iii)

Again due to the impact tangential components of velocity is not altered as there is no force in that direction, the rim being smooth.

$\therefore$ At B, we have $v\sin\beta=u\sin\alpha$ ...(vi)

and at C, we have $w\sin\gamma=v\sin\beta$ ...(v)

From (iv) and (v) we get $u\sin\alpha=w\sin\gamma$

or $\dfrac{w}{u}=\dfrac{\sin\alpha}{\sin\gamma}=\sin\alpha\operatorname{cosec}\gamma=\sin\alpha\sqrt{(1+\cot^2\gamma)}$

$=\sin\alpha\sqrt{[1+e^4\cot^2\alpha]}$, since from (ii) $\cot\gamma=e^2\cot\alpha$

or $w/u=\sin\alpha\,[1+e^4\cot^2\alpha]$ (vi)

From (iii), $\tan^2\beta=\dfrac{e^3}{1+e+e^2}$ or $\dfrac{\sin^2\alpha}{\cos^2\alpha}=\dfrac{e^3}{1+e+e^2}$

or $\dfrac{\sin^2\alpha}{e^3}=\dfrac{\cos^2\alpha}{1+e+e^2}=\dfrac{\sin^2\alpha+\cos^2\alpha}{(1+e+e^2)+e^3}$ or $\sin^2\alpha=\dfrac{e^3}{(1+e+e^2+e^3)}$

$\therefore$ from (iv) we get

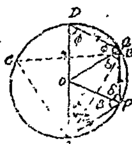
$$\frac{w}{u}=\frac{e^{3/2}}{\sqrt{[1+e+e^2+e^3]}}\sqrt{\left[1+\frac{e^4(1+e+e^2)}{e^3}\right]},\text{ from (iii)}$$

$$=\frac{e^{3/2}}{\sqrt{[1+e+e^2+e^3]}}\sqrt{(1+e+e^2+e^3)}=e^{3/2}$$

i.e. $\dfrac{\text{velocity of return to }A}{\text{original velocity}}=\dfrac{w}{u}=\dfrac{e^{3/2}}{1}$ Hence proved.

Ex 3. **Two particles start from the same point in circumference of a smooth circular table with raised edges. After each has been reflected twice at the circumference, the one returns to the point of projection and the other arrives at point diametrically opposite. Prove that if α and β be the incinations of their directions of projection to the radius to the point, then $\tan\alpha\tan\beta=e^3$.**

Sol. Let A be the point of start of both the particles. One particle returns to A after two reflections at B and C (this path has been shown in dotted lines) and the other particle reflect at P and Q and comes to D, a point diametrically opposite to A.



(Fig. 33)

If the direction of projection of the first particle, which has impacts at B and C, makes an angle α with the radius at A then as in last example we can prove that

$\tan^2\alpha=e^3/(1+e+e^2)$. (i)

For the second particle the direction of projection makes an angle β (given) with the radius at A. Let δ and ϕ be the angles of

reflection at P and Q. Then the angles of incidence at P and Q are β and δ respectively.

∵ cotangent of angle of reflection $= e \times$ cotangent of angle of incidence.

At P, $\cot\delta = e\cot\beta$ or $\tan\delta = (1/e)\tan\beta$ ...(ii)

and at Q, we have $\cot\phi = e\cot\delta$ or $\tan\phi = (1/e)\tan\delta$

or $\tan\phi = (1/e^2)\tan\beta$ ...(iii)

Also for the quadrilateral $APQD$, we know that the sum of the angles A, P, Q and D is equal to 2π

or $2\beta + 2\delta + 2\phi = 2\pi$ (see figure)

or $\beta + \delta = \pi - \phi$ or $\tan(\beta+\delta) = \tan(\pi-\phi) = -\tan\phi$

or $\dfrac{\tan\beta + \tan\delta}{1 - \tan\beta\tan\delta} = -\tan\phi$

or $\dfrac{\tan\beta + (1/e)\tan\beta}{1 - \tan\beta\,(1/e)\tan\beta} = -\dfrac{\tan\beta}{e^2}$, from (ii) and (iii)

or $\dfrac{e+1}{e - \tan^2\beta} = -\dfrac{1}{e^2}$ or $(e+1)e^2 = -(e - \tan^2\beta)$

or $\tan^2\beta = e^3 + e^2 + e$ ...(iv)

∴ from (i) and (iv) we get

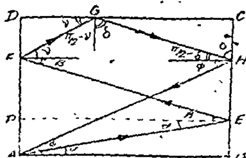
$\tan^2\alpha\tan^2\beta = [e^3/(1+e+e^2)].(e^3+e^2+e) = e^4$

or $\tan\alpha\tan\beta = e^2$ Hence proved.

***Ex. 4. From one corner A of a rectangular billard table ABCD a ball is projected in a direction making an angle α with the side AB ; it strikes first the side BC, then AD, then DC, then BC again, and then returns to A. Prove that if e be the coefficient of restitution, then AB/AD $=(e^4\cot\alpha)/(1-e^2)$.**

Sol. The ball is projected from A at an angle α to AB.

It then strikes at E, F, G, H and finally returns to A. Let the angles of reflection at E, F, G and H be β, γ, δ and ϕ respectively, and the angles of incidence are α, β, $(\frac{1}{2}\pi-\gamma)$ and $(\frac{1}{2}\pi-\delta)$ respectively.



(Fig. 34)

Since cotangent of angle of reflection $= e \times$ cotangent of angle of incidence, so we have

At E, $\cot\beta = e\cot\alpha$ or $\tan\beta = (1/e)\tan\alpha$ ...(i)

At F, $\cot\gamma = e\cot\beta$ or $\tan\gamma = (1/e)\tan\beta$

or $\tan\gamma = (1/e^2)\tan\alpha$ ...(ii)

At G, $\cot\delta = e\cot(\frac{1}{2}\pi - \gamma) = e\tan\gamma = e(1/e^2)\tan\alpha$, from (ii)

or $\cot\delta = (1/e)\tan\alpha$ .. (iii)

At H, $\cot\phi = e\cot(\frac{1}{2}\pi - \delta) = e\tan\delta$

or $\tan\phi = (1/e)\cot\delta = (1/e^2)\tan\alpha$, from (iii) ...(iv)

Now from the figure we can see that

$AD = AP + PF + FD = BE + PF + FD$, since $AP = BE$

$= AB\tan\alpha + PE\tan\beta + GD\tan\gamma$

$= AB\tan\alpha + AB\tan\beta + (AB - GC)\tan\gamma$, since $PE = AB = DC$

$= AB(\tan\alpha + \tan\beta + \tan\gamma) - GC\tan\gamma$

$= AB\{1 + (1/e) + (1/e^2)\}\tan\alpha - (CH\tan\delta)\tan\gamma$, from (i) and (ii)

$= AB(1 + 1/e + 1/e^2)\tan\alpha - (CB - HB)\tan\delta\tan\gamma$

$= AB(1 + 1/e + 1/e^2)\tan\alpha - (AD - AB\tan\phi)\tan\delta\tan\gamma$

$= AB(1 + 1/e + 1/e^2)\tan\alpha - AD\tan\delta\tan\gamma + AB\tan\phi\tan\delta\tan\gamma$

$= AB(1 + 1/e + 1/e^2)\tan\alpha - AD\,e\cot\alpha\,(1/e^2)\tan\alpha$

$+ AB(1/e^2)\tan\alpha\cot\alpha\,(1/e^2)\tan\alpha$, from (ii), (iii) and (iv)

or $AD = AB(1 + 1/e + 1/e^2 + 1/e^3)\tan\alpha - AD(1/e)$

or $AD[1 + (1/e)] = AB[(e^3 + e^2 + e + 1)/e^3]\tan\alpha$

or $\dfrac{AD}{AB} = \dfrac{(e^3 + e^2 + e + 1)\tan\alpha}{e^3(e+1)} = \dfrac{(e^2+1)(e+1)\tan\alpha}{e^3(e+1)}$

or $\dfrac{AB}{AD} = \dfrac{e^3}{(e^2+1)\tan\alpha} = \dfrac{e^3\cot\alpha}{(1+e^2)}$. Hence proved.

Ex. 5. A square table ABCD, whose side is a, has raised edges. A particle of elasticity e is projected from a point P in AB, and hits the sides BC, CD, DA in Q, R, S respectively. Prove that PQ and RS are parallel.

If α be the angle QPB and BP $= x$, prove that if the particle returns to P, then $x(1-e) = a(1 - e\cot\alpha)$

Sol. The ball is projected from P at an angle α to AB. Let the angle of reflection at Q, R and S be β, γ and δ respectively. then the angle of incidence at these points are α, $\frac{1}{2}\pi - \beta$ and $\frac{1}{2}\pi - \gamma$ respectively.

Since cotangent of angle of reflection $= e \times$ contangent of angle of incidence, so we have $\cot\beta = e\cot\alpha$

or $\tan\beta = (1/e)\tan\alpha$ ...(i)

At R, $\cot\gamma = e\cot(\frac{1}{2}\pi - \beta)$

or $\cot\gamma = e\tan\beta = \tan\alpha$, from (i)

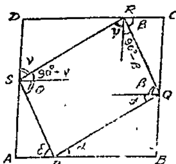
or $\tan(\frac{1}{2}\pi - \gamma) = \tan\alpha$

or $\frac{1}{2}\pi - \gamma = \alpha$ ...(ii)

Hence PQ is parallel to RS

Again length of each side of the square $= a$ (given)

$BP = x$, and so $AP = a - x$



(Fig. 35)

Also for the impact at S, we have

$$\cot \delta = e \cot (\tfrac{1}{2}\pi - \gamma) = e \tan \gamma \quad \text{...(iii)}$$

Also from the figure we see that

$$BQ = BR \tan \alpha = x \tan \alpha;$$

$$CQ = BC - BQ = a - x \tan \alpha,$$

In Δ RCQ, $RS = CQ \cot \beta = (a - x \tan \alpha) \cot \beta$ ...(iv)

In Δ APS; $AS = AP \tan \delta = (a - x) \tan \delta$

and in Δ RDS, $RD = DS \tan \gamma = (AD - AS) \tan \gamma$

$$= [a - (a - x) \tan \delta] \tan \gamma \quad \text{...(v)}$$

Now $\quad DC = DR + RC$

or $\quad a = [a - (a - x) \tan \delta] \tan \gamma + [(a - x \tan \alpha) \cot \beta]$, from (iv) and (v)

$= [a - (a - x) \{1/(e \tan \gamma)\}] \tan \gamma + [(a - x \tan \alpha) e \cot \alpha]$; from (i) and (ii)

$= a \tan \gamma - [(a - x)/e] + ae \cot \alpha - ex$

or $\quad a - a \tan \gamma + (a/e) - ae \cot \alpha = (x/e) - ex$

or $\quad a[1 - \cot \alpha + (1/e) - e \cot \alpha] = x(1 - e^2)/e$, from (ii)

or $\quad a[e - e \cot \alpha + 1 - e^2 \cot \alpha] = x(1 - e^2)$

or $\quad a(1 + e)(1 - e \cot \alpha) = (1 - e^2) x$

or $\quad x(1 - e) = a(1 - e \cot \alpha)$. Hence proved.

**Ex 6. A ball of elasticity e, is projected along a horizontal plane from the middle point of one side at the sides of an isosceles right angled triangle, so as, after reflection at the hypotenuse and remaining side; it returns to the point of projection, prove that the contangents of angles of reflection are $(e+1)$ and $(e+2)$ respectively.

Solution Let the ball be projected from P, the middle point of the side AB of Δ ABC. It hits the hypotenuse BC at Q and after being reflected from Q hits the side AC at R and then returns to P Let at P and Q the angles of reflection be β and δ and angles



Also for the impact at S, we have

$$\cot \delta = e \cot (\tfrac{1}{2}\pi - \gamma) = e \tan \gamma \quad \text{...(iii)}$$

Also from the figure we see that

$$BQ = BR \tan \alpha = x \tan \alpha ;$$

$$CQ = BC - BQ = a - x \tan \alpha.$$

In $\triangle RCQ$, $RS = CQ \cot \beta = (a - x \tan \alpha) \cot \beta$ ...(iv)

In $\triangle APS$; $AS = AP \tan \delta = (a - x) \tan \delta$

and in $\triangle RDS$, $RD = DS \tan \gamma = (AD - AS) \tan \gamma$

$$= [a - (a-x) \tan \delta] \tan \gamma \quad \text{...(v)}$$

Now $\quad DC = DR + RC$

or $\quad a = [a - (a-x) \tan \delta] \tan \gamma + [(a - x \tan \alpha) \cot \beta]$, from (iv) and (v)

$= [a - (a-x) \{1/(e \tan \gamma)\}] \tan \gamma + [(a - x \tan \alpha) e \cot \alpha]$; from (i) and (ii)

$= a \tan \gamma - [(a-x)/e] + ae \cot \alpha - ex$

or $\quad a - a \tan \gamma + (a/e) - ae \cot \alpha = (x/e) - ex$

or $\quad a[1 - \cot \alpha + (1/e) - e \cot \alpha] = x (1 - e^2)/e$, from (ii)

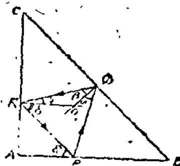
or $\quad a [e - e \cot \alpha + 1 - e^2 \cot \alpha] = x (1 - e^2)$

or $\quad a (1 + e) (1 - e \cot \alpha) = (1 - e^2) x$

or $\quad x (1 - e) = a (1 - e \cot \alpha)$. Hence proved.

Ex 6. A ball of elasticity e, is projected along a horizontal plane from the middle point of one side at the sides of an isosceles right angled triangle, so as, after reflection at the hypotenuse and remaining side; it returns to the point of projection, prove that the contangents of angles of refleciton are $(e+1)$ and $(e+2)$ respectively.

Solution Let the ball be projected from P, the middle point of the side AB of $\triangle ABC$. It hits the hypotenuse BC at Q and after being reflected from Q hits the side AC at R and then returns to P. Let at P and Q the angles of reflection be β and δ and angles



(Fig. 6)

of incidence be α and γ respectively. Since contangent of reflection $=e\times$cotangent of angle of incidence, therefore we have at Q,

$$\cot\beta=e\cot\alpha \quad \text{(i)}$$

and at R,

$$\cot\delta=e\cot\gamma \quad \text{(ii)}$$

From the figure, we see that in quadrilaterial $CROQ$,

$\angle ROQ+\angle C=\pi$, since $\angle ORC=\angle OQC=\frac{1}{2}\pi$

or $\{\pi-(\beta+\gamma)\}+\frac{1}{2}\pi=\pi$ $\because$ $\angle C=\angle B=\frac{1}{2}\pi$

or

$$\beta+\gamma=\tfrac{1}{2}\pi$$

In ΔAPR, $AP=RP.\cos\delta$ and in ΔBPQ,

$$\frac{BP}{PQ}=\frac{\sin\angle BQP}{\sin\angle PBQ}=\frac{\sin(\frac{1}{2}\pi-\alpha)}{\sin\frac{1}{2}\pi}=\sqrt{2}\cos\alpha$$

or $BP=PQ\sqrt{2}\cos\alpha$

But $AP=BP$, $\because$ P is the mid-point of AB

$\therefore$ $RP\cos\delta=PQ.\sqrt{2}\cos\alpha$ or $\dfrac{RP}{PQ}=\dfrac{\sqrt{2}\cos\alpha}{\cos\delta}$ .. (iv)

Again from ΔPQR, $\dfrac{RP}{PQ}=\dfrac{\sin(\alpha+\beta)}{\sin(\gamma+\delta)}$

or $\dfrac{RP}{PQ}=\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\gamma\cos\delta+\cos\gamma\sin\delta}=\dfrac{\sin\alpha\sin\beta(\cot\beta+\cot\alpha)}{\sin\gamma\sin\delta(\cot\delta+\cot\gamma)}$

$=\dfrac{\sin\alpha\sin\beta\,[e\cot\alpha+\cot\alpha]}{\sin\gamma\cos\delta\,[e\cot\gamma+\cot\gamma]}$, from (i) and (ii)

$=\dfrac{\sin\alpha\sin\beta(\cot\alpha)(e+1)}{\sin\gamma\sin\delta(\cot\gamma)(e+1)}=\dfrac{\sin\beta\cos\alpha}{\sin\delta\cos\gamma}$ ...(v)

$\therefore$ from (iv) and (v) we get $\dfrac{\sqrt{2}\cos\alpha}{\cos\delta}=\dfrac{\sin\beta\cos\alpha}{\sin\delta\cos\gamma}$

or $\dfrac{\sqrt{2}}{\cot\delta}=\dfrac{\sin\beta}{\cos\gamma}$ or $\dfrac{\sqrt{2}}{e\cot\gamma}=\dfrac{\sin\beta}{\cos\gamma}$, from (ii)

or $e\sin\beta=\sqrt{2}\sin\gamma=\sqrt{2}\sin(\frac{1}{2}\pi-\beta)$, from (iii)

$=\sqrt{2}\,[\sin\frac{1}{4}\pi\cos\beta-\cos\frac{1}{4}\pi\sin\beta]=\cos\beta-\sin\beta$

or $\cos\beta=(e+1)\sin\beta$ or $\cot\beta=e+1$ ...(vi)

And from (ii), $\cot\delta=e\cot\gamma=e\cot(\frac{1}{2}\pi-\beta)$, from (iii)

$=\dfrac{e}{\tan(\frac{1}{4}\pi-\beta)}=\dfrac{e(1+\tan\beta\tan\frac{1}{4}\pi)}{\tan\frac{1}{4}\pi-\tan\beta}$

$=\dfrac{e[1+\{1/(e+1)\}]}{[1-\{1/(e+1)\}]}=\dfrac{e(e+2)}{e}=e+2$ ...(vii)

From (vi) and (vii) we have the required results.

*§ 10. Impact on an Inclined plane.

A particle is projected from O with a velocity u making an angle α with the plane which is inclined to the horizon at an angle β; to discuss motion.

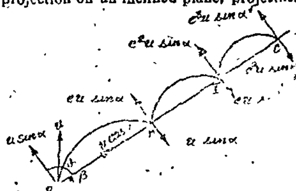
The particle is projected from O. The successive points of impact are A, B, C,. etc. Initially *i e.* at O the resolved parts of the velocity of projection u along and perpendicular to the inclined plane are $u \cos \alpha$ and $u \sin \alpha$ respectively.

Due to impacts at A, B, C, ..the velocity components perpendicular to the inclined plane will only be altered and the velocities parallel to the inclined plane will not be affected due to impact.

$\therefore$ At A, B, C ..etc. the velocity components perpendicular to the inclined plane after impacts will be $eu \sin \alpha$, $e^2u \sin \alpha$, $e^3u \sin \alpha$, ..etc.

And times of flight of the successive parabolic paths from O to A, A to B, .etc will be $\frac{2u \sin \alpha}{g \cos \beta}$, $\frac{2eu \sin \alpha}{g \cos \beta}$, $\frac{2e^2u \sin \alpha}{g \cos \beta}$,....etc.

(see projection on an inclined plane, projectiles)



(Fig. 37)

$\therefore$ Time of flight upto rth rebound

$$\frac{2u \sin \alpha}{g \cos \beta} + \frac{2eu \sin \alpha}{g \cos \beta} + \frac{2eu^2 \sin \alpha}{g \cos \beta} + \dots$$

$$= \frac{2e \sin \alpha}{g \cos \beta} (1 + e + e^2 + \dots \text{to } r \text{ terms})$$

Total time and distance till the parti

Rebounding ceases after an infinite be the required time till the particle

$$T = \frac{2u \sin \alpha}{g \cos \beta} + \frac{2eu \sin \alpha}{g \cos \beta} + \frac{2e^2u \sin}{g}$$

$$= \frac{2u \sin \alpha}{g \cos \beta} (1 + e + e^2 + \dots \text{ad. inf})$$

If S be total distance moved along ing ceases; then considering the mo inclined plane from "$s = ut + ft^2$". we

$$S=(u\cos\alpha)\,T-\tfrac{1}{2}(g\sin\beta)\,T^2$$
$$=[(u\cos\alpha)-\tfrac{1}{2}(g\sin\beta).T]\,T \qquad \text{(Note)}$$
$$=\left[u\cos\alpha-\tfrac{1}{2}(g\sin\beta).\frac{2u\sin\alpha}{g\cos\beta}.\frac{1}{(1-e)}\right]\times\frac{2u\sin\alpha}{g\cos\beta}.\frac{1}{(1-e)}$$
$$=\frac{2u^2\sin\alpha}{g(1-e)\cos\beta}\left[\cos\alpha-\frac{\sin\alpha\sin\beta}{(1-e)\cos\beta}\right]$$
$$=\frac{2u^2\sin\alpha}{g(1-e)^2\cos^2\beta}\,[(1-e)\cos\alpha\cos\beta-\sin\alpha\sin\beta]$$
$$=\frac{2u^2\sin\alpha}{g(1-e)^2\cos^2\beta}\,[(\cos\alpha\cos\beta-\sin\alpha\sin\beta)-e\cos\alpha\cos\beta]$$

or $$S=\frac{2u^2\sin\alpha}{g(1-e)^2\cos^2\beta}[\cos(\alpha+\beta)-e\cos\alpha\cos\beta] \qquad \text{...(iii)}$$

Total time, if after rth rebound, the particle returns to the point of projection.

The time for each jump depends upon the initial velocity component of that jump perpendicular to the inclined plane and as there have been $(r+1)$ jumps till the particle returns to the point of projection O, therefore

total time of $(r+1)$ jumps

$$=\frac{2u\sin\alpha}{g\cos\beta}+\frac{2eu\sin\alpha}{g\cos\beta}+\frac{2e^2u\sin\alpha}{g\cos\beta}+\text{...to } (r+1) \text{ terms}$$
$$=\frac{2u\sin\alpha}{g\cos\beta}\,[1+e+e^2+\text{..to } (r+1) \text{ terms}]$$
$$=\frac{2u\sin\alpha}{g\cos\beta}.\left(\frac{1-e^{r+1}}{1-e}\right) \qquad \text{...(iv)}$$

If the particle at the rth impact strikes the plane normally, then prove that $2\tan\alpha\tan\beta(1-e^r)=1-e$

In whatever manner the particle may strike the inclined plane at the rth impact, the time of flight upto rth rebound

$$=\frac{2u\sin\alpha}{g\cos\alpha}\cdot\frac{1-e^r}{1-e}, \text{ from (i)}$$

Now if the particle strikes the plane normally at the rth impact then the velocity along the plane is destroyed during this time.

∴ Considering motion of the particle along the inclined plane from "$v=u+ft$" we have

$$0=u\cos\alpha-\sin\beta\,\frac{2u\sin\alpha}{g\cos\beta}\cdot\frac{(1-e^r)}{(1-e)}$$

or $$(1-e)\cos\alpha=2\sin\alpha\tan\beta(1-e^r)$$

or [illegible]

If the pa[illegible] impact and r[illegible] prove that $e^{\text{[illegible]}}$

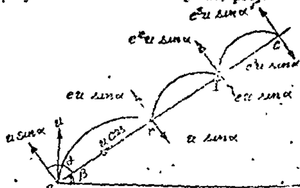
The particle is projected from O. The successive points of impact are $A, B, C,$. etc. Initially i.e. at O the resolved parts of the velocity of projection u along and perpendicular to the inclined plane are $u \cos \alpha$ and $u \sin \alpha$ respectively.

Due to impacts at $A, B, C,$ .the velocity components perpendicular to the inclined plane will only be altered and the velocities parallel to the inclined plane will not be affected due to impact.

$\therefore$ At A, B, C ..etc. the velocity components perpendicular to the inclined plane after impacts will be $eu \sin \alpha$, $e^2u \sin \alpha$, $e^3u \sin \alpha$, . etc.

And times of flight of the successive parabolic paths from O to A, A to B, .etc will be $\frac{2u \sin \alpha}{g \cos \beta}$, $\frac{2eu \sin \alpha}{g \cos \beta}$, $\frac{2e^2u \sin \alpha}{g \cos \beta}$,...etc.

(see projection on an inclined plane, projectiles)



(Fig. 37)

$\therefore$ Time of flight upto rth rebound

$$\frac{2u \sin \alpha}{g \cos \beta}+\frac{2eu \sin \alpha}{g \cos \beta}+\frac{2eu^2 \sin \alpha}{g \cos \beta}+\ldots\text{to } r \text{ terms.}$$

$$=\frac{2e \sin \alpha}{g \cos \beta}(1+e+e^2+\ldots\text{to } r \text{ terms})=\frac{2u \sin \alpha}{g \cos \beta}\cdot\left(\frac{1-e^r}{1-e}\right) \quad ..(i)$$

Total time and distance till the particle ceases to rebound.

Rebounding ceases after an infinite number of jumps i.e. if T be the required time till the particle ceases to rebound, then

$$T=\frac{2u \sin \alpha}{g \cos \beta}+\frac{2eu \sin \alpha}{g \cos \beta}+\frac{2e^2u \sin \alpha}{g \cos \beta}+\ \text{ad. inf.}$$

$$=\frac{2u \sin \alpha}{g \cos \beta}(1+e+e^2+\ldots\text{ad. inf})=\frac{2u \sin \alpha}{g \cos \beta}\left(\frac{1}{1-e}\right) \quad ..(ii)$$

If S be total distance moved along the plane till the rebounding ceases; then considering the motion of the particle along the inclined plane from "$s=ut+ft^2$", we have

$S=(u\cos\alpha)\,T-\frac{1}{2}(g\sin\beta)\,T^2$ (Note)

$=[(u\cos\alpha)-\frac{1}{2}(g\sin\beta).T]\,T$

$$=\left[u\cos\alpha-\frac{1}{2}(g\sin\beta).\frac{2u\sin\alpha}{g\cos\beta}.\frac{1}{(1-e)}\right]\times\frac{2u\sin\alpha}{g\cos\beta}.\frac{1}{(1-e)}$$

$$=\frac{2u^2\sin\alpha}{g(1-e)\cos\beta}\left[\cos\alpha-\frac{\sin\alpha\sin\beta}{(1-e)\cos\beta}\right]$$

$$=\frac{2u^2\sin\alpha}{g(1-e)^2\cos^2\beta}\,[(1-e)\cos\alpha\cos\beta-\sin\alpha\sin\beta]$$

$$=\frac{2u^2\sin\alpha}{g(1-e)^2\cos^2\beta}\,[(\cos\alpha\cos\beta-\sin\alpha\sin\beta)-e\cos\alpha\cos\beta]$$

or $$S=\frac{2u^2\sin\alpha}{g(1-e)^2\cos^2\beta}[\cos(\alpha+\beta)-e\cos\alpha\cos\beta] \quad ...(iii)$$

Total time, if after rth rebound, the particle returns to the point of projection.

The time for each jump depends upon the initial velocity component of that jump perpendicular to the inclined plane and as there have been [illegible] till the particle returns to the

[illegible]

$$=\frac{2u\sin\alpha}{g\cos\beta}+\frac{2eu\sin\alpha}{g\cos\beta}+\frac{2e^2u\sin\alpha}{g\cos\beta}+...\text{to }(r+1)\text{ terms}$$

$$=\frac{2u\sin\alpha}{g\cos\beta}\,[1+e+e^2+..\text{to }(r+1)\text{ terms}]$$

$$=\frac{2u\sin\alpha}{g\cos\beta}.\left(\frac{1-e^{r+1}}{1-e}\right) \quad ...(iv)$$

If the particle at the rth impact strikes the plane normally, then prove that $2\tan\alpha\tan\beta\,(1-e^r)=1-e$

In whatever manner the particle may strike the inclined plane at the rth impact, the time of flight upto rth rebound

$$=\frac{2u\sin\alpha}{g\cos\alpha}.\frac{1-e^r}{1-e},\text{ from (i)}$$

Now if the [illegible]

pac [illegible]

from "$v=u+ft$" we have [illegible] particle along the inclined plane

$$0=u\cos\alpha-\sin\beta\,\frac{2u\sin\alpha}{g\cos\beta}.\frac{(1-e^r)}{(1-e)}$$

or $(1-e)\cos\alpha=2\sin\alpha\tan\beta\,(1-e^r)$

or [illegible] ...(v)

If the pa[illegible] angles at rth impact and re[illegible] projection at the nth impact prove that $e^n-2e^r+1=0$.

In all there are n jumps, out of which r jumps are up the plane and $(n-r)$ jumps are down the plane. Also the time for each jump depends upon the initial velocity component of that jump at right angles to the inclined plane.

$\therefore$ Time up the plane for r jumps

$$=\frac{2u \sin \alpha\,(1-e^r)}{g \cos \beta\,(1-e)}, \text{ from (i)}$$

And time for $(n-r)$ jumps down the plane

$$=\frac{2e^r u \sin \alpha}{g \cos \beta}+\frac{2e^{r+1} u \sin \alpha}{g \cos \beta}+\ldots+\text{to } (n-r) \text{ terms}$$

(Note)

$$=\frac{2u \sin \alpha}{g \cos \beta}.e^r\left[1+e+e^2+\ldots+\text{to } (n-r) \text{ terms}\right]$$

$$=\frac{2u \sin \alpha}{g \cos \beta}.e^r\,\frac{(1-e^{n-r})}{(1-e)}=\frac{2u \sin \alpha\,(e^r-e^n)}{g \cos \beta\,(1-e)}$$

Since the particle strikes the inclined plane normally at the rth impact, so the velocity component parallel to the inclined plane is destroyed at the rth impact. The condition down the plane being the same the particle will recover the same velocity $u \cos \alpha$ when it reaches O in equal time

$\therefore$ Time up the plane for r jumps,

= time down the plane for r jumps

i.e. $$\frac{2u \sin \alpha}{g \cos \beta}.\frac{(1-e^r)}{(1-e)}=\frac{2u \sin \alpha}{g \cos \beta}.\frac{(e^r-e^n)}{(1-e)}$$

or $$1-e^r=e^r-e^n \quad \text{or} \quad e^n-2e^r+1=0.$$

Total time and distance till the particle ceases to go up.

The particle ceases to go up the inclined plane when its velocity along the inclined plane vanishes. There are two stages for this. Firstly the particle ceases to rebound and then it slides along and up the inclined plane till its velocity becomes zero.

For both the stages viz. rebounding and sliding the velocity along the plane is affected only by gravity and therefore if t be the total time till the particle ceases to go up, considering the motion along the plane from "$v=u+ft$" we have

$$0=u \cos \alpha-(g \sin \beta)\,t \quad \text{or} \quad t=(u \cos \alpha/g \sin \beta) \qquad \ldots(vi)$$

And total distance along the plane till the particle ceases to go up $=$"$ut+\frac{1}{2}ft^2$"$=(u \cos \alpha)\,t-\frac{1}{2}g \sin \beta\; t^2$

$$=u \cos \alpha\left(\frac{u \cos \alpha}{g \sin \beta}\right)-\tfrac{1}{2}g \sin \beta\left(\frac{u \cos \alpha}{g \sin \beta}\right)^2=\frac{u^2 \cos^2 \alpha}{2g \sin \beta}$$

Solved Examples on § 10.

Ex. 1. A particle is projected from the foot of a plane of inclination α in a direction making an angle θ with the plane and rebounds vertically at the first impact. Show that

$\cot\theta=(2+e)\tan\alpha$, where e is the coefficient of restitution.

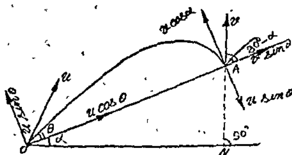
Sol. Let the particle be projected from O with a velocity u making an angle θ with the inclined plane. Let the first impact be at A.

$\therefore$ Time of flight from O to A.

$$=\frac{\text{"}2u\sin\alpha\text{"}}{g\cos\beta}=\frac{2u\sin\theta}{g\cos\alpha}=t \text{ (say)} \qquad (i)$$

The ball strikes at A with a velocity whose component perpendicular to the inclined plane is $u\sin\theta$ (see figure) and after impact at A its velocity component perpendicular to the plane will become $eu\sin\theta$

Let v be the velocity of the ball after impact at A. Since the ball rebounds vertically at A so its velocity at A will make an angle $(90°-\alpha)$ with the inclined plane. Hence its velocity components at A perpendicular and along the inclined plane will be $v\cos\alpha$ and $v\sin\alpha$ respectively.



(Fig. 38)

$$\therefore \quad v\cos\alpha=eu\sin\theta \qquad ...(ii)$$

in the direction perpendicular to the plane.

And $v\sin\alpha=u\cos\theta-g\sin\alpha.t$ from "$v=u+ft$"

$=u\cos\theta-g\sin\alpha\,\{(2u\sin\theta)/(g\cos\alpha)\}$, from (i)

$=(u/\cos\alpha)(\cos\theta\cos\alpha-2\sin\theta\sin\alpha) \qquad ...(iii)$

Dividing (iii) by (ii) we get

$$\tan\alpha=\frac{\cos\theta\cos\alpha-2\sin\theta\sin\alpha}{e\cos\alpha\sin\theta}=(1/e)(\cot\theta-2\tan\alpha)$$

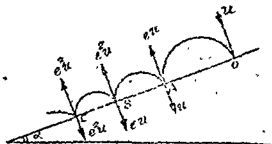
or $\cot\theta=(e+2)\tan\alpha$. Hence proved.

Ex. 2. A ball is projected with velocity u from a point of a plane inclined at an angle α to the horizontal, the direction of projection is at right angles to the plane. Prove that before ceasing to rebound, it will have described a length

$$(2u^2\sin\alpha)/[g(1-e)^2\cos^2\alpha]$$

Sol. Let the ball be projected from O with a velocity u at right angles to the inclined plane. Let afterwards the successive impacts be at A, B, C, etc. Then the velocity components perpendicular to the plane after impact at A, B, C, ..etc, will be eu, e^2u, e^3u, ..etc. (see figure).

And the times of these jumps from O to A, A to B, B to C... etc. are $\frac{2u}{g\cos\alpha}, \frac{2eu}{g\cos\alpha}, \frac{2e^2u}{g\cos\alpha}$, ...etc.



(Fig. 39)

∴ Total time taken by the by the ball before it ceases to rebound

$$=\frac{u}{g\cos\alpha}+\frac{2eu}{g\cos\alpha}+\frac{2e^2u}{g\cos\alpha}+\text{...ad. inf.}$$

$$=[2u/(g\cos\alpha)]\,(1+e+e^2+\text{...ad. inf.})=[2u/(g\cos\alpha)]\,\{1/(1-e)\}$$

Also considering the motion along the plane, from "$s=ut+\frac{1}{2}ft^2$" we have

$$\text{Required length}=0+\tfrac{1}{2}(g\sin\alpha)\left[\frac{2u}{g\cos\alpha}\cdot\frac{1}{1-e}\right]^2,$$

initial velocity along the plane being zero and acc. $=g\sin\alpha$

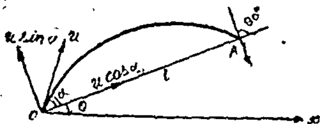
$=(2u^2\sin\alpha)/[g(1-e)^2\cos^2\alpha]$ Hence proved.

Ex. 3. A perfectly elastic ball is thrown from the foot of a plane inclined at an angle θ to the horizon. If after striking the plane at a distance l from the point of projection it rebounds and retraces its former path, show that the velocity of projection is

$$\sqrt{(gl(1+3\sin^2\theta)/(2\sin\theta))}.$$

Sol. The particle is projected from O and strikes the inclined plane at A, such that $OA=l$.

Now the particle can retrace its path provided it strikes the inclined plane at right angles at A. (Note)



(Fig. 40)

∴ Component of velocity along the plane vanishes at A.

Hence from "$v=u+ft$" for the motion along the plane from O to A we have $0=u\cos\alpha-(g\sin\theta)\,t$, since the acceleration along the plane is $-g\sin\theta$

or $$t=(u\cos\alpha)/(g\sin\theta) \quad \text{...(i)}$$

∴ During this time distance moved along the plane is l.

∴ from "$s=ut+\frac{1}{2}ft^2$" for the motion along the plane from O to A, we get

$$l=u\cos\alpha\left(\frac{u\cos\alpha}{g\sin\theta}\right)-\frac{1}{2}g\sin\theta\left(\frac{u\cos\alpha}{g\sin\theta}\right)^2=\frac{u^2\cos^2\alpha}{2g\sin\theta}$$

or $$u^2\cos^2\alpha=2gl\sin\theta \quad \text{...(ii)}$$

Also during this time the distance moved perpendicular to the inclined plane with an acceleration $-g\cos\theta$ is zero.

From the motion perpendicular to the inclined plane from O to A, we get

$$0=\sin\alpha\left(\frac{u\cos\alpha}{g\sin\theta}\right)-\frac{1}{2}g\cos\theta\left(\frac{u\cos\alpha}{g\sin\theta}\right)^2, \text{ from } s=ut+\tfrac{1}{2}ft^2$$

or $$2u\sin\alpha=u\cos\alpha\cot\theta$$

or $$4u^2\sin^2\alpha=(u^2\cos^2\alpha)\cot^2\theta=2gl\sin\theta\cot^2\theta, \text{ from (ii)}$$

or $$u^2\sin^2\alpha=gl\cos^2\theta/(2\sin\theta) \quad \text{...(iii)}$$

Adding (ii) and (iii), we get

$$u^2=gl\left[2\sin\theta+\frac{\cos^2\theta}{2\sin\theta}\right]=\frac{gl}{2\sin\theta}(4\sin^2\theta+\cos^2\theta)$$

$$=gl(1+3\sin^2\theta)/2\sin\theta$$

or $$u=\sqrt{\{gl(1+3\sin^2\theta)/(2\sin\theta)\}}.$$ Hence proved.

Ex 4. An imperfectly elastic ball is projected from the foot of a plane inclined at an angle β to the horizon, with a given velocity u at an angle α to the plane. (a) Find the time that elapses before the ball has ceased rebounding and the distance described by it parallel to the plane in this time. (b) If this distance is maximum, prove that $\tan 2\alpha = (1-e) \cot \beta$, where e is the coefficient of restitution.

Sol **Part (a)** See § 10. Results (ii) and (iii) Page 47.

Part (b). If S be the total distance moved along the inclined plane till the rebounding ceases then we can prove as in § 10 result (iii) Page 47 that

$$S = [2u^2 \sin \alpha / \{g (1-e)^2 \cos^2 \beta\}] [\cos (\alpha+\beta) - e \cos \alpha \cos \beta]$$

or $$S = [u^2 / \{g (1-e)^2 \cos^2 \beta\}] [2 \sin \alpha \cos (\alpha+\beta) - e \sin 2\alpha \cos \beta].$$

If S is maximum, then $\dfrac{dS}{d\alpha} = 0$ and $\dfrac{d^2S}{d\alpha^2} =$ negative

Now $$dS/d\alpha = [u^2 / \{g (1-e)^2 \cos^2 \beta\}] [-2 \sin \alpha \sin (\alpha+\beta) + 2 \cos \alpha \cos (\alpha+\beta) - 2e \cos 2\alpha \cos \beta]$$

$$= [u^2 / \{g (1-e)^2 \cos^2 \beta\}] [2 \cos (2\alpha+\beta) - 2e \cos 2\alpha \cos \beta]$$

$\therefore$ $dS/d\alpha = 0$ gives $\cos (2\alpha+\beta) - e \cos 2\alpha \cos \beta = 0$

or $\cos 2\alpha \cos \beta - \sin 2\alpha \sin \beta - e \cos 2\alpha \cos \beta = 0$

or $(1-e) \cos 2\alpha \cos \beta = \sin 2\alpha \sin \beta$

or $(1-e) \cot \beta = \tan 2\alpha$ Hence proved.

Ex. 5 A particle of elasticity e is dropped from a vertical height a upon the highest point of a plane, which is of length b and is inclined at an angle α to the horizon and descends to the bottom in three jumps prove that $b = 4ae (1+e) (1+e+e^2) (1+e^2) \sin \alpha$.

Sol. Let the particle strike the inclined plane at O after falling vertically a height a.

$\therefore$ The velocity of the particle before impact at O

$$= \sqrt{(2ga)} = u \text{ (say)}, \quad \text{.. (i)}$$

acting in the vertically downward direction.

Hence its components along the perpendicular to the inclined plane are $u \cos (90°-\alpha)$ and $u \sin (90°-\alpha)$ or $u \sin \alpha$ and $u \cos \alpha$ respectively.

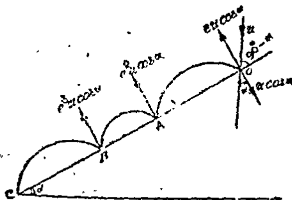
The particle afterwards takes three jumps striking the plane at A, B and C. The velocity components perpendicular to the inclined plane after impact at O, A and B are $eu \cos \alpha$, $e^2u \cos \alpha$ and $e^3u \cos \alpha$ respectively.

Times of flight of three jumps from O to A, A to B and B to C are $\frac{2eu\cos\alpha}{g\cos\alpha}$, $\frac{2e^2u\cos\alpha}{g\cos\alpha}$ and $\frac{2e^3u\cos\alpha}{g\cos\alpha}$ respectively.

i.e. $(2eu/g)$, $(2e^2u/g)$ and $(2e^3u/g)$ respectively.

$\therefore$ Total time taken in moving from O to C

$$=\frac{2eu}{g}+\frac{2e^2u}{g}+\frac{2e^3u}{g}=\frac{2eu}{g}(1+e+e^2)=T \text{ (say)} \qquad \text{...(iii)}$$



(Fig. 41)

Throughout the motion from O to C, the velocity component parallel to the inclined plane is unaffected due to impact and is affected only by gravity. Also the acceleration parallel to the inclined plane is $-g\sin\alpha$ upwards.

$\therefore$ Considering motion from O to C parallel to the plane, we have

$$OC=u\sin\alpha \cdot T+\tfrac{1}{2}(g\sin\alpha).\ T^2$$

or

$$b=(u+\tfrac{1}{2}gT)\ T\sin\alpha, \text{ where } T \text{ is given by (ii)}$$

$$=[u+eu(1+e^2)]\ (2eu/g)\ (1+e+e^2)\sin\alpha$$

$$=(2/g)\ eu^2(1+e+e^2+e^3)\ (1+e+e^2)\sin\alpha$$

$$=(2/g)\ e\ (2ag)\ (1+e+e^2+e^3)\ (1+e+e^2)\sin\alpha, \text{ from (i)}$$

$$=4ae\ (1+e)\ (1+e^2)\ (1+e+e^2)\sin\alpha \qquad \text{Hence proved}$$

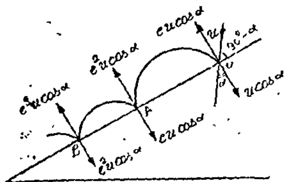
Ex. 6. A particle after falling from rest through a distance h strikes a smooth plane inclined at an angle α to the horizon. Show that the distance between the first two points of impact is $4he(1+e)\sin\alpha$, and the whole range on the inclined plane when the particle ceases to rebound is $4he\sin\alpha/(1-e)^2$

Sol. Let the particle strike the inclined plane at O after falling vertically a height h.

∴ The velocity of the particle before impact at $O=\sqrt{(2gh)}=u$ (say)...(i) acting vertically downwards. Hence its components along and perpendicular to inclined plane are $u \sin \alpha$ and $u \cos \alpha$ respectively. The particle afterwards strikes at A, B etc. The velocity components perpendicular to the inclined plane after impact at O, A, B, etc are $eu \cos \alpha$, $e^2u \cos \alpha$, $e^3u \cos \alpha$,...etc respectively.

∴ Time of flight from O to A i.e. the first two points of impact $=\dfrac{2eu \cos \alpha}{g \cos \alpha}=\dfrac{2eu}{g}$.

∴ Considering motion from O to A along the inclined plane from "$s=ut+\frac{1}{2}ft^2$" we have



(Fig. 42)

$OA=u \sin \alpha\, (2eu/g)+\frac{1}{2}\,(g \sin \alpha)\,(2eu/g)^2$

$=\dfrac{2eu^2 \sin \alpha}{g}\,[1+e]=\dfrac{2e\,(2gh)\,(1+e) \sin \alpha}{g}$, from (i)

$=4eh\,(1+e) \sin \alpha$.

Also the time for successive trajectories starting from O are

$$\frac{2eu \cos \alpha}{g \cos \alpha},\ \frac{2e^2u \cos \alpha}{g \cos \alpha},\ \frac{2e^3u \cos \alpha}{g \cos \alpha},\ \ldots \text{etc.}$$

or $(2eu/g)$, $(2e^2u/g)$, $(2e^3u/g)$, ..etc.

∴ Total time till the particle ceases to rebound

$$=\frac{2eu}{g}+\frac{2e^2u}{g}+\frac{2e^3u}{g}+\ldots\text{ad. inf.}=\frac{2eu}{g\,(1-e)}=T \text{ (say)}$$

∴ Whole range on the plane till the particle ceases to rebound

$=(u \sin \alpha).\ T+\frac{1}{2}(g \sin \alpha) T^2=(u+\frac{1}{2}gT).\ T \sin \alpha$

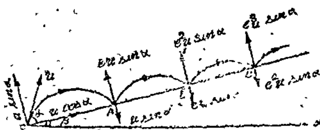
$$=\left[u+\frac{eu}{(1-e)}\right]\frac{2eu}{g(1-e)} \sin \alpha, \text{ since } T=\frac{2eu}{g(1-e)}$$

$$=\frac{2eu^2 \sin \alpha}{g(1-e)^2}=\frac{4eh \sin \alpha}{(1-e)^2}, \text{ since from (i) } u^2=2gh.$$

Ex. 7. A ball makes a series of rebounds on a given inclined plane. Show that angle θ of the nth rebound is given by $e^n \cos \theta = \cot \alpha - 2 \tan \beta \times \{(1-e^n)/(1-e)\}$, where β is the inclination of the plane to the horizon and α the angle which the direction of the projection makes with the inclined plane.

Sol Let the ball be projected from O with a velocity u making an angle α with the inclined plane. Then the resolved parts of this velocity along and perpendicular to the inclined plane are $u \cos \alpha$ and $u \sin \alpha$ respectively. The velocity of the ball perpendicular to the plane is affected by impacts and the velocity along the plane is unaffected due to impacts and affected by gravity only. Let the ball strike the plane at $A, B, C,$ etc.

Then the velocity components perpendicular to the inclined plane after impacts at A, B, C etc. are $eu \sin \alpha$, $e^2u \sin \alpha$, $e^3u \sin \alpha$, ...etc. Therefore the velocity perpendicular to the inclined plane after $(n-1)$th rebound $=e^{n-1} u \sin \alpha$ and after nth rebound $=e^n u \sin \alpha$.



(Fig 43)

Also the time of describing n trajectories are

$$\frac{2u \sin \alpha}{g \cos \beta}, \frac{2eu \sin \alpha}{g \cos \beta}, \frac{2eu^2 \sin \alpha}{g \cos \beta}, \ldots, \frac{2e^{n-1} u \sin \alpha}{g \cos \beta}$$

$\therefore$ Total time for the first n trajectories

$$=\frac{2u \sin \alpha}{g \cos \beta}+\frac{2eu \sin \alpha}{g \cos \beta}+\ldots+\frac{2e^{n-1} u \sin \alpha}{g \cos \beta}$$

$$=\frac{2u \sin \alpha}{g \cos \beta}[1+e+e^2+\ldots+e^{n-1}]=\frac{2u \sin \alpha}{g \cos \beta} \cdot \left(\frac{1-e^n}{1-e}\right)$$

$= T$ (say). ...(i)

Also considering the motion along the plane, velocity of the ball along the plane after nth rebound *i.e.* after time T

$= u \cos \alpha - g \sin \beta . T$ from "$v = u + ft$"

And we have already proved that velocity perpendicular to the plane after nth rebound $= e^n u \sin \alpha$.

∴ If after the nth rebound, the direction of particle makes an angle θ with the inclined plane, then

$$\cot \theta = \frac{\text{velocity along the plane after } n\text{th rebound}}{\text{velocity perpendicular to the plane after } n\text{th rebound}}$$

$$= \frac{u \cos \alpha - g \sin \beta \, T}{e^n u \sin \alpha}$$

or $\quad e^n \cot \theta = \cot \alpha - \{(g \sin \beta)/(u \sin \alpha)\} T$

$$= \cot \alpha - \frac{g \sin \beta}{u \sin \alpha} \cdot \frac{2u \sin \alpha}{g \cos \beta} \cdot \left(\frac{1-e^n}{1-e}\right), \text{ from (i)}$$

or $\quad e^n \cot \theta = \cot \alpha - 2 \tan \beta \{(1-e^n)/(1-e)\}$. Hence proved.

MISCELLANEOUS SOLVED EXAMPLES

Ex 1. *Two equal spheres A, B are in a smooth horizontal circular groove at opposite ends of a diameter. A is projected along the groove and at the end of time T impinges on B. Show that the second impact will occur after a further time 2T/e, where e is the coefficient of elasticity.*

Sol. Let the sphere A be projected with a velocity u. Then in time T the distance travelled by $A = u T$. Also in time T the sphere A has moved through half the circumference of the circular groove of radius a (say). So that distance moved by the sphere A in time $T = \pi a$.

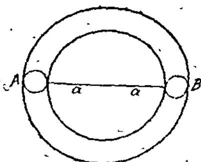
∴ $u T = \pi a$ ...(i)

Also if v and v' be the velocities of the spheres A and B after 1st impact, then from Newton's Experimental Law, we get

$$v' - v = -e \, (0 - u)$$

or $\quad v' - v = eu \quad$ (ii)

$(v' - v)$ is the velocity of sphere B relative to A and the second impact will occur when B strikes A after making a complete round with a velocity $(v' - v)$.



(Fig. 44)

Solved Examples on § 10.

Ex. 1. **A particle is projected from the foot of a plane of inclination α in a direction making an angle θ with the plane and rebounds vertically at the first impact. Show that**

$\cot\theta=(2+e)\tan\alpha$, where e is the coefficient of restitution.

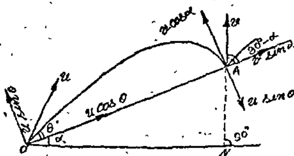
Sol. Let the particle be projected from O with a velocity u making an angle θ with the inclined plane. Let the first impact be at A.

$\therefore$ Time of flight from O to A.

$$= \frac{\text{"}2u\sin\alpha\text{"}}{g\cos\beta}=\frac{2u\sin\theta}{g\cos\alpha}=t \text{ (say)} \quad \text{(i)}$$

The ball strikes at A with a velocity whose component perpendicular to the inclined plane is $u\sin\theta$ (see figure) and after impact at A its velocity component perpendicular to the plane will become $eu\sin\theta$.

Let v be the velocity of the ball after impact at A. Since the ball rebounds vertically at A so its velocity at A will make an angle $(90°-\alpha)$ with the inclined plane. Hence its velocity components at A perpendicular and along the inclined plane will be $v\cos\alpha$ and $v\sin\alpha$ respectively.



(Fig. 38)

$$\therefore \quad v\cos\alpha=eu\sin\theta \quad \text{...(ii)}$$

in the direction perpendicular to the plane.

And $v\sin\alpha=u\cos\theta-g\sin\alpha.t$ from "$v=u+ft$"

$=u\cos\theta-g\sin\alpha\,\{(2u\sin\theta)/(g\cos\alpha)\}$, from (i)

$=(u/\cos\alpha)(\cos\theta\cos\alpha-2\sin\theta\sin\alpha)$ ...(iii)

Dividing (iii) by (ii) we get

$$\tan\alpha=\frac{\cos\theta\cos\alpha-2\sin\theta\sin\alpha}{e\cos\alpha\sin\theta}=(1/e)(\cot\theta-2\tan\alpha)$$

or $\cot\theta = (e+2)\tan\alpha$. Hence proved.

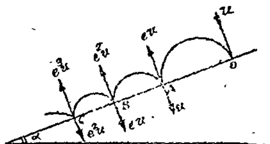
Ex. 2. A ball is projected with velocity u from a point of a plane inclined at an angle α to the horizontal, the direction of projection is at right angles to the plane. Prove that before ceasing to rebound, it will have described a length

$$(2u^2 \sin\alpha)/[g(1-e)^2 \cos^2\alpha]$$

Sol. Let the ball be projected from O with a velocity u at right angles to the inclined plane. Let afterwards the successive impacts be at A, B, C, etc. Then the velocity components perpendicular to the plane after impact at A, B, C, ..etc, will be eu, e^2u, e^3u, ..etc. (see figure).

And the times of these jumps from O to A, A to B, B to C...

etc are $\dfrac{2u}{g\cos\alpha}, \dfrac{2eu}{g\cos\alpha}, \dfrac{2e^2u}{g\cos\alpha}, \ldots$etc.



(Fig. 39)

∴ Total time taken by the by the ball before it ceases to rebound

$$= \frac{u}{g\cos\alpha} + \frac{2eu}{g\cos\alpha} + \frac{2e^2u}{g\cos\alpha} + \ldots\text{ad. inf.}$$

$$=[2u/(g\cos\alpha)]\,(1+e+e^2+\ldots\text{ad. inf.}) = [2u/(g\cos\alpha)]\,\{1/(1-e)\}$$

Also considering the motion along the plane, from "$s = ut + \frac{1}{2}ft^2$" we have

$$\text{Required length} = 0 + \tfrac{1}{2}(g\sin\alpha)\left[\frac{2u}{g\cos\alpha}\cdot\frac{1}{1-e}\right]^2,$$

Initial velocity along the plane being zero and acc. $= g\sin\alpha$

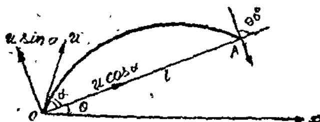
$$= (2u^2\sin\alpha)/\{g(1-e)^2\cos^2\alpha\}$$ Hence proved.

Ex. 3. *A perfectly elastic ball is thrown from the foot of a plane inclined at an angle θ to the horizon. If after striking the plane at a distance l from the point of projection it rebounds and retraces its former path, show that the velocity of projection is*

$$\sqrt{(gl(1+3\sin^2\theta)/(2\sin\theta))}.$$

Sol. The particle is projected from O and strikes the inclined plane at A, such that $OA=l$.

Now the particle can retrace its path provided it strikes the inclined plane at right angles at A. (Note)



(Fig. 40)

∴ Component of velocity along the plane vanishes at A.

Hence from "$v=u+ft$" for the motion along the plane from O to A we have $0=u \cos \alpha-(g \sin \theta)\, t$, since the acceleration along the plane is $-g \sin \theta$

or $$t=(u \cos \alpha)/(g \sin \theta) \qquad \text{...(i)}$$

During this time distance moved along the plane is l.

∴ from "$s=ut+\frac{1}{2} ft^2$" for the motion along the plane from O to A, we get

$$l=u \cos \alpha \left(\frac{u \cos \alpha}{g \sin \theta}\right)-\tfrac{1}{2} g \sin \theta \left(\frac{u \cos \alpha}{g \sin \theta}\right)^2=\frac{u^2 \cos^2 \alpha}{2g \sin \theta}$$

or $$u^2 \cos^2 \alpha=2gl \sin \theta \qquad \text{...(ii)}$$

Also during this time the distance moved perpendicular to the inclined plane with an acceleration $-g \cos \theta$ is zero.

∴ From the motion perpendicular to the inclined plane from O to A, we get

$$0=\sin \alpha \left(\frac{u \cos \alpha}{g \sin \theta}\right)-\tfrac{1}{2} g \cos \theta \left(\frac{u \cos \alpha}{g \sin \theta}\right)^2, \text{ from } s=ut+\tfrac{1}{2} ft^2$$

or $$2u \sin \alpha=u \cos \alpha \cot \theta$$

or $$4u^2 \sin^2 \alpha=(u^2 \cos^2 \alpha) \cot^2 \theta=2gl \sin \theta \cot^2 \theta, \text{ from (ii)}$$

or $$u^2 \sin^2 \alpha=gl \cos^2 \theta/(2 \sin \theta) \qquad \text{...(iii)}$$

Adding (ii) and (iii), we get

$$u^2=gl\left[2 \sin \theta+\frac{\cos^2 \theta}{2 \sin \theta}\right]=\frac{gl}{2 \sin \theta}(4 \sin^2 \theta+\cos^2 \theta)$$

$$=gl\,(1+3 \sin^2 \theta)/2 \sin \theta$$

or $$u=\sqrt{\{gl\,(1+3 \sin^2 \theta)/(2 \sin \theta)\}}.$$ Hence proved.

Ex 4. An imperfectly elastic ball is projected from the foot of a plane inclined at an angle β to the horizon, with a given velocity u at an angle α to the plane. (a) Find the time that elapses before the ball has ceased rebounding and the distance described by it parallel to the plane in this time. (b) If this distance is maximum, prove that $\tan 2\alpha=(1-e)\cot\beta$, where e is the coefficient of restitution.

Sol Part (a) See § 10. Results (ii) and (iii) Page 47.

Part (b). If S be the total distance moved along the inclined plane till the rebounding ceases then we can prove as in § 10 result (iii) Page 47 that

$$S=[2u^2\sin\alpha/\{g(1-e)^2\cos^2\beta\}][\cos(\alpha+\beta)-e\cos\alpha\cos\beta]$$

or $$S=[u^2/\{g(1-e)^2\cos^2\beta\}][2\sin\alpha\cos(\alpha+\beta)-e\sin 2\alpha\cos\beta].$$

If S is maximum, then $\frac{dS}{d\alpha}=0$ and $\frac{d^2S}{d\alpha^2}=$ negative

Now $$dS/d\alpha=[u^2/\{g(1-e)^2\cos^2\beta\}][-2\sin\alpha\sin(\alpha+\beta)$$
$$+2\cos\alpha\cos(\alpha+\beta)-2e\cos 2\alpha\cos\beta]$$
$$=[u^2/\{g(1-e)^2\cos^2\beta\}][2\cos(2\alpha+\beta)-2e\cos 2\alpha\cos\beta]$$

$\therefore$ $dS/d\alpha=0$ gives $\cos(2\alpha+\beta)-e\cos 2\alpha\cos\beta=0$

or $\cos 2\alpha\cos\beta-\sin 2\alpha\sin\beta-e\cos 2\alpha\cos\beta=0$

or $(1-e)\cos 2\alpha\cos\beta=\sin 2\alpha\sin\beta$

or $(1-e)\cot\beta=\tan 2\alpha$ Hence proved.

Ex. 5 A particle of elasticity e is dropped from a vertical height a upon the highest point of a plane, which is of length b and is inclined at an angle α to the horizon and descends to the bottom in three jumps prove that $b=4ae(1+e)(1+e+e^2)(1+e^2)\sin\alpha$.

Sol. Let the particle strike the inclined plane at O after falling vertically a height a.

$\therefore$ The velocity of the particle before impact at O

$$=\sqrt{(2ga)}=u \text{ (say)}, \quad \ldots(i)$$

acting in the vertically downward direction.

Hence its components along the perpendicular to the inclined plane are $u\cos(90°-\alpha)$ and $u\sin(90°-\alpha)$ or $u\sin\alpha$ and $u\cos\alpha$ respectively.

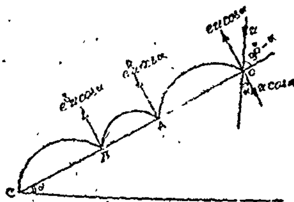
The particle afterwards takes three jumps striking the plane at A, B and C. The velocity components perpendicular to the inclined plane after impact at O, A and B are $eu\cos\alpha$, $e^2u\cos\alpha$ and $e^3u\cos\alpha$ respectively.

Times of flight of three jumps from O to A, A to B and B to C are $\frac{2eu\cos\alpha}{g\cos\alpha}$, $\frac{2e^2u\cos\alpha}{g\cos\alpha}$ and $\frac{2e^3u\cos\alpha}{g\cos\alpha}$ respectively.

i.e. $(2eu/g)$; $(2e^2u/g)$ and $(2e^3u/g)$ respectively.

∴ Total time taken in moving from O to C

$$=\frac{2eu}{g}+\frac{2e^2u}{g}+\frac{2e^3u}{g}=\frac{2eu}{g}(1+e+e^2)=T \text{ (say)} \quad ...(iii)$$



(Fig. 41)

Throughout the motion from O to C, the velocity component parallel to the inclined plane is unaffected due to impact and is affected only by gravity. Also the acceleration parallel to the inclined plane is $-g\sin\alpha$ upwards.

∴ Considering motion from O to C parallel to the plane, we have $OC=u\sin\alpha \cdot T+\frac{1}{2}(g\sin\alpha) \cdot T^2$

or $b=(u+\frac{1}{2}gT)\,T\sin\alpha$, where T is given by (ii)

$=[u+eu(1+e^2)](2eu/g)(1+e+e^2)\sin\alpha$

$=(2/g)\,eu^2(1+e+e^2+e^3)(1+e+e^2)\sin\alpha$

$=(2/g)\,e\,(2ag)(1+e+e^2+e^3)(1+e+e^2)\sin\alpha$, from (i)

$=4ae(1+e)(1+e^2)(1+e+e^2)\sin\alpha$ Hence proved

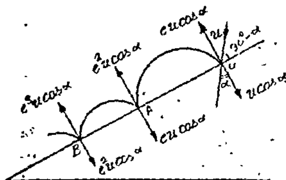
Ex. 6. A particle after falling from rest through a distance h strikes a smooth plane inclined at an angle α to the horizon. Show that the distance between the first two points of impact is $4he(1+e)\sin\alpha$, and the whole range on the inclined plane when the particle ceases to rebound is $4he\sin\alpha/(1-e)^2$

Sol. Let the particle strike the inclined plane at O after falling vertically a height h.

∴ The velocity of the particle before impact at $O=\sqrt{(2gh)}=u$ (say)...(i) acting vertically downwards. Hence its components along and perpendicular to inclined plane are $u \sin \alpha$ and $u \cos \alpha$ respectively. The particle afterwards strikes at A, B etc. The velocity components perpendicular to the inclined plane after impact at O, A, B, etc are $eu \cos \alpha$, $e^2u \cos \alpha$, $e^3u \cos \alpha$,...etc respectively.

∴ Time of flight from O to A i.e. the first two points of impact$=\dfrac{2eu \cos \alpha}{g \cos \alpha}=\dfrac{2eu}{g}$.

∴ Considering motion from O to A along the inclined plane from "$s=ut+\frac{1}{2}ft^2$" we have



(Fig. 42)

$OA=u \sin \alpha\,(2eu/g)+\frac{1}{2}\,(g \sin \alpha)\,(2eu/g)^2$

$=\dfrac{2eu^2 \sin \alpha}{g}\,[1+e]=\dfrac{2e\,(2gh)\,(1+e) \sin \alpha}{g}$, from (i)

$=4eh\,(1+e) \sin \alpha$.

Also the time for successive trajectories starting from O are

$$\frac{2eu \cos \alpha}{g \cos \alpha},\ \frac{2e^2u \cos \alpha}{g \cos \alpha},\ \frac{2e^3u \cos \alpha}{g \cos \alpha},\ ..\text{etc},$$

or $(2eu/g)$, $(2e^2u/g)$, $(2e^3u/g)$,...etc.

∴ Total time till the particle ceases to rebound

$$=\frac{2eu}{g}+\frac{2e^2u}{g}+\frac{2e^3u}{g}+\text{...ad. inf.}=\frac{2eu}{g\,(1-e)}=T\ \text{(say)}$$

∴ Whole range on the plane till the particle ceases to rebound

$=(u \sin\alpha).\ T+\frac{1}{2}(g \sin\alpha)\,T^2=(u+\frac{1}{2}gT).\ T\sin\alpha$

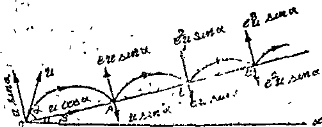
$$=\left[u+\frac{eu}{(1-e)}\right]\frac{2eu}{g\,(1-e)}\sin\alpha,\ \text{since } T=\frac{2eu}{g\,(1-e)}.$$

$$=\frac{2eu^2\sin\alpha}{g\,(1-e)^2}=\frac{4eh\sin\alpha}{(1-e)^2},\ \text{since from (i) } u^2=2gh.$$

Ex. 7. A ball makes a series of rebounds on a given inclined plane. Show that angle θ of the nth rebound is given by $e^n \cos\theta = \cot\alpha - 2\tan\beta\times\{(1-e^n)/(1-e)\}$, where β is the inclination of the plane to the horizon and α the angle which the direction of the projection makes with the inclined plane.

Sol Let the ball be projected from O with a velocity u making an angle α with the inclined plane. Then the resolved parts of this velocity along and perpendicular to the inclined plane are $u\cos\alpha$ and $u\sin\alpha$ respectively. The velocity of the ball perpendicular to the plane is affected by impacts and the velocity along the plane is unaffected due to impacts and affected by gravity only. Let the ball strike the plane at A, B, C, etc.

Then the velocity components perpendicular to the inclined plane after impacts at A, B, C etc. are $eu\sin\alpha$, $e^2u\sin\alpha$, $e^3u\sin\alpha$, ...etc. Therefore the velocity perpendicular to the inclined plane after $(n-1)$th rebound$=e^{n-1}u\sin\alpha$ and after nth rebound$=e^n u \sin\alpha$.



(Fig 43)

Also the time of describing n trajectories are

$$\frac{2u\sin\alpha}{g\cos\beta},\ \frac{2eu\sin\alpha}{g\cos\beta},\ \frac{2e^2u\sin\alpha}{g\cos\beta},\ \ldots,\ \frac{2e^{n-1}u\sin\alpha}{g\cos\beta}$$

$\therefore$ Total time for the first n trajectories,

$$=\frac{2u\sin\alpha}{g\cos\beta}+\frac{2eu\sin\alpha}{g\cos\beta}+\ldots+\frac{2e^{n-1}u\sin\alpha}{g\cos\beta}$$

$$=\frac{2u\sin\alpha}{g\cos\beta}[1+e+e^2+\ldots+e^{n-1}]=\frac{2u\sin\alpha}{g\cos\beta}\cdot\left(\frac{1-e^n}{1-e}\right)$$

$=T$ (say). ...(i)

Also considering the motion along the plane, velocity of the ball along the plane after nth rebound *i.e.* after time T

$=u \cos \alpha - g \sin \beta . T$ from "$v=u+ft$"

And we have already proved that velocity perpendicular to the plane after nth rebound $=e^n u \sin \alpha$.

$\therefore$ If after the nth rebound, the direction of particle makes an angle θ with the inclined plane, then

$$\cot \theta = \frac{\text{velocity along the plane after } n\text{th rebound}}{\text{velocity perpendicular to the plane after } n\text{th rebound}}$$

$$= \frac{u \cos \alpha - g \sin \beta\, T}{e^n u \sin \alpha}$$

or $\quad e^n \cot \theta = \cot \alpha - \{(g \sin \beta)/(u \sin \alpha)\}\, T$

$$= \cot \alpha - \frac{g \sin \beta}{u \sin \alpha} \cdot \frac{2u \sin \alpha}{g \cos \beta} \cdot \left(\frac{1-e^n}{1-e}\right); \text{ from (i)}$$

or $\quad e^n \cot \theta = \cot \alpha - 2 \tan \beta\, [(1-e^n)/(1-e)]$. Hence proved.

MISCELLANEOUS SOLVED EXAMPLES

Ex. 1. Two equal spheres A, B are in a smooth horizontal circular groove at opposite ends of a diameter. A is projected along the groove and at the end of time T impinges on B. Show that the second impact will occur after a further time 2T/*e*, where *e* is the coefficient of elasticity.

Sol. Let the sphere A be projected with a velocity u. Then in time T the distance travelled by $A = u\,T$. Also in time T the sphere A has moved through half the circumference of the circular groove of radius a (say). So that distance moved by the sphere A in time $T = \pi a$.

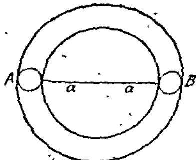
$\therefore \quad u\,T = \pi a$ ...(i)

Also if v and v' be the velocities of the spheres A and B after 1st impact, then from Newton's Experimental Law, we get

$$v' - v = -e\,(0-u)$$

or $\quad v' - v = eu$ (iii)

$(v'-v)$ is the velocity of sphere B relative to A and the second impact will occur when B strikes A after making a complete round with a velocity $(v'-v)$.



(Fig. 44)

If the second impact occurs after a further time t,

then $(v'-v)\ t=2\ \pi a$, circumference of the circle

or $t=2\pi a/(v'-v)=2Tu/(eu)$, from (i) and (ii)

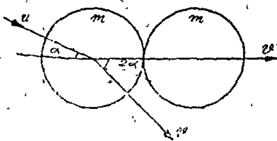
or $t=2T/e$. Hence proved.

Ex. 2. A smooth ball impinges on another smooth equal ball at rest in a direction that bisects the angle between the subsequent direction of balls. If the direction of motion of the impinging ball before impact makes an angle α with the line of centres, prove that $\tan \alpha=\sqrt{e}$

Sol. Let the impinging ball move with a velocity u making an angle α with the line of centres before impact. After impact the other ball which was at rest will move along the line of centres with a velocity v', say. Therefore impinging ball will move after impact in a direction making an angle 2α with the line of centres. Let v be its velocity after impact. If m be the mass of each ball we have by the Principle of conservation of momentum—

$$mv'+m.v \cos 2\alpha=m\ 0+mu \cos \alpha$$

or $$v'+v \cos 2\alpha=u \cos \alpha \quad \text{...(iv)}$$



(Fig. 45)

Also by Newton's Experimental Law, we get

$$v'-v \cos 2\alpha=-e\ (0-u \cos \alpha) \text{ or } v'-v \cos 2\alpha=eu \cos \alpha \quad \text{(ii)}$$

Subtracting (ii) from (i) we get

$$2v \cos 2\alpha=(1-e)\ u \cos \alpha \quad \text{.. (iii)}$$

Also as the spheres are smooth, there is no force acting perpendicular to the line of centres during impact, hence velocity of the balls perpendicular to the line of centres will remain unaltered, i.e.

$$v \sin 2\alpha=u \sin \alpha \quad \text{(iv)}$$

Dividing (iv) by (iii) we get

$\frac{1}{2}\tan 2\alpha=\frac{\tan\alpha}{(1-e)}$ or $\frac{1}{2}\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{\tan\alpha}{(1-e)}$

or $1-e=1-\tan^2\alpha$ or $\tan^2\alpha=e$ or $\tan\alpha=\sqrt{e}$. Hence proved.

***Ex. 3.** **Particles are projected horizontally from different points in a tower of height h_1, each with a velocity due to the height of the tower above the point of projection. Show that they will cease rebounding from the horizontal plane through *the foot of the* tower within or on a circle of radius $[(1+e)/(1-e)]\, h$, with its centre at the foot of the tower**

Sol Let PQ be the tower of height h. Let O be a point on it at a height x from the foot Q of the tower, then its depth below the top P is $(h-x)$.

$\therefore$ according to the problem a particle is projected horizontally from O with a velocity $\sqrt{\{2g(h-x)\}}$.

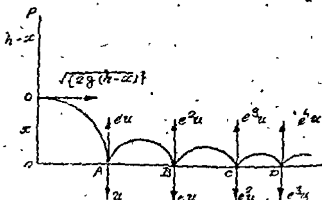
It strikes the horizontal plane through Q at $A, B, C,$...etc. in the subsequent motion.

At O the vertical component of velocity of the particle is zero.

$\therefore$ When the particle reaches A its vertical velocity *before* impact is

$$\sqrt{(2gx)}=u \text{ (say)} \qquad ...(i)$$

Since only the vertical component of velocity of the particle will be affected due to impacts on a horizontal plane, so *after* impact at $A, B, C,$ ... etc. the vertical velocities will be eu, e^2u, e^3u, e^4u etc.



(Fig. 46)

And time of describing successive trajectories starting from A are $(2eu/g)$, $(2e^2u/g)$, $(2e^3u/g)$, ..etc.

∴ Total time taken by this particle from A till rebounding ceases

$$=(2eu/g)+(2e^2u/g)+(2e^3u/g)+\ldots\text{ad. inf.}$$

$$=\frac{2eu}{g}(1+e+e^2+\ldots\text{ad. inf.})=\frac{2eu}{g(1-e)}=\frac{2e\sqrt{(2gx)}}{g(1-e)},\text{ from (i)}$$

Also the time taken by this particle moving from O to A is given by "$s=ut+\frac{1}{2}ft^2$" or $x=0+\frac{1}{2}gt^2$ or $t=\sqrt{(2x/g)}$.

Total time taken by this particle in moving from O till the rebounding ceases $=\sqrt{\left(\frac{2x}{g}\right)}+\frac{2e\sqrt{(2gx)}}{g(1-e)}=\sqrt{\left(\frac{2x}{g}\right)}\left[1+\frac{2e}{(1-e)}\right]$

$$=\sqrt{(2x/g)}\,[(1+e)/(1-e)] \quad \ldots\text{(ii)}$$

Through this time horizontal velocity remains constant and equal to $\sqrt{\{2g(h-x)\}}$. Hence total distance moved by this particle on the horizontal plane (from the foot O till the rebounding ceases)

= horizontal velocity × total time given by (ii)

$$=\sqrt{\{2g(h-x)\}}\sqrt{\left(\frac{2x}{g}\right)}\cdot\left(\frac{1+e}{1-e}\right)=\left(\frac{1+e}{1-e}\right)\sqrt{\{x(h-x)\}} \quad \ldots\text{(iii)}$$

The maximum value of this distance is obtained when $x(h-x)$ is maximum

i.e. when first differential of $x(h-x)$ with respect to x vanishes

i.e. when $h-2x=0$ *i.e.* when $x=\frac{1}{2}h$.

∴ From (iii), maximum distance reached by any particle on the horizontal plane through O

$=2\,[(1+e)/(1-e)]\sqrt{\{\frac{1}{2}h\,(h-\frac{1}{2}h)\}}$, putting $x=\frac{1}{2}h$ in (iii)

$=[(1+e)/(1-e)]\,h$

As the particles are projected in all directions from different points of tower, so they lie in the horizontal plane within or on a circle of radius $[(1+e)/(1-e)]\,h$ with the foot O of the tower as centre.

**Ex. 4 A ball is dropped from the top of a tower of height h and at the same time another ball of equal weight is projected upwards from the base of the tower with velocity just sufficient to take it to the top of the tower and collides directly with the falling ball. Prove that the falling ball will in the rebound rise to a height short of the top of the tower by $\frac{1}{4}h(1-e^2)$.

Sol. Let A and B be the top and base of the tower and let balls collide at C. The lower ball is projected upwards with velo-

city just sufficient to take it to the top hence its velocity of projection u_1 (say) is given by

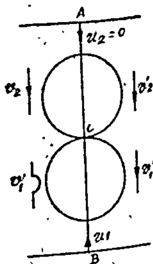
$0=u_1^2-2gh$ or $u_1=\sqrt{(2gh)}$ ...(i)

Since the upper ball is dropped from A, hence its initial velocity u_2 (say) is zero. Again since both the balls start moving simultaneously, hence both of them take the same time t (say) to reach the point C.

$h=AC+BC=\frac{1}{2}gt^2+(u_1t-\frac{1}{2}gt^2)$

or $h=u_1t$ or $t=h/u_1=\sqrt{(h/2g)}$.(ii)

putting the values of velocities of the lower and upper balls before colliding at C as v_1 and v_2



(Fig 47)

Then $v_1=u_1-gt=\sqrt{(2g)}-g\sqrt{(h/2g)}=\sqrt{(\frac{1}{2}gh)}$

and $v_2=u_2+gt=0+g\sqrt{(h/2g)}=\sqrt{(\frac{1}{2}gh)}$

If their respective velocities after the impact be v_1' and v_2' and both downwards (say), then from the principle of conservation of momentum and Newton's experimental law we have

$(w/g)(-v_1)+(w/g)v_2=(w/g)v_1'+(w/g)v_2'$,

where w is the weight of each ball. .. (iii)

or $v_1'+v_2'=0$, $\because$ $v_1=v_2$

and $v_1'-v_2'=-e\{(-v_1)-v_2\}=e(v_1+v_2)$ .. (iv)

$=e[\sqrt{(\frac{1}{2}gh)}+\sqrt{(\frac{1}{2}gh)}]=e\sqrt{(2gh)}$

$\therefore$ Subtracting (iv) from (iii), we get $v_2'=-e\sqrt{(\frac{1}{2}gh)}$, which being negative shows that the falling ball, after the impact, moves upward with velocity $e\sqrt{(\frac{1}{2}gh)}$ Suppose it rebounds upto D, then we have $0=[e\sqrt{(\frac{1}{2}gh)}]^2-2g\,CD$ or $CD=\frac{1}{4}he^2$.

$\therefore$ In the rebound it rises to a height short of the top of the tower by AD which is given by-

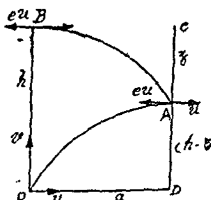
$AD=AC-CD=\frac{1}{2}gt^2-\frac{1}{4}he^2$

$=\frac{1}{2}g(h/2g)-\frac{1}{4}he^2=\frac{1}{4}h(1-e^2)$ Hence proved.

Ex 5 Two parallel vertical walls of height h stand on a horizontal plane. A ball is projected from the foot of one wall towards the other and after impact it just clears the top of the first wall. Prove that the point of impact with the second wall is at a depth $h/(1+e)^2$ below its top, where e is the coefficient of restitution.

Sol. OB and CB are the walls at a distance a (say) apart. Let the ball be projected from O with u and v as horizontal and vertical

velocities. In the subsequent motion of the ball, let it strike the second wall at A and afterwards just reach the top B of the first wall.



(Fig. 48)

Let the depth of A below the top C of the second wall be z.

Due to the impact with wall only the horizontal velocity will be affected and therefore the horizontal velocity of the ball after impact at $A = eu$.

Time taken by the ball in moving from O to $A = (a/u)$ and time taken by the ball in moving from A to $B = (a/eu)$.

$\therefore$ Total time taken by the ball in moving from O to B

$$= \frac{a}{u} + \frac{a}{eu} = \frac{a}{u}\left(1 + \frac{1}{e}\right) = \frac{a(e+1)}{eu} \quad \ldots(i)$$

Also as the ball just reaches B, $\therefore$ the vertical velocity at B is zero and vertical velocity is also not affected due to impact at B. Hence considering the vertical motion of the ball from O to B from "$v = u + ft$" we have

$0 = v - gt$ where t is the time taken by the ball in moving from O to B

or $t = (v/g)$, which is the same as the time given by (i), hence

$$\frac{v}{g} = \frac{a(e+1)}{eu} \text{ or } \frac{a}{u} = \frac{ev}{g(e+1)} \quad \ldots(ii)$$

Also for the vertical motion from O to B, from "$v^2 = u^2 + 2fs$" we have $0 = v^2 - 2gh$ or $v^2 = 2gh$ $\quad \ldots(iii)$

And for vertical motion from O to A, from "$s = ut + \frac{1}{2}ft^2$" we have $(a - z) = vt - \frac{1}{2}gt^2$, where t is the time from O to A,

or $(h - z) = v\,(a/u) - \frac{1}{2}g\,(a/u)^2$, $\because$ $t = au$ = time from O to A

$$= v\left[\frac{ev}{g(e+1)}\right] - \frac{g}{2}\left[\frac{ev}{g(e+1)}\right]^2, \text{ from (ii)}$$

$$= \frac{ev^2}{g(e+1)}\left[1 - \frac{e}{2(e+1)}\right] = \frac{e.2gh}{g(e+1)}\left[\frac{e+2}{2(e+1)}\right], \text{ from (iii)}$$

$= [e\,(e+2)\,h/(e+1)^2]$

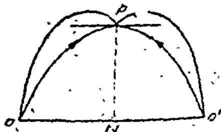
or $z = h - [e\,(e+2)\,h/(e+1)^2] = h/(e+1)^2$ Hence proved.

Ex. 6. Two equal elastic balls are projected towards each other at the same instant in the same vertical plane, v being the velocity

and α the elevation in each case. Show that after impact they will return to the points of projection if $ga(1+e)=ev^2 \sin 2\alpha$, where $2a$ is the distance between the points of projection.

Sol. Let O and O' be the points of projection of the two balls. Since their velocity and angle of projection is the same, so they would describe equal parabolas. Let them strike each other at P, then P is vertically above N, where N is middle point of $OO'=2a$ (given).

After striking each other at P, the balls, return to their points of projection, hence the motion of each ball is the same as if it strikes a vertical wall at a distance a from its point of projection and then return back to its point of projection.



(Fig. 49)

Due to the impact at P, the vertical component of velocity of each ball is not affected.

∴ Time of flight of each ball, since the ball returns to the point of projection $=2(v \sin \alpha)/g$ ...(i)

Also as the horizontal velocity $v \cos \alpha$ remains constant for the motion from O to P of the ball projected from O (similar is the case of 2nd ball), so the time from O to $P=(a/v \cos \alpha)$ ...(ii)

After impact at P, the horizontal velocity of this ball becomes $ev \cos \alpha$, which remains consta t for its motion from P to O.

∴ the time taking by this ball in moving from P to O

$$=a/(ev \cos \alpha) \quad \text{...(iii)}$$

∴ Total time taken by each ball to return to its point of projection $=\dfrac{a}{v \cos \alpha}+\dfrac{a}{ev \cos \alpha}$, from (ii) and (iii)

$$=a(e+1)/(ev \cos \alpha)$$

Since time given by (i) and (iv) are equal, so we get

$$\frac{2v \sin \alpha}{g}=\frac{a(e+1)}{ev \cos \alpha} \text{ or } ev^2 \sin 2\alpha=ag(1+e)$$ Hence proved.

*Ex. 7. A sphere of mass m falls on a smooth hemisphere of mass M resting with its plane face on smooth horizontal table, so

that at the moment of impact line of centres makes an angle α with the vertical. Find the equations for determining the velocities of the bodies after impact, the velocity of the sphere just before impact is u and e is the coefficient of restitution.

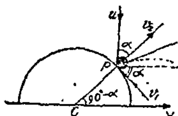
Sol. Let the sphere strike the hemisphere at P. Let the velocity of the sphere after impact be v whose components along the tangent and normal at P to the hemisphere are v_1 and v_2 respectively.

After impact the hemisphere moves on the horizontal table with a velocity V (say).

On the system as a whole there is no horizontal force acting during the period of impact, hence change in momentum of the system in the horizontal direction remains unaltered.

i.e. Total momenta after impact = total momenta before impact

or $MV+mv_1\cos\alpha+mv_2\cos(90°-\alpha)=M\,0+mu\cos 90°$



(Fig. 50)

or
$$MV+mv_1\cos\alpha+mv_2\sin\alpha=0 \qquad ...(i)$$

Also by Newton's Experimental Law in the direction of common normal at P, we have

$$v_2-V\cos(90°-\alpha)=-e\,[(-u\cos\alpha)-0]$$

or
$$v_2-V\sin\alpha=eu\cos\alpha. \qquad ...(ii)$$

Also as the velocity of the sphere perpendicular to the common normal remains unaltered due to impact, so we have

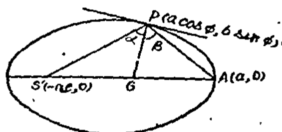
$$v_1=u\sin\alpha \qquad ...(iii)$$

Equations (i), (ii), and (iii) will give the values of V, v_1 and v_2.

**Ex. 8. A ball at the focus of an ellipse whose eccentricity is e, receives a blow, and after one impact on the elliptic periphery passes through the other end of the major axis. Find the point of impact on the ellipse and show that coefficient of restitution can not be greater than $2e/(1+e^2)$.

Sol. The ball is projected from the focus $S'(-ae, 0)$, strikes the elliptic periphery at $P\,(a\cos\phi,\, b\sin\phi)$ and then reaches the

other end $A(a, 0)$ of the major axis. Here the equation of the ellipse has been taken as $(x^2/a^2)+(y^2/b^2)=1$.



Let PG be the normal to the ellipse at P.

Let α and β be the angles of incidence and reflection at P. Then if e' be the coefficient of restitution, we have

$$\cot \beta = e' \cot \alpha \text{ or } \tan \alpha = e' \tan \beta \quad \text{...(i)}$$

Also the slope of the normal $PG = -\text{"}\dfrac{dx}{dy}\text{"} = \dfrac{a \sin \phi}{b \cos \phi}$,

the slope of the line $S'P = \text{"}\dfrac{y_2 - y_1}{x_2 - x_1}\text{"} = \dfrac{b \sin \phi - 0}{a \cos \phi - (-ae)}$

$$=(b \sin \phi)/\{a(e+\cos \phi)\};$$

and the slope of the line $PA = \dfrac{b \sin \phi - 0}{a \cos \phi - a} = \dfrac{b \sin \phi}{a(\cos \phi - 1)}$

Since α is the angle between $S'P$ and PG so we have

$$\tan \alpha = \text{"}\frac{m_1 - m_2}{1+m_1 m_2}\text{"} = \frac{\dfrac{a \sin \phi}{b \cos \phi} - \dfrac{b \sin \phi}{a(e+\cos \phi)}}{1+\dfrac{a \sin \phi}{b \cos \phi}\,\dfrac{b \sin \phi}{a(e+\cos \phi)}}$$

$$= \frac{a^2 \sin \phi (e+\cos \phi) - b^2 \sin \phi \cos \phi}{ab \cos \phi (e+\cos \phi) + ab \sin^2 \phi}$$

$$= \frac{a^2 e (e \cos \phi + 1) \sin \phi}{ab(1+e \cos \phi)}, \because b^2 = a^2 - a^2 e^2$$

or

$$\tan \alpha = (ae \sin \phi)/b \quad \text{...(ii)}$$

Also β is the angle between PG and PA, so we have

$$\tan \beta = \frac{[(b \sin \phi)/\{a(\cos \phi - 1)\}] - [(a \sin \phi)/(b \cos \phi)]}{1+[(b \sin \phi)/\{a(\cos \phi - 1)\}]\,[(a \sin \phi)/(b \cos \phi)]}$$

$$= \frac{b^2 \sin \phi \cos \phi + a^2 \sin \phi (1 - \cos \phi)}{ab \cos \phi (\cos \phi - 1) + ab \sin^2 \phi}$$

$$= \frac{a^2 \sin \phi - (a^2 - b^2) \sin \phi \cos \phi}{ab - ab \cos \phi}$$

or $\tan \beta = \dfrac{a(1 - e^2 \cos \phi) \sin \phi}{b(1 - \cos \phi)}$, $\because b^2 = a^2 - a^2 e^2$ ...(iii)

If the second impact occurs after a further time t,

then $(v'-v)\ t=2\pi a$, circumference of the circle

or $t=2\pi a/(v'-v)=2Tu/(eu)$, from (i) and (ii)

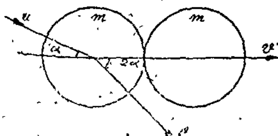
or $t=2T/e$. Hence proved.

Ex. 2. A smooth ball impinges on another smooth equal ball at rest in a direction that bisects the angle between the subsequent direction of balls. If the direction of motion of the impinging ball before impact makes an angle α with the line of centres, prove that $\tan \alpha=\sqrt{e}$

Sol. Let the impinging ball move with a velocity u making an angle α with the line of centres before impact. After impact the other ball which was at rest will move along the line of centres with a velocity v', say. Therefore impinging ball will move after impact in a direction making an angle 2α with the line of centres. Let v be its velocity after impact. If m be the mass of each ball we have by the Principle of conservation of momentum—

$$mv'+m.v\cos 2\alpha=m.0+mu\cos\alpha$$

or $$v'+v\cos 2\alpha=u\cos\alpha \quad ...(iv)$$



(Fig. 45)

Also by Newton's Experimental Law, we get

$v'-v\cos 2\alpha=-e.(0-u\cos\alpha)$ or $v'-v\cos 2\alpha=eu\cos\alpha$ (ii)

Subtracting (ii) from (i) we get

$$2v\cos 2\alpha=(1-e)\ u\cos\alpha \quad ..(iii)$$

Also as the spheres are smooth, there is no force acting perpendicular to the line of centres during impact, hence velocity of the balls perpendicular to the line of centres will remain unaltered.

i.e. $$v\sin 2\alpha=u\sin\alpha \quad (iv)$$

Dividing (iv) by (iii) we get

$\frac{1}{2}\tan 2\alpha=\frac{\tan\alpha}{(1-e)}$ or $\frac{1}{2}\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{\tan\alpha}{(1-e)}$

or $1-e=1-\tan^2\alpha$ or $\tan^2\alpha=e$ or $\tan\alpha=\sqrt{e}$. Hence proved.

*Ex. 3. *Particles are projected horizontally from different points in a tower of height h_1, each with a velocity due to the height of the tower above the point of projection. Show that they will cease rebounding from the horizontal plane through the foot of the tower within or on a circle of radius $[(1+e)/(1-e)]\,h$, with its centre at the foot of the tower*

Sol Let PQ be the tower of height h. Let O be a point on it at a height x from the foot Q of the tower, then its depth below the top P is $(h-x)$.

$\therefore$ according to the problem a particle is projected horizontally from O with a velocity $\sqrt{\{2g(h-x)\}}$.

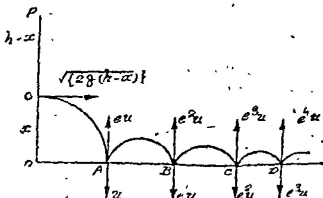
It strikes the horizontal plane through Q at $A, B, C,$...etc. in the subsequent motion.

At O the vertical component of velocity of the particle is zero.

$\therefore$ When the particle reaches A its vertical velocity before impact is

$$\sqrt{(2gx)}=u \text{ (say)} \quad ...(1)$$

Since only the vertical component of velocity of the particle will be affected due to impacts on a horizontal plane, so after impact at $A, B, C,$ .. ..etc. the vertical velocities will be eu, e^2u, e^3u, e^4u etc.



(Fig. 46)

And time of describing successive trajectories starting from A are $(2eu/g)$, $(2e^2u/g)$, $(2e^3u/g)$, ..etc.

$\therefore$ Total time taken by this particle from A till rebounding ceases

$=(2eu/g)+(2e^2u/g)+(2e^3u/g)+...$ad. inf.

$=\frac{2eu}{g}(1+e+e^2+...\text{ad. inf.})=\frac{2eu}{g(1-e)}=\frac{2e\sqrt{(2gx)}}{g(1-e)}$, from (i)

Also the time taken by this particle moving from O to A is given by "$s=ut+\frac{1}{2}ft^2$" or, $x=0+\frac{1}{2}gt^2$ or $t=\sqrt{(2x/g)}$.

Total time taken by this particle in moving from O till the rebounding ceases $=\sqrt{\left(\frac{2x}{g}\right)}+\frac{2e\sqrt{(2gx)}}{g(1-e)}=\sqrt{\left(\frac{2x}{g}\right)}\left[1+\frac{2e}{(1-e)}\right]$

$=\sqrt{(2x/g)}\,[(1+e)/(1-e)]$ ...(ii)

Through this time, horizontal velocity remains constant and equal to $\sqrt{\{2g(h-x)\}}$. Hence total distance moved by this particle on the horizontal plane (from the foot Q till the rebounding ceases)

$=$horizontal velocity $\times$ total time given by (ii)

$=\sqrt{\{2g(h-x)\}}\sqrt{\left(\frac{2x}{g}\right)}\cdot\left(\frac{1+e}{1-e}\right)=\left(\frac{1+e}{1-e}\right)\sqrt{\{x(h-x)\}}$ ..(iii)

The maximum value of this distance is obtained when $x(h-x)$ is maximum

i.e. when first differential of $x(h-x)$ with respect to x vanishes

i.e. when $h-2x=0$ i.e. when $x=\frac{1}{2}h$.

$\therefore$ From (iii), maximum distance reached by any particle on the horizontal plane through Q

$=2[(1+e)/(1-e)]\sqrt{\{\frac{1}{2}h(h-\frac{1}{2}h)\}}$, putting $x=\frac{1}{2}h$ in (iii)

$=[(1+e)/(1-e)]\,h$

As the particles are projected in all directions from different points of tower, so they lie in the horizontal plane within or on a circle of radius $[(1+e)/(1-e)]\,h$ with the foot Q of the tower as centre.

****Ex. 4** A ball is dropped from the top of a tower of height h and at the same time another ball of equal weight is projected upwards from the base of the tower with velocity just sufficient to take it to the top of the tower and collides directly with the falling ball. Prove that the falling ball will in the rebound rise to a height short of the top of the tower by $\frac{1}{4}h(1-e^2)$.**

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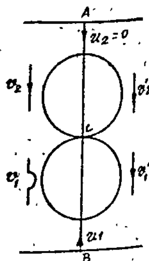
$0=u_1^2-2gh$ or $u_1=\sqrt{(2gh)}$ ...(i)

Since the upper ball is dropped from A, hence its initial velocity u_2 (say) is zero. Again since both the balls start moving simultaneously, hence both of them take the same time t (say) to reach the point C.

$h=AC+BC=\frac{1}{2}gt^2+(u_1t-\frac{1}{2}gt^2)$

or $h=u_1t$ or $t=h/u_1=\sqrt{(h/2g)}$. (ii)

putting the values of velocities of the lower and upper balls before colliding at C as v_1 and v_2.



(Fig 47)

Then $v_1=u_1-gt=\sqrt{(2g)}-g\sqrt{(h/2g)}=\sqrt{(\frac{1}{2}gh)}$

and $v_2=u_2+gt=0+g\sqrt{(h/2g)}=\sqrt{(\frac{1}{2}gh)}$

If their respective velocities after the impact be v_1' and v_2' and both downwards (say), then from the principle of conservation of momentum and Newton's experimental law we have

$(w/g)(-v_1)+(w/g)v_2=(w/g)v_1'+(w/g)v_2'$,

where w is the weight of each ball.

or $v_1'+v_2'=0$, $\because$ $v_1=v_2$ .. (iii)

and $v_1'-v_2'=-e\{(-v_1)-v_2\}=e(v_1+v_2)$

$=e[\sqrt{(\frac{1}{2}gh)}+\sqrt{(\frac{1}{2}gh)}]=e\sqrt{(2gh)}$ .. (iv)

$\therefore$ Subtracting (iv) from (iii), we get $v_2'=-e\sqrt{(\frac{1}{2}gh)}$, which being negative shows that the falling ball, after the impact, moves upward with velocity $e\sqrt{(\frac{1}{2}gh)}$. Suppose it rebounds upto D, then we have $0=[e\sqrt{(\frac{1}{2}gh)}]^2-2g\,CD$ or $CD=\frac{1}{4}he^2$.

$\therefore$ In the rebound it rises to a height short of the top of the tower by AD which is given by

$AD=AC-CD=\frac{1}{2}gt^2-\frac{1}{4}he^2$

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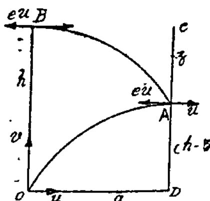
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Let the depth of A below the top C of the second wall be z.

Due to the impact with wall only the horizontal velocity will be affected and therefore the horizontal velocity of the ball after impact at $A = eu$.



(Fig. 48)

Time taken by the ball in moving from O to $A = (a/u)$ and time taken by the ball in moving from A to $B = (a/eu)$.

∴ Total time taken by the ball in moving from O to B

$$= \frac{a}{u} + \frac{a}{eu} = \frac{a}{u}\left(1 + \frac{1}{e}\right) = \frac{a(e+1)}{eu} \quad ..(i)$$

Also as the ball just reaches B, ∴ the vertical velocity at B is zero and vertical velocity is also not affected due to impact at B. Hence considering the vertical motion of the ball from O to B from "$v = u + ft$" we have

$0 = v - gt$ where t is the time taken by the ball in moving from O to B

or $t = (v/g)$, which is the same as the time given by (i), hence

$$\frac{v}{g} = \frac{a(e+1)}{eu} \text{ or } \frac{a}{u} = \frac{ev}{g(e+1)} \quad ..(ii)$$

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And for vertical motion from O to A, from "$s = ut + \frac{1}{2}ft^2$" we have $(a - z) = vt - \frac{1}{2}gt^2$, where t is the time from O to A,

or $(h - z) = v(a/u) - \frac{1}{2}g(a/u)^2$, ∵ $t = au =$ time from O to A

$$= v\left[\frac{ev}{g(e+1)}\right] - \frac{g}{2}\left[\frac{ev}{g(e+1)}\right]^2, \text{ from (ii)}$$

$$= \frac{ev^2}{g(e+1)}\left[1 - \frac{e}{2(e+1)}\right] = \frac{e.2gh}{g(e+1)}\left[\frac{e+2}{2(e+1)}\right], \text{ from (iii)}$$

$= [e(e+2)h/(e+1)^2]$

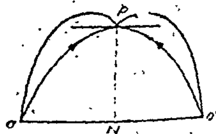
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Sol. Let O and O' be the points of projection of the two balls. Since their velocity and angle of projection is the same, so they would describe equal parabolas. Let them strike each other at P, then P is vertically above N, where N is middle point of $OO'=2a$ (given).

After striking each other at P, the balls, return to their points of projection, hence the motion of each ball is the same as if it strikes a vertical wall at a distance a from its point of projection and then return back to its point of projection.



(Fig. 49)

Due to the impact at P, the vertical component of velocity of each ball is not affected.

$\therefore$ Time of flight of each ball, since the ball returns to the point of projection $=2(v\sin\alpha)/g$ ...(i)

Also as the horizontal velocity $v\cos\alpha$ remains constant for the motion from O to P of the ball projected from O (similar is the case of 2nd ball), so the time from O to $P=(a/v\cos\alpha)$ ...(ii)

After impact at P, the horizontal velocity of this ball becomes $ev\cos\alpha$, which remains constant for its motion from P to O.

$\therefore$ the time taking by this ball in moving from P to O

$$=a/(ev\cos\alpha) \quad ...(iii)$$

$\therefore$ Total time taken by each ball to return to its point of projection$=\dfrac{a}{v\cos\alpha}+\dfrac{a}{ev\cos\alpha}$, from (ii) and (iii)

$$=a(e+1)/(ev\cos\alpha)$$

Since time given by (i) and (iv) are equal, so we get

$\dfrac{2v\sin\alpha}{g}=\dfrac{a(e+1)}{ev\cos\alpha}$ or $ev^2\sin 2\alpha=ag(1+e)$ Hence proved.

*Ex. 7. A sphere of mass m falls on a smooth hemisphere of mass M resting with its plane face on smooth horizontal table, so

that at the moment of impact line of centres makes an angle α with the vertical. Find the equations for determining the velocities of the bodies after impact, the velocity of the sphere just before impact is u and e is the coefficient of restitution.

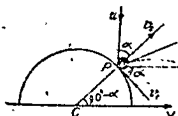
Sol. Let the sphere strike the hemisphere at P. Let the velocity of the sphere after impact be v whose components along the tangent and normal at P to the hemisphere are v_1 and v_2 respectively.

After impact the hemisphere moves on the horizontal table with a velocity V (say).

On the system as a whole there is no horizontal force acting during the period of impact, hence change in momentum of the system in the horizontal direction remains unaltered

i.e. Total momenta after impact = total momenta before impact

or $\quad MV+mv_1\cos\alpha+mv_2\cos(90^\circ-\alpha)=M\,0+mu\cos 90^\circ$



(Fig. 50)

or
$$MV+mv_1\cos\alpha+mv_2\sin\alpha=0 \qquad ..(i)$$

Also by Newton's Experimental Law in the direction of common normal at P, we have

$$v_2-V\cos(90^\circ-\alpha)=-e[(-u\cos\alpha)-0]$$

or
$$v_2-V\sin\alpha=eu\cos\alpha. \qquad ..(ii)$$

Also as the velocity of the sphere perpendicular to the common normal remains unaltered due to impact, so we have

$$v_1=u\sin\alpha \qquad ...(iii)$$

Equations (i), (ii), and (iii) will give the values of V, v_1 and v_2.

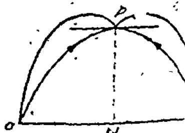
**Ex. 8. A ball at the focus of an ellipse whose eccentricity is e, [illegible] be greater than $2e/(1+e^2)$.

Sol. The ball is projected from the focus $S'(-ae, 0)$, strikes the elliptic periphery at P $(a\cos\phi, b\sin\phi)$ and then reaches the

and α the elevation in each case. Show that after impact they return to the points of projection if $ga(1+e)=ev^2 \sin 2\alpha$, wher is the distance between the points of projection.

Sol. Let O and O' be the points of projection of the two Since their velocity and angle of projection is the same, so would describe equal parabolas. Let them strike each other then P is vertically above N, where N is middle point of OO' (given).

After striking each other at P, the balls, return to their points of projection, hence the motion of each ball is the same as if it strikes a vertical wall at a distance a from its point of projection and then return back to its point of projection.



(Fig. 49)

Due to the impact at P, the vertical component of veloci each ball is not affected.

∴ Time of flight of each ball, since the ball returns to point of projection $=2(v \sin \alpha)/g$

Also as the horizontal velocity $v \cos \alpha$ remains constant for motion from O to P of the ball projected from O (similar is case of 2nd ball), so the time from O to $P=(a/v \cos \alpha)$

After impact at P, the horizontal velocity of this ball beco $ev \cos \alpha$, which remains consta t for its motion from P to O.

∴ the time taking by this ball in moving from P to O

$$=a/(ev \cos \alpha)$$

∴ Total time taken by each ball to return to its poin projection $=\dfrac{a}{v \cos \alpha}+\dfrac{a}{ev \cos \alpha}$, from (ii) and (iii)

$$=a(e+1)/(ev \cos \alpha)$$

Since time given by (i) and (iv) are equal, so we get

$\dfrac{2v \sin \alpha}{g}=\dfrac{a(e+1)}{ev \cos \alpha}$ or $ev^2 \sin 2\alpha=ag(1+e)$ Hence pro

*Ex. 7. A sphere of mass m falls on a smooth hemisphe mass M resting with its plane face on smooth horizontal table

Substituting values of tan α and tan β from (ii) and (iii) in (i), we get $\frac{ae \sin \phi}{b} = e' \cdot \frac{a(1-e^2 \cos \phi) \sin \phi}{b(1-\cos \phi)}$

or $$e' = [e(1-\cos \phi)/(1-e^2 \cos \phi)] \quad \ldots(iv)$$

which give the value of ϕ and hence the position of P

The value of e' given by (iv) is maximum when

$$\frac{de'}{d\phi} = 0 \text{ and } \frac{d^2e'}{d\phi^2} = \text{negative.}$$

From (iv), $de'/d\phi = 0$ gives.

$$\frac{(1-e^2 \cos \phi)(e \sin \phi) - e(1-\cos \phi)(e^2 \sin \phi)}{(1-e^2 \cos \phi)^2} = 0$$

or $e(1-e^2) \sin \phi = 0$ or $\sin \phi = 0$ or $\phi = 0$ and π

But $\phi = 0$ corresponds to the point A, but P can not take the position of A as it is against hypothesis.

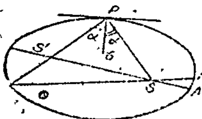
Hence $\phi = \pi$ and so form (iv) we get $e' = \frac{e(1-\cos \pi)}{1-e^2 \cos \pi} = \frac{2e}{1+e^2}$. which is max. value of e', the coefficient of resistution.

Ex. 9. A perfectly elastic ball is at the focus of an elliptic billiard table, show that the ball, in whatever manner struck, will ultimately be moving along the major axis.

Sol. The ball is projected from the focus S of the ellipse strikes the elliptic periphery at P, making an angle α with the normal PG to the ellipse at P. Since the ball is perfectly elastic *i.e.* $e=1$ so after impact at P, the angle of reflection will be equal to the angle of incidence. ($\because$ cot $\beta = e$ cot α gives $\beta = \alpha$ when $e=1$)

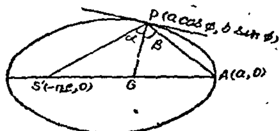
Also in an ellipse the normal at any point bisects the focal radii, so after impact at P, the ball will pass through the other focus S'. Let it strike the elliptic periphery again at Q and after impact at Q it will as before pass through the focus S and meet the elliptic periphery at D (say).

Since $\angle DSA$ is a part of $\angle PSA$, so $\angle DSA$ is less than $\angle PSA$. So in this way we observe that every time the ball returns to S after striking the elliptic periphery and passing throgh S', the point D will come nearer to A and ultimately after an infinite number of impacts D will coincide with A *i.e.* ultimately the ball will be moving along the major axis.



(Fig. 52)

other end $A\ (a, 0)$ of the major axis. Here the equation of the ellipse has been taken as $(x^2/a^2)+(y^2/b^2)=1$.



Let PG be the normal to the ellipse at P.

Let α and β be the angles of incidence and reflection at P. Then if e' be the coefficient of restitution, we have

$$\cot \beta = e' \cot \alpha \text{ or } \tan \alpha = e' \tan \beta \quad \ldots(i)$$

Also the slope of the normal $PG = -\dfrac{dx}{dy} = \dfrac{a \sin \phi}{b \cos \phi}$,

the slope of the line $S'P = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{b \sin \phi - 0}{a \cos \phi - (-ae)}$

$$= (b \sin \phi)/\{a\ (e + \cos \phi)\};$$

and the slope of the line $PA = \dfrac{b \sin \phi - 0}{a \cos \phi - a} = \dfrac{b \sin \phi}{a\ (\cos \phi - 1)}$.

Since α is the angle between $S'P$ and PG so we have

$$\tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\dfrac{a \sin \phi}{b \cos \phi} - \dfrac{b \sin \phi}{a\ (e + \cos \phi)}}{1 + \dfrac{a \sin \phi}{b \cos \phi}\,\dfrac{b \sin \phi}{a\ (e + \cos \phi)}}$$

$$= \frac{a^2 \sin \phi\ (e + \cos \phi) - b^2 \sin \phi \cos \phi}{ab \cos \phi\ (e + \cos \phi) + ab \sin^2 \phi}$$

$$= \frac{a^2 e\ (e \cos \phi + 1) \sin \phi}{ab\ (1 + e \cos \phi)}, \because \quad b^2 = a^2 - a^2 e^2$$

or

$$\tan \alpha = (ae \sin \phi)/b \quad \ldots(ii)$$

Also β is the angle between PG and PA, so we have

$$\tan \beta = \frac{[(b \sin \phi)/\{a\ (\cos \phi - 1)\}] - [(a \sin \phi)/(b \cos \phi)]}{1 + [(b \sin \phi)/\{a\ (\cos \phi - 1)\}]\ [(a \sin \phi)/(b \cos \phi)]}$$

$$= \frac{b^2 \sin \phi \cos \phi + a^2 \sin \phi\ (1 - \cos \phi)}{ab \cos \phi\ (\cos \phi - 1) + ab \sin^2 \phi}$$

$$= \frac{a^2 \sin \phi - (a^2 - b^2) \sin \phi \cos \phi}{ab - ab \cos \phi}$$

or $$\tan \beta = \frac{a\ (1 - e^2 \cos \phi) \sin \phi}{b\ (1 - \cos \phi)}, \because \quad b^2 = a^2 - a^2 e^2 \quad \ldots(iii)$$

Substituting values of tan α and tan β from (ii) and (iii) in (i), we get $\frac{ae \sin \phi}{b} = e' \cdot \frac{a (1-e^2 \cos \phi) \sin \phi}{b (1-\cos \phi)}$

or $$e' = [e (1-\cos \phi)/(1-e^2 \cos \phi)] \qquad ...(iv)$$

which give the value of ϕ and hence the position of P

The value of e' given by (iv) is maximum when

$$\frac{de'}{d\phi} = 0 \text{ and } \frac{d^2e'}{d\phi^2} = \text{negative}.$$

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$$\frac{(1-e^2 \cos \phi) (e \sin \phi) - e (1-\cos \phi) (e^2 \sin \phi)}{(1-e^2 \cos \phi)^2} = 0$$

or $e(1-e^2) \sin \phi = 0$ or $\sin \phi = 0$ or $\phi = 0$ and π

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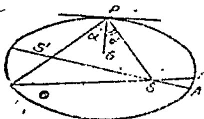
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(Fig. 52)

this sphere of m' in that direction

i.e. $I \cos 60° = m'(v'-0)$ or $I = 2m'v'$ ...(iii)

Eliminating I between (ii) and (iii) we get

$2m'v'\sqrt{3} = m(v-u)$ or $mv - 2m'v'\sqrt{3} = mu$ ...(iv)

Solving (i) and (iv) we get $v = (m+6em')\, u/(m-6m')$

And $v' = m(1+e)\, u\sqrt{3}/(m-6m')$, which gives the velocities of the spheres after impact.

After impact the impulse on one of the lower spheres in the vertical direction is $I \cos 30°$, which is balanced by the tension in the string. Hence at the time of impact, we have.

Impulsive tension $= I \cos 30° = \frac{1}{2} I\sqrt{3} = \frac{1}{2}(2m'v')\sqrt{3}$, from (iii)

$= 3mm'(1+e)\, u/(m-6m')$, substituting the value of v'.

MISCELLANEOUS EXECISES ON IMPACT

Ex. 1. Two inelastic balls of equal size, but the masses m and m', lie in contact on a smooth table. The former receives a blow in a direction through its centre making an angle α with the line of centres. Show that the kinetic energy of the balls is

$$m'(m+m'\sin^2\alpha)/m(m'+m\sin^2\alpha)$$

of what it would have been if the balls had interchanged and m had received the blow.

*Ex 2. At what angle must a smooth ball strike a horizontal plane so that after impact its direction may be at right angles to its former path.

[Hint. Here angle of reflection $= 90° -$ (angle of incidence)

∴ Substitute $90° - \alpha$ for θ in $\cot\theta = e \cot\alpha$]. Ans. $\tan^{-1}(\sqrt{e})$

Ex. 3. An elastic body is projected from a given point with a given velocity u and after hitting a vertical wall returns to the point from which it started Show that the distance of the point from the wall must be less than $eu^2/g(1+e)$, where e is the coefficient of restitution.

[Hint. From result (iv) of Ex. 8 Page 37 we have

$a = \frac{u^2}{g} \cdot \frac{e}{(1+e)} \cdot \sin 2\alpha < \frac{u^2}{g} \cdot \frac{e}{(1+e)}$ as $\sin 2\alpha < 1$]

Ex. 4. In a certain game a ball is rolled along a horizontal plane until it strikes an inclined plane of inclination θ from which it rebounds. The object of the game is to make the ball after rebounding fall into a hole in the inclined plane at a distance d from

its junction with the horizontal plane. Show that in order that the ball may enter the hole, its velocity of projection V must be given by $gd = 2eV^2 \sin(1 - e \tan^2 \theta)$.

Ex. 5. Three small spheres A, B, C whose masses are $8m$, m, $7m$, are at rest in a line, and $AB = BC = a$. The middle sphere B is projected towards C with velocity u. Assuming all the spheres to be perfectly elastic, determine the subsequent motion.

Ex. 6. A wedge of mass M is capable of moving freely on a smooth horizontal plane and a sphere of mass m is dropped upon its face and rebounds, show that the intial velocity of the wedge is $m(1+e) u \sin \alpha \cos \alpha/(M + m \sin^2 \alpha)$, where u is the velocity of the sphere on striking the face of the wedge, α is the inclination of face of the wedge and e is the coefficient of restitution.

Impulse, Work and Energy

§ 1. Work.

Definition. A force acting on a body is said to do work when its point of application moves through some distance. (*Kanpur* 85)

The work done by a constant force is measured by the product of the force and the distance through which the point of application moves in the direction of the force. Work done is a scalar quantity and is generally denoted by the letter W.

Let due to the application of a force

...tion OB of the force.

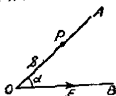


Fig. 1 (a)

Then the amount of work done
$= F.s.\cos\alpha$ or $F\cos\alpha.s.$

We can take either the force multiplied by ... the direction of resolved part of

If the directions of the displacement and the force are same, then the work done is positive. If their directions are opposite then the work done is negative *i.e.* the work is done against the force.

By Vector Method :

Let due to the application of a constant force **F**, the particle move from A to B and let the displacement vector from A to B be denoted by **d** then the work done by the constant force **F** is defined by W, where $W = \mathbf{F}.\mathbf{d}$. ...(i)

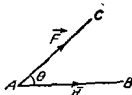


Fig. (b)

If θ be the angle between the line of action of **F** and AB, then in scalar notation, the work done by the constant force **F** from (i) is given by $W = Fd\cos\theta$, ...(ii) where $F = |\mathbf{F}|$ and $d = |\mathbf{d}|$.

From (ii) we find that if θ is acute the work done is positive, if θ is obtuse the work done is negative and if θ is a right angle the work done is zero.

Example. Consider the motion of a particle of mass m under the action of the force due to gravity only. Now let us consider three following cases :—

Case I. If the particle is moving **vertically downwards** *i.e.* falls through a distance h, then work done by the weight of the particle $=(mg)\ h=mgh$, as $\theta=0$ in this case.

Case II. If the particle is moving **vertically upwards** *i.e.* lifted through a distance h, then the work done by the weight of the particle $=-(mg)\ h=-mgh$, as $\theta=\pi$ in this case.

Case III. If the particle is moving **horizontally** through a distance h, then the work done by the weight of the particle $=0$, as $\theta=\frac{1}{2}\pi$ in this case. Hence no work is done in this case.

Units of Work.

In the C. G. S. system the unit of work is known as an *erg*. It is the work done by a force of one dyne in displacing its point of application through one centimetre in the direction of the force.

In the F. P. S. system the unit of work is **foot-poundal.** It is the work done by a force of one poundal in displacing its point of application through one foot in the direction of the force.

In the M.K.S. system the unit of work is joule. It is the work [illegible] plicat- [illegible]

The units of work given above are absolute units.

The gravitational unit of work in C.G.S., F.P.S. and M.K.S. systems are *gm-cm., foot-pound* and *kg-m.* respectively and these systems are related as follows :—

One gm-cm. $=981$ ergs.

One foot-pound $=32$ foot poundals.

One kg.-cm. $=9\cdot8$ joules.

and One joule $=$ One newton $\times 1$ metre $=10^7$ ergs.

[Note : One newton $=100,000$ dynes $=(1/9\cdot8)$ kg.-weight *i.e.* one newton is that force which can produce an acceleration of 1 metre/sec^2 in a body of mass one kilogram.]

The units of force are given as :—

One gm-weight $=g$ dynes $=981$ dynes

One kg-weight $=g$ newtons $=9\cdot8$ newtons

One lb-weight $=g$ poundals $=32$ poundals.

§ 2. Work done by a variable force.

Case 1. Let OAB be the straight path upon which the point of application of the force F moves. This force **F is varying in magnitude only.**

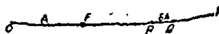


Fig. 2 (a)

Let $OA=a$ and $OB=b$. We are to find work done by the force F as the particle moves from A to B. Let after time t the particle be at P, such that $OP=s$. Let $PQ=\delta s$.

In vector notation.

We know if $\mathbf{v}$ be the velocity vector of a particle at the instant t, then the displacement $\delta\mathbf{r}$ of a particle during a short interval of time δt is given by $\delta\mathbf{r}=\mathbf{v}\ \delta t$. ...(i)

Also from § 2 Page 2 we know that the work done by a force $\mathbf{F}$ in this short interval of time δt is given by

$$\delta W=\mathbf{F}\cdot\delta\mathbf{r}. \quad ..(ii)$$

$\therefore$ From (i) and (ii), we get $\delta W=\mathbf{F}\cdot\mathbf{v}\ \delta t$

or $$\delta W/\delta t=\mathbf{F}\cdot\mathbf{v}.$$

Hence P, the rate per unit time at which a force $\mathbf{F}$ is working at any instant is given by

$$P=\lim_{\delta t\to 0}\frac{\delta W}{\delta t}=\frac{dW}{dt}=\mathbf{F}\cdot\mathbf{v}. \quad ...(iii)$$

This P is called the power of the force.

From (iii), we have $dW=P\ dt$.

$\therefore$ From (ii) of § 2 Page 3 of this chapter, we get

$$W=\int_A^B dW=\int_A^B P\ dt, \quad ..(iv)$$

which gives another form of expressing W.

If however the particle is moving in a straight line and the direction of $\mathbf{F}$ and $\mathbf{v}$ are the same *i.e.* $\theta=0$, then $\mathbf{F}\cdot\mathbf{v}=Fv$, where $v=|\mathbf{v}|$ and $F=|\mathbf{F}|$. And from (iii), we have $P=Fv$. ...(v)

Units of Power.

In the C. G. S. system, the unit of power (see definition of power) is one erg/sec. In F. P. S. system it is one foot poundal/sec and in M.K.S. system it is one joule/sec.

The units given above are absolute units of power.

The gravitational (or practical) units of power are one *watt* in M.K.S. system and one-*horse*-*power* in F.P.S. system and the two systems of units are related as follows :—

One *watt*=one *joule/sec*$=10^7$ *ergs/sec.*

One *Horse power* (or H.P.)=550 *foot-pounds/sec.*

One metric horse power=4500 m-kg per minute
=735 joule/sec=735 watt,
=(735/746) horse power.

Solved Examples on work.

Note : First two examples are for the practice of units only.

Ex. 1. *A train of 120 metric tonnes which is running at a speed of 60 kms/hr. is stopped in 12 seconds by applying brakes.*

What is the distance covered by the train during this time and what is the force applied by the brakes in ton-weight ?

Solution. Let f m/sec² be the retardation produced by the brakes.

Here initial velocity 'u' = 60 km./hr. = $\frac{50}{3}$ m./sec., 't' = 12 sec., final velocity 'v' = 0 and 'f' = $-f$ m/sec².

∴ From "$v=u+ft$", we get

$$0=\frac{50}{3}-f\times 12 \quad \text{or} \quad f=\frac{25}{18}\ \text{m/sec}^2.$$

If the train covers a distance of x metres in 12 seconds, then from "$s=ut+\frac{1}{2}ft^2$", we get

$$x=\frac{50}{3}\times 12-\frac{1}{2}\times\frac{25}{18}\times(12)^2=100\ \text{metres.}$$ **Ans.**

Also from "$P=mf$", we get

the required force $=(120\times 1000)\times\frac{25}{18}$ newtons (Note)

$$=\frac{120\times 1000\times 25}{18\times 9{\cdot}8}\ \text{kg-wt}=\frac{120\times 1000\times 25}{18\times 9{\cdot}8\times 1000}\ \text{m. ton-wt.}$$

$$=\frac{1200\times 25}{18\times 98}=17\ \text{metric-ton weight nearly.}$$ **Ans.**

Ex. 2. **A man is standing on the weighing machine placed inside a lift. When the lift is stationary the machine indicates the weight of the man as 75 kgs. If the lift rises with an acceleration 196 cm/sec², what weight of the man will be indicated by the machine ? Here g=980 cm/sec².**

Solution. Here mass of the man $=m=75$ kgs $=75\times 1000$ gms.

∴ The lift is ascending, so the equation of motion is

$mf=R-mg$, where R is the reaction of the lift

or $R=mg+mf=m\,(g+f)=75\times 1000\,(980+196)$ dynes

$$=\frac{75\times 1000\times 1176}{980}\ \text{gm. wt.}=\frac{75\times 1000\times 1176}{980\times 1000}\ \text{kg wt}$$

$=90$ kg-wt.

∴ The weight of the man indicated by the machine when the lift is ascending $=90$ kg-wt. **Ans.**

Ex. 3 (a). **For uniformly accelerated motion of a particle of unit mass its velocities at two instants are u_1 ft/sec and u_2 ft/sec. Find the work done by the force during this time interval.**

(Calcutta 85)

(b) *Find the work done by the force in part (a) above if $u_1=3$ ft/sec and $u_2=5$ ft/sec.*

Solution (a). Mass of the particle=1 lb.

Let f be the acceleration produced by the force.

Then from "$v^2=u^2+2fs$" we get $u_2^2=u_1^2+2fs$

or $2fs=u_2^2-u_1^2$,

where s is the distance moved in this time interval.

$\therefore$ Required work done$=(mf)\times s=\frac{1}{2}m\,(2fs)$

$=\frac{1}{2}\,(1)\,(u_2^2-u_1^2)$, $\because$ $2fs=u_2^2-u_1^2$, $m=1$

$=\frac{1}{2}\,(u_2^2-u_1^2)$ ft. poundal. **Ans.**

(b) Do as part (a) above. **Ans.** 8 ft. poundal.

Ex. 4 (a). *What is the work done by gravity on a mass of m lbs. during the t th second of its fall?* *(Calcutta 85)*

(b) *What is the work done by gravity on a stone of mass 70 gm. during the 10th second of its fall?*

Solution. (a). Distance moved in t th second of the fall

$=$"$u+\frac{1}{2}\,(2n-1)\,f$"$=0+\frac{1}{2}\,(2t-1)\,g$

$=\frac{1}{2}\,(2t-1)\,(32)$, $\because$ $g=32$ ft/sec^2

$=16\,(2t-1)$ feet.

$\therefore$ Required work done$=$(weight of the particle)

$\times$(distance moved in t th second)

$=mg\times 16\,(2t-1)=m\times 32\times 16\,(2t-1)$

$=512\,m\,(2t-1)$ foot-poundals. **Ans.**

(b). Do as part (a) above.

Here $g=981$ cm/sec^2, $t=10$ and $m=70$ gm.

$\therefore$ Distance moved in 10th second$=\frac{1}{2}\,(20-1)\times 981$ cms.

Weight of the body$=70g$ dynes$=70\times 981$ dynes.

$\therefore$ Required work done$=(70\times 981)\times\frac{1}{2}\,(19\times 981)$

$=35\times 19\times (981)^2$ gm. cms. **Ans.**

Ex. 5. *How many cubic metres of water will be taken out from a depth of 50 metres by a 75 kilowatt engine? (mass of one cubic metre of water=1 metric ton).*

Solution. Suppose the engine draws x cubic metres of water per minute.

mass of x cubic metres of water$=x\times 1000$ kg.

$\therefore$ weight of x cubic metres of water$=x\times 1000\times 9{\cdot}8$ newton

$\therefore$ Work done per second in drawing water by the engine

$=\frac{1}{60}\times x\times 1000\times 9{\cdot}8\times 50$ joule *(Note)*

But the engine is of 75 kilowatt, so the work done per second by engine $=75\times1000$ joule, $\because$ 1 kilowatt $=1000$ joule/sec

$$\therefore \frac{1}{60}\times x\times1000\times9{\cdot}8\times50=75\times1000$$

or $$x=\frac{75\times60}{9{\cdot}8\times50}=\frac{450}{49}=9\frac{9}{49} \text{ cubic metre.}$$ **Ans.**

Ex. 6. A labourer is throwing 12 bricks per minute from the ground and these just reach the roof which is 3·3 metres high. If the weight of each brick be 3·75 kg., then find the horse power with which the labourer is working.

Solution. Since the labourer is throwing 12 bricks per minute, so the weight of the bricks thrown by the labourer per minute

$$=12\times3{\cdot}75=45 \text{ kg-wt.}$$

The bricks are thrown to a height $=3{\cdot}3$ metres

$\therefore$ Work done by the labourer per minute

$$=45\times3{\cdot}3 \text{ metre-kilogram} \quad ...(i)$$

If x be the required horse power, then the metric horse power of the labourer $=x\times\frac{746}{735}$

$\therefore$ The work done by the labourer per minute

$$=\left(x\times\frac{746}{735}\right)\times4500 \text{ metre-kilogram} \quad ...(ii)$$

$\therefore$ From (i) and (ii), equating the values of work done per minute by the labourer, we get

$$x\times\frac{746}{735}\times4500=45\times3{\cdot}3 \quad \text{or} \quad x=\frac{45\times3{\cdot}3\times735}{746\times4500}$$

$$x=\frac{33\times735}{746\times1000}=0{\cdot}0325$$

Hence the required horse power of the labourer

$$=0\ 0325.$$ **Ans.**

Ex. 7. Find the H.P. of an engine which can project vertically wards 10,000 lbs. of water per minute with a velocity of 80 ft/sec.

Solution. Work done by the engine per second

$=$(weight of water)$\times$(distance moved per second)

$=\left(\frac{10{,}000}{60}\times32\right)\times80$ ft. poundals

$=(128{,}000)/3$ ft. poundals

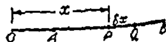
$=(128{,}000/3\times32)$ ft. pounds $=(4000)/3$ ft. lbs.

$\therefore$ Required H.P. $=\frac{4000}{3}\times\frac{1}{550}$ $\quad$ ...See units of power on Page 4

$=80/33=2{\cdot}4$ neraly **Ans.**

**Ex. 8 (a). Prove that the work done against the tension in streching a light extensible string is equal to the product of its extension and the mean of the initial and final tensions. (*Ranchi* 85)

Solution. Let l be the natural length of the string. Let it be extended from A to B, such that $OA=a$ and $OB=b$. We are to find the work done against the tension in stretching the string from A to B. Let λ be the moduls of elasticity of the string.



(Fig. 3)

Let $OP=x$ and $PQ=\delta x$. This distance PQ being small, the tensions at P and Q can be taken to remain practically the same.

Tension in the string when it is strtched upto $P=(\lambda/l)(x-l)$ acting in the direction PQ (by Hooke's Law).

$\therefore$ Work done in stretching the string from P to Q against the tension$=\{(\lambda/l)(x-l)\}\,\delta x$. (Note)

$\therefore$ Work done against tension in stretching from A to B

$$=\int_{x=a}^{b}(\lambda/l)(x-l)\,dx=(\lambda/l)\left[\tfrac{1}{2}(x-l)^2\right]_a^b$$

$$=\tfrac{1}{2}(\lambda/l)[(b-l)^2-(a-l)^2] \qquad \text{...(i)}$$

$$=\tfrac{1}{2}(\lambda/l)[\{(b-l)-(a-l)\}\{(b-l)+(a-l)\}]$$

$$=\tfrac{1}{2}(\lambda/l)(b-a)\{(b-l)+(a-l)\}$$

$$=(b-a)\tfrac{1}{2}\left[\frac{\lambda}{l}(b-l)+\frac{\lambda}{l}(a-l)\right] \qquad \text{...(ii)}$$

Also if T_1 and T_2 be the tensions in the string at A and B *i.e.* if T_1 and T_2 be the initial and final tensions, then we have

$$T_1=\frac{\lambda}{l}(a-l) \text{ and } T_2=\frac{\lambda}{l}(b-l) \qquad \text{...(iii)}$$

Also extension produced$=AB=b-a$.

$\therefore$ From (ii) we find that work done against the tension in stretching from A to B

=(extension produced)$\times\frac{1}{2}(T_2+T_1)$
=(extension produced)$\times$(mean of T_1 and T_2)
=(extension produced)$\times$(mean of initial and final tensions).

[Also from (i) we find that work done against the tension in stretching from A to $B=\frac{1}{2}(l/\lambda)(T_2^2-T_1^2)$].

Ex. 8 (b). Find the work done in extending a light elastic string to double its length.

Solution. Do as Ex. 8 (a) above. Here the string is extended

from $x=l$ to $x=2l$ where l is the natural length of the string. (Note)

$\therefore$ Required work done

$$=\int_{x=l}^{2l} \frac{\lambda}{l}(x-l)\,dx, \text{ see part (a) above.}$$

$$=\tfrac{1}{2}(\lambda/l)\left[(x-l)^2\right]_l^{2l}=\tfrac{1}{2}(\lambda/l)[l^2-0]=\tfrac{1}{2}\lambda l.$$ Ans.

***Ex. 9. A spider hangs from the ceiling by a thread of modulus of elasticity equal to its weight. Show that it can climb to the ceiling with an expenditure of work equal to only three quarters of what would be required if the thread were inelastic.**

Solution. Let l be natural length of the string and a be its extended length when the spider hangs in equilibrium. Let m be the mass of the spider.

In this position of equilibrium, we have

Tension in the string = weight of the spider

or $\lambda(a-l)/l=mg$

or $mg(a-l)/l=mg$, since $\lambda=mg$ (given)

or $a-l=l$ or $a=2l$. ...(i)

$\therefore$ If the string were inelastic, then magnitude of work done by the spider in climbing to the celling

$=mg.a=2mgl$, since $a=2l$. ...(ii)

If the string is elastic the work done in stretching it to a length a *i.e.* $2l$ from Ex. 8 (a) above,

=(mean of initial and final tensions) × extension produced

$=\tfrac{1}{2}(0+mg)\,l=\tfrac{1}{2}mgl.$ ...(iii)

When the spider reaches the celling and the string is elastic, the thread shrinks from its final to intial form, so the magnitude of work done against the tension is the same as given by (iii).

$\therefore$ The actual work done in climbing the ceiling when the string is elastic $=2mgl-\tfrac{1}{2}mgl=\tfrac{3}{2}mgl$

$=\tfrac{3}{4}(2mgl)=\tfrac{3}{4}$ (work done if the string is inelastic).

Hence proved.

Ex. 10. A man of weight W hangs at the end of a light extensible rope whose modulus is nW, the other end being fastened to a fixed point. He proceeds to climb up the rope. Prove that when he reached the fixed point he has done $\dfrac{2n+1}{2n+2}$ times the work he would have to do in climbing the same distance up an inextensible rope.

Solution. Let l be the natural length of the rope and a be its extended length when the man is hanging in equilibrium. In the position of equilibrium, tension in the rope = weight of the man

i.e. $\frac{\lambda}{l}(a-l)=W$, where $\lambda=nW$ (given)

or $n(a-l)=l$ or $a=l(1+1/n)$. ...(i)

If the rope were inelastic, the magnitude of work done by the man in climbing to the fixed point $=Wa$.

If the rope were elastic, the amount of work done in the process of shrinking of the rope

$=$(mean of initial and final tensions)$\times$extension produced

$$=\frac{1}{2}\left[\frac{\lambda(a-l)}{l}+0\right]\times(a-l)=\frac{\lambda(a-l)^2}{2l}, \text{ where } \lambda=nW$$

$$=\frac{nW(a-l)^2}{2l}=\frac{nW}{2l}(l/n)^2, \text{ from (i)}$$

$$=Wl/2n.$$

$\therefore$ The actual work done by the man in reaching the fixed point, when the rope is elastic $=Wa-Wl/2n$

$$=W(l+l/n)-(Wl/2n), \text{ from (i)}$$

$$=Wl\left(1+\frac{1}{2n}\right)=\frac{Wl(2n+1)}{2n}=\frac{(2n+1)}{(2n+2)}\left[\frac{W(2n+2)l}{2n}\right]$$

$$=\left(\frac{2n+1}{2n+2}\right)\left[Wl\left(1+\frac{1}{n}\right)\right]=\left(\frac{2n+1}{2n+2}\right).Wa, \text{ from (i)}$$

$$=\left(\frac{2n+1}{2n+2}\right).\text{(work done if the rope were inelastic).}$$

Hence proved.

Ex. 11. A uniform elastic string has length a_1 when the tension is T_1 and length a_2 when the tension is T_2. Show that its natural length is $(a_2T_1-a_1T_2)/(T_1-T_2)$, and that the amount of work done in stretching it from the natural length to (a_1+a_2) is

$$\frac{1}{2}\cdot\frac{(a_1T_1-a_2T_2)^2}{(T_1-T_2)(a_1-a_2)}.$$

Solution. Let l be the natural length of the *string* and λ be its modulus of elasticity. Then according to the problem,

$$T_1=\frac{\lambda}{l}(a_1-l) \text{ and } T_2=\frac{\lambda}{l}(a_2-l)$$

Dividing $\frac{T_1}{T_2}=\frac{(a_1-l)}{(a_2-l)}$ or $T_1(a_2-l)=T_2(a_1-l)$

or $l(T_2-T_1)=a_1T_2-a_2T_1$

or $l=\frac{a_1T_2-a_2T_1}{T_2-T_1}=\frac{a_2T_1-a_1T_2}{T_1-T_2}$, ...(i)

which give the natural leng h of the string.

Also from $T_1=(\lambda/l)(a_1-l)$, we have $(\lambda/l)=T_1/(a_1-l)$.

or $\dfrac{\lambda}{l}=T_1\Big/\left[a_1-\left(\dfrac{a_2T_1-a_1T_2}{T_1-T_2}\right)\right]$, from (i)

or $(\lambda/l)=(T_1-T_2)/(a_1-a_2)$. ...(ii)

Now the required amount of work done

$=$(mean of initial and final tensions)$\times$extension produced

$=\frac{1}{2}\,[0+(\lambda/l)\,\{(a_1+a_2)-l\}]\,(a_1+a_2-l)$

$=\frac{1}{2}\,\dfrac{(T_1-T_2)}{(a_1-a_2)}\,(a_1+a_2-l)^2$, from (ii)

$=\frac{1}{2}\cdot\dfrac{(T_1-T_2)}{(a_1-a_2)}\left[a_1+a_2-\dfrac{a_2T_1-a_1T_2}{T_1-T_2}\right]^2$, from (i).

$=\frac{1}{2}\,\dfrac{(T_1-T_2)}{(a_1-a_2)}\times\dfrac{(a_1T_1-a_2T_2)^2}{(T_1-T_2)^2}=\dfrac{(a_1T_1-a_2T_2)^2}{2\,(T_1-T_2)\,(a_1-a_2)}$.

Hence proved.

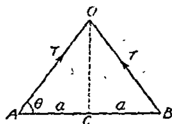
***Ex. 12. If a light elastic string, whose natural length is that of a uniform rod, be attached to rod at both ends and suspended by the middle point, show that the rod will descend until each of the two portions of the string is inclined to the horizon at an angle θ, given by the equation $\cot^3(\theta/2)-\cot(\theta/2)=2n$, the modulus of elasticity of the string being n times the weight of the rod.**

Solution. AB is the rod of length $=2a$ (say). O is the middle point of the string AOB, whose natural length $=$ length of the rod $2a$.

Initially the rod AB was at rest with its middle point C coinciding with O, then rod moves downwards till OA and OB are inclined at an angle θ to AB.

Thus vertical distance moved by the rod $AB=OC=a\tan\theta$.

Hence work done by the weight mg of the rod $=mg.a\cdot\tan\theta$. .. (i)



(Fig. 4)

For equation, work done by the weight of the rod $=$work done by the tension of the string. ...(ii)

Also from the figure, we have $OA=OB=a\sec\theta$.

∴ Work done by the string AOB in stretching from a length $2a$ to $2a\sec\theta$.

$=$(mean of initial and final tensions)$\times$extension produced

$=\frac{1}{2}\,[0+(\lambda/2a)\,(2a\sec\theta-2a)]\times(2a\sec\theta-2a)$, where $\lambda=nmg$

$=(nmg/4a)\,(2a\sec\theta-2a)^2=nmga\,(\sec\theta-1)^2$.

∴ from (ii) and (i), we get

$mg.a\tan\theta=nmga\,(\sec\theta-1)^2$

or $\tan\theta=n\,(\sec\theta-1)^2=n\,(1-\cos\theta)^2/\cos^2\theta$

or $\sin\theta\cos\theta=n\,(1-\cos\theta)^2$

or $2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\,(\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta)=n\,(2\sin^2\frac{1}{2}\theta)^2$

or $2\sin\frac{1}{2}\theta\cos^3\frac{1}{2}\theta-2\sin^3\frac{1}{2}\theta\cos\frac{1}{2}\theta=4n\sin^4\frac{1}{2}\theta$

or $\cot^3\frac{1}{2}\theta-\cot\frac{1}{2}\theta=2n.$ Hence proved.

Ex. 13 (a). Find the horse power of an engine which draws a train at a uniform rate of V ft./sec against a resistance of P lbs., wt. *(Calcutta 85)*

(b) Find the H. P. of an engine which draws a train at a uniform rate of 45 miles/hour against a resistance of 900 lbs. wt.

Solution (a) Distance moved per second $=V$ feet.

Pull of the engine = Resistance of P lbs. wt., as the engine is drawing the train at a uniform rate.

$\therefore$ Work done by the engine in one second $=P\times V$ ft. lbs.

$\therefore$ Required horse power $=\dfrac{P\times V}{550}$ **Ans.**

(b) Distance moved per second $=\dfrac{45\times 1760\times 3}{6\times 60}=66$ feet,

as 1 mile $=1760\times 3$ feet

Pull of the engine = Resistance = 900 lbs. wt.

$\therefore$ Work done by the engine in one second $=900\times 66$ ft. lbs.

$\therefore$ Required horse power $=\dfrac{900\times 66}{550}=108.$ **Ans.**

***Ex. 14. A motor car weighing 8 *quintals* and running at 10 metres/sec is brought to rest in 20 metres, by the application of its brakes. Find the work done by the force of resistance due to its brakes.**

Solution. Let the resistance due to brakes of the car be uniform and let f be its retardation.

Then from "$v^2=u^2+2fs$" we get $0=(10)^2-2f(20)$

or $f=(5/2)$ m/sec².

Also we are given mass of the car $=8\times 100=800$ kg.

$\therefore$ Force of resistance $=800\times\frac{5}{2}=2000$ newtons.

Hence the required work done $=2000\times 20$ joules

$=\dfrac{2000\times 20}{9\cdot 8}$ kg-m. $=4081\cdot 6$ kg. m. **Ans.**

Ex. 1. The earth's attraction on a particle varies inversely as the square of its distance from the earth's centre. A particle whose weight on the surface of the earth is W, falls to the surface of the earth from a height 5a above it. Show that the work done by the earth's attraction is (5/6) aW, where a is the radius of the earth.

Solution. At a height x above the centre of the earth, let the earth's attraction $=\lambda/x^2$.

On the surface of the earth $x=a$ and this attraction $=W$.

$\therefore$ $W=\lambda/a^2$ or $\lambda=a^2W$

∴ Attraction at a height x above the centre of the earth
$=a^2W/x^2$,

∴ Work done by the particle in moving a distance $-\delta x$
$=-(a^2W/x^2)\,\delta x$

∴ Required work done$=a^2W\int_{x=6a}^{a}-\frac{dx}{x^2}=a^2W\left(\frac{1}{x}\right)_{6a}^{a}$
$=a^2W\,[(1/a)-(1/6a)]=(5/6)\,aW.$ Hence proved.

Ex. 16. **A cyclist and his machine together are of mass M lbs. If he rides without pedalling down an incline of 1 in m, with an uniform speed v ft/sec, show that to go up an incline of 1 in n at the same rate he must work at $M\left(\frac{1}{m}+\frac{1}{n}\right)\frac{v}{550}$ H.P.**

Solution. Incline of 1 in m means a plane inclined to the horizontal at an angle α, such that $\sin\alpha=1/m$.

Now the cyclist rides without pedalling down this incline with a uniform speed, this means that resistance to the motion=resolved part of the weight parallel to the plane$=Mg\sin\alpha$
$=Mg\,(1/m)$. ...(i)

Now when the cyclist moves up an incline of 1 in n, then the resolved part of the weight parallel to the plane also resist the motion.

∴ Total resistance to the motion
=Resolved part of the weight+resistance due to friction etc.
$=Mg\,(1/n)+Mg\,(1/m)$, from (i)

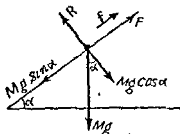
$M=\left(\frac{1}{n}+\frac{1}{m}\right)g$ poundals$=M\left(\frac{1}{n}+\frac{1}{m}\right)$ lbs. wt.

∴ If x be the H.P. of the man, then
$x.550$=Total resistance × distance moved in one second

or $x.550=M\left(\frac{1}{n}+\frac{1}{m}\right).v$ or $x=M\left(\frac{1}{n}+\frac{1}{m}\right).\frac{v}{550}$ H. P.

Hence proved.

***Ex. 17.** **A train of mass M lbs. is ascending a smooth incline of 1 in n and when the velocity of the train is v ft/sec, its acceleration is f ft/sec², prove that the effective horse-power of the engine is $Mv\,(nf+g)/550\,mg$.**



(Fig. 5)

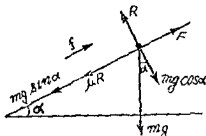
Solution. Let α be the inclination of the plane to the horizontal, then

$\sin\alpha=1/n$. ...(i)

Let the pulling force of the engine be F poundals and f be the acceleration of the train.

From Newton's second law of motion we have parallel to the inclined plane

a velocity v in t seconds, show that the average H.P. at which the engine has worked is $\frac{mv}{1100}\left[\frac{v}{gt}+\mu\cos\alpha+\sin\alpha\right]$.



(Fig. 6)

Solution. Let f be the acceleration of the engine. Then from "$v=u+ft$", we have

$v=0+ft$

or $\quad f=v/t.$ ...(i)

Let the pulling force of the engine be F poundals and R be the normal reaction between the engine and the rails. Then force due to friction$=\mu R=\mu mg\cos\alpha$, since resolving the force perpendicular to the inclined plane, we have

$R=mg\cos\alpha,$

∴ From Newton's second law of motion, we have

$m.f=F-mg\sin\alpha-\mu R=F-mg\sin\alpha-\mu mg\cos\alpha$

or $\quad F=m\,[f+g\sin\alpha+\mu g\cos\alpha]$ poundals

or $\quad F=m\,[(v/gt)+\sin\alpha+\mu\cos\alpha]$ lbs. wt., from (i),

Also distance moved in t seconds$=\frac{1}{2}ft^2=\frac{1}{2}vt$, from (i),

Work done in t seconds$=F\times\frac{1}{2}v$ ft. pounds.

or Average work done in one second$=F\times\frac{1}{2}v$ ft. pounds.

∴ Required H.P. of the engine$=F\times\frac{1}{2}v/550$.

$=\frac{v}{1100}\times F=\frac{mv}{1100}\left[\frac{v}{gt}+\sin\alpha+\mu\cos\alpha\right]$. Hence proved.

***Ex. 16. A body of mass M lbs. starts with a speed μ ft./sec. The body moves under the action of a force which does work at the constant rate of H horse-power. Prove that the equation which determines the speed v at a time t is $Mv\,dv/dt=550\,g$H.**

Write down the initial acceleration of the body in terms of M, H and u and prove that the time which the acceleration takes to the one quarter of its initial value is $3Mu^2/220\,g$H.

Solution. Let F poundals be the force which acts on the body and does work at the constant rate of H horse-power.

Then $H\times 550=(F/g).\,v$, where v is the velocity of the body at time t

or $\quad F=550g\,H/v.$ ...(i)

Also from Newton's second law of motion, we have

mass×acceleration=force acting.

∴ From (i), $M\frac{dv}{dt}=F=\frac{550\,gH}{v}$ or $Mv\frac{dv}{dt}=500\,gH$ ...(ii)

Hence proved.

$$M.f=F-Mg\sin\alpha$$

or $F=M[f+g\sin\alpha]$

or $=M[f+g(1/n)]$, *from* (i)

$=M(fn+g)/n$ poundals.

or $F=M(fn+g)/ng$ lbs. wt.

∴ If x be the required horse-power of the engine when it is moving with a velocity v ft./sec., we have

$$x.550=\left[\frac{M(fn+g)}{ng}\right]\times v \quad \text{or} \quad x=\frac{Mv(fn+g)}{550\,ng} \text{ H.P.}$$

Ex. 18. **An engine works at the constant rate of *H* horse power in drawing a train of total mass M tons up an incline of 1 in n, frictional resistance being r lb. wt. *per ton*, where r is a constant. Prove that the maximum *speed that can be generated is***

(550 nH)/[M (2240+nr)] *ft. per sec.*

Solution. If α be the inclination of the slope to the horizontal, then $\sin\alpha=1/n$. ...(i)

mass of the train $=M$ tons $=M\times2240$ lbs.

∴ The resolved part of the weight of the train down the inclined plane $=M\times2240\times g\sin\alpha$ poundals

$=M\times2240\times32\times(1/n)$ poundals.

The resistance due to friction $=r\times M$ pounds weight

$=rM\,g$ poundals $=rM\times32$ poundals

∴ Total resistance to the motion of the train which is moving up the inclined plane

$=[M\times2240\times32\times(1/n)+32\,r\,M]$ poundals

$=32M\,[2240\times(1/n)+r]$ poundals

Let v ft./sec be the maximum speed of the train up the plane, then the work done by train in one second

$=32M\,[2240\times(1/n)+r]\times v$ ft. poundals.

$=\dfrac{32M\,[2240\times(1/n)+r]\times v}{g}$ ft. pounds

$=\dfrac{M(2240+nr)\,v}{n}$ ft. lbs.

Now it is given that the engine is working at the constant rate of H horse power, so we have

$$H=\frac{M(2240+nr)\,v}{n+550} \text{ or } v=\frac{550\,Hn}{M(2240+nr)}$$ Hence proved.

Ex. 19. **A locomotive engine draws a load of *m* lbs. up an incline of α to the horizon, the coefficient of friction being μ. If starting from rest and moving with uniform acceleration it acquired**

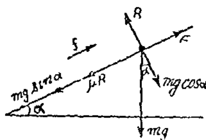
a velocity v in t seconds, show that the average H.P. at which the engine has worked is $\frac{mv}{1100}\left[\frac{v}{gt}+\mu\cos\alpha+\sin\alpha\right]$.

Solution. Let f be the acceleration of the engine. Then from "$v=u+ft$", we have

$v=0+ft$

or $f=v/t$. ...(i)

Let the pulling force of the engine be F poundals and R be the normal reaction between the engine and the rails. Then force due to friction $=\mu R=\mu mg\cos\alpha$, since resolving the force perpendicular to the inclined plane, we have



(Fig. 6)

$R=mg\cos\alpha$,

∴ From Newton's second law of motion, we have

$m.f=F-mg\sin\alpha-\mu R=F-mg\sin\alpha-\mu mg\cos\alpha$

or $F=m[f+g\sin\alpha+\mu g\cos\alpha]$ poundals

or $F=m[(v/gt)+\sin\alpha+\mu\cos\alpha]$ lbs. wt., from (i),

Also distance moved in t seconds $=\frac{1}{2}ft^2=\frac{1}{2}vt$, from (i),

Work done in t seconds $=F\times\frac{1}{2}v$ ft. pounds.

or Average work done in one second $=F\times\frac{1}{2}v$ ft. pounds.

∴ Required H.P. of the engine $=F\times\frac{1}{2}v/550$.

$=\frac{v}{1100}\times F=\frac{mv}{1100}\left[\frac{v}{gt}+\sin\alpha+\mu\cos\alpha\right]$. *Hence proved.*

***Ex. 16.** **A body of mass M lbs. starts with a speed μ ft./sec. The body moves under the action of a force which does work at the constant rate of H horse-power. Prove that the equation which determines the speed v at a time t is $Mv\,dv/dt=550\,g$H.**

Write down the initial acceleration of the body in terms of M, H and u and prove that the time which the acceleration takes to the one quarter of its initial value is $3Mu^2/220\,g$H.

Solution. Let F poundals be the force which acts on the body and does work at the constant rate of H horse-power.

Then $H\times 550=(F/g).\,v$, where v is the velocity of the body at time t

or $F=550g\,H/v$. ...(i)

Also from Newton's second law of motion, we have

mass × acceleration = force acting.

∴ From (i), $M\frac{dv}{dt}=F=\frac{550\,gH}{v}$ or $Mv\frac{dv}{dt}=500\,gH$ ...(ii)

Hence proved.

From (ii) acceleration at time $t = \frac{dv}{dt} = \frac{550\, gH}{Mv}$

$\therefore$ Initial acceleration $= 550\, gH/Mu$, where u is the initial velocity of the body.

Now from (ii), $v\frac{dv}{dt} = \frac{550\, gH}{M}$.

Integrating, $v^2 = (1100\, gH/M)\, t + C$, where C is constant of integration.

Initially $v = u$ and $t = 0$ $\therefore$ $C = u^2$.

$\therefore$ $v^2 = (1100\, gH/M)\, t + u^2$ or $v = [(1100\, gH/M)\, t + u^2]^{1/2}$

Differentiating $dv/dt = \frac{1}{2}[(1100\, gH/M)\, t + u^2]^{-1/2}.(110\, gH/M)$ ...(iii)

Let time $t = T$ when acceleration $dv/dt = \frac{1}{4}$ (initial acceleration) $= \frac{1}{4}[550\, gH/Mu]$.

$\therefore$ From (iii), $\frac{1}{4}[550\, gH/Mu]$

$= \frac{1}{2}[(1100\, gH/M)\, T + u^2]^{-1/2}.(1100\, gH/M)$

or $[(1100\, gH/M)\, T + u^2]^{1/2} = 4u$

or $(1100\, gH/M).T = 16u^2 - u^2$, squ[aring] and transposing

or $$T = \frac{15u^2 M}{1100\, gH} = \frac{3u^2\, M}{220\, gH}.$$ Hence proved.

Exercises on Work

Ex. 1. A train of total mass 250 tons is drawn by an engine working at 550 H.P. If at a certain instant the total resistance is 10 lbs. wt. per ton the weight of the train, and the velocity 30 m.p.h., what is the train's acceleration measured in miles per hour per second.

Ex. 2. A train of mass 40 tons is kept moving at the uniform rate of 40 m p.h. on the level, the resistance of air, friction, etc. being 30 lbs. wt. per ton. Find the H.P. of the engine.

Ex. 3. An engine is raising water from a depth of 55 ft. and discharging 960 gallons a minute with a velocity of 44 ft./sec. Find the H.P. at which the engine is working. One gallon of water weights 10 lbs.)

Ex. 4. State whether the following statement is true or false :

'The work done by a force is a vector quantity.'

Ans. False.

§ 4. Energy. Energy of a body is its capacity of doing work. (*Kanpur* 85)

The energy is of two types. **Kinetic and Potential.**

The energy of a body which it possesses by virtue of its motion is known as **Kinetic Energy** and which it possesses by virtue of its position is known as **Potential Energy.**

§ 5. **Kinetic Energy.** Kinetic Energy is measured by the amount of work which the body can perform against some resistance till it is reduced to rest.

Let the mass of the body be m and its velocity be V. To bring this body to rest there must be some opposing force. Let this force be F which reduces the velocity of this body from V to zero.

Then from Newton's second law of motion, we get

$$m.v\frac{dv}{ds}=-F. \qquad \text{...(i)}$$

$\therefore$ Work done by the force F in moving through a distance δs

$$=F.\delta s$$

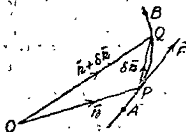
$\therefore$ Kinetic energy $=\int_{v=V}^{0} F\,ds=\int_{v=V}^{0} -mv\frac{dv}{ds}.ds$, from (ii)

$$=-m\int_{V}^{0} v\,dv=-m\left[\tfrac{1}{2}v^2\right]_{V}^{0}=\tfrac{1}{2}mV^2.$$

$\therefore$ **Kinetic Energy $=\frac{1}{2}mV^2$.**

In vector notations :—

Let a particle of mass m be under the action of a variable force , which reduces the particle to rest.



(Fig. 7)

Let **r** be the position vector of the position P of the particle at time referred to some fixed point O as origin. Let **v** be the velocity vector of the particle at P.

Then the equation of motion of the particle at P is $\quad m\frac{d\mathbf{v}}{dt}=-\mathbf{F}$, $\qquad$ ...(i)

where **F** is the resisting force which reduces the particle to rest at B (say) hence taken negative.

Let Q be the position vector of Q referred to O.

Then the work done by the force **F** acting on the particle in moving from P to $Q=\mathbf{F}\cdot\delta\mathbf{r}$.

$\therefore$ Work done by the force **F** acting on the particle in moving from P to B

$$=\int_P^B \mathbf{F}\cdot d\mathbf{r}=-\int_P^B m\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{r}}{dt}\,dt, \text{ from (i)} \qquad \textbf{(Note)}$$

$$=-\int_P^B\left(m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)dt; \quad \because \mathbf{v}=\frac{d\mathbf{r}}{dt}$$

$$=-\int_P^B\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right)dt=\int_B^P\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right)dt \quad \because v=|\mathbf{v}|$$

Examples of conservative and non-conservative forces.

(i) Earth's gravitational field.

The work done by gravity (*i.e.* weight force) when a particle of mass m ascends to height h and thencomes back to its original position is

$$-mgh+mgh=0.$$

∴ The force due to ... of this type ... Other examples

(ii) F ...

If a body is dragged against a constant frictional force F through a distance s on a horizontal plane, then the work done by the frictional force F is $-Fs$. If this body is brought back to its original position on the same path, the amount of work done is $-Fs$, again.

∴ Total work done in bringing the body back to its original position $=-Fs-Fs=-2Fs\neq 0$.

Thus frictional force is not a conservative force.

***§ 7. Principle of Energy. (Energy equation)**

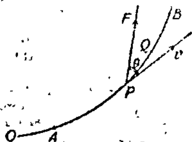
In any displacement of a particle the change in the kinetic energy is equal to the work done by the forces (which must be conservative) acting on the particle.

A particle is moving along a curved path under the action of given forces.

Let P be its position at time t.

Let F be the resultant of all the forces acting at P, inclined at an angle θ to the tangent at P.

Considering motion along the tangent at P, we have the equation of motion as $mv\,(dv/ds)=F\cos\theta$



(Fig. 9)

$$mv\,dv=F\cos\theta\,ds \quad \text{...(i)}$$

Let arc be measured from the point O on the path.

Let arc $OA=s_1$ and arc $OB=s_2$ and the velocity at A and B be u_1 and u_2 respectively.

Integrating (i), we have $m\int_{v=u_1}^{u_2} v\,dv=\int_{s=s_1}^{s_2} F\cos\theta.ds$

or $\frac{1}{2}m\left[v^2\right]_{v=u_1}^{u_2}=\int_{s_1}^{s_2} F\cos\theta.ds=$ work done by F in moving the particle from $s=s_1$ to $s=s_2$ *i.e.* from A to B.

or $\frac{1}{2}mu_2^2-\frac{1}{2}mu_1^2=$ work done by the forces in moving the particle from A to B.

Hence change in kinetic energy = work done by the forces in moving the particle from A to B.

$$=\left[\tfrac{1}{2}mv^2\right]_0^v=\tfrac{1}{2}mv^2=\text{K.E. of the particle at } P$$

(according to definition given on Page 17).

Vis-viva. Two times the kinetic energy of a particle is called "vis-viva."

Units of kinetic energy. According to definition kinetic energy is measured by the amount of work done in a particular way hence its units are also the same as those for work done as given in § 1 Page 2 of this chapter.

*§ 6. Conservative Forces.

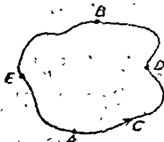
If a variable force F is acting on a particle which moves along the curve C on which the points A and B lie (See Fig. 7 Page 17), then the work done by the force F in moving from A to $B=\int_A^B \mathbf{F}\cdot d\mathbf{r}$.

This expression appears to depend on the extreme positions of the particle but it may also depend on the path C along which the particle moves.

If the force F is such that the expression $\int \mathbf{F}\cdot d\mathbf{r}$ depends upon the extreme positions A and B of the particle only and not on the path C followed by the particle, then the force F is said to be **conservative** and if this expression depends on the path C also then the force F is said to be non-conservative.

A field of force. If a force F is uniquely defined at each and every point of a region of space, then the totality of all such vector forces F defined throughout the space is known *as a field of force.*

Another Definition of conservative force.



(Fig. 8)

If a force acting on a particle be such that the total work done by it on the particle is zero as the particle returns to its original position after describing any path in the field of force, then the force is said to be conservative.

If a force F be acting on a particle which moves in a round trip from A [illegible]

F is conservative, the total work done in moving in this way

$$=\underset{ADBEA}{\oint}\mathbf{F}\cdot d\mathbf{r}=\underset{(\text{along } ADB)}{\int_A^B}\mathbf{F}\cdot d\mathbf{r}+\underset{(\text{along } BEA)}{\int_B^A}\mathbf{F}\cdot d\mathbf{r} \qquad \text{(Note)}$$

$$=\underset{(\text{along } ADB)}{\int_A^B}\mathbf{F}\cdot d\mathbf{r}-\underset{(\text{along } AEB)}{\int_A^B}\mathbf{F}\cdot d\mathbf{r}=0. \qquad \text{(Note)}$$

Examples of conservative and non-conservative forces.

(i) Earth's gravitational field.

The work done by gravity (*i.e.* weight force) when a particle of mass m ascends to height h and thencomes back to its original position is $-mgh+mgh=0$.

$\therefore$ The force due to gravity is conservative. Other examples of this type are tension, reaction etc.

(ii) Fractional force.

If a body is dragged against a constant frictional force F through a distance s on a horizontal plane, then the work done by the frictional force F is $-Fs$. If this body is brought back to its original position on the same path, the amount of work done is $-Fs$, again.

$\therefore$ Total work done in bringing the body back to its original position$=-Fs-Fs=-2Fs\neq 0$.

Thus frictional force is not a conservative force.

***§ 7. Principle of Energy. (Energy equation)**

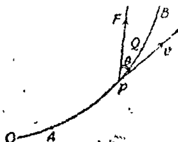
In any displacement of a particle the change in the kinetic energy is equal to the work done by the forces (which must be conservative) acting on the particle.

A particle is moving along a curved path under the action of given forces.

Let P be its position at time t.

Let F be the resultant of all the forces acting at P, inclined at an angle θ to the tangent at P.

Considering motion along the tangent at P, we have the equation of motion as $mv\,(dv/ds)=F\cos\theta$



(Fig. 9)

or $$mv\,dv=F\cos\theta\,ds \quad ...(i)$$

Let arc be measured from the point O on the path.

Let arc $OA=s_1$ and arc $OB=s_2$ and the velocity at A and B be u_1 and u_2 respectively.

Integrating (i), we have $m\int_{v=u_1}^{u_2} v\,dv=\int_{s=s_1}^{s_2} F\cos\theta.ds$

or $\frac{1}{2}m\left[v^2\right]_{v=u_1}^{u_2}=\int_{s_1}^{s_1} F\cos\theta.ds=$work done by F in moving the particle from $s=s_1$ to $s=s_2$ *i.e.* from A to B.

or $\frac{1}{2}mu_2^2-\frac{1}{2}mu_1^2=$work done by the forces in moving the particle from A to B.

Hence change in kinetic energy = work done by the forces in moving the particle from A to B.

In vector notations :—(Refer Fig. 7 Page 17 of this chapter).

Let P be the position of the particle, which is moving along a curved path under the action of the given forces, at time t. Let $\mathbf{F}$ be the resultant of all the forces acting on the particle at P and let $\mathbf{v}$ be the velocity vector of the particle at P.

Then the equation of motion of the particle is $m\dfrac{d\mathbf{v}}{dt}=\mathbf{F}$. ...(i)

Now the work done by the force $\mathbf{F}$ acting on the particle in moving from P to $Q=\mathbf{F}\cdot\delta\mathbf{r}$, where $\mathbf{r}$ and $\mathbf{r}+\delta\mathbf{r}$ are the position vectors of the neighbouring points P and Q referred to some origin O.

$\therefore$ Work done by $\mathbf{F}$ acting on the particle in moving from A to $B=\displaystyle\int_A^B \mathbf{F}\cdot d\mathbf{r}=\int_A^B\left(m\frac{d\mathbf{v}}{dt}\right)\cdot d\mathbf{r}$, from (i)

$$=\int_A^B\left(m\frac{d\mathbf{v}}{dt}\right)\cdot\frac{d\mathbf{r}}{dt}\,dt=\int_A^B m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,dt,\quad\because\ \mathbf{v}=\frac{d\mathbf{r}}{dt}$$

$$=\int_A^B\left(m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right)dt=\int_A^B\frac{d}{dt}\left(\tfrac{1}{2}m\mathbf{v}^2\right)dt=\left[\tfrac{1}{2}m\mathbf{v}^2\right]_A^B$$

$$=\left[\tfrac{1}{2}mv^2\right]_A^B,\ \text{as } v=|\mathbf{v}|\text{ and }\mathbf{v}^2=v^2$$

$=\tfrac{1}{2}mv_B^2-\tfrac{1}{2}mv_A^2$, where v_A and v_B are the velocities of the particle at A and B respectively

$=$(Kinetic energy at B)$-$(Kinetic energy at A)

$=$Change in kinetic energy of the particle as it moves from A to B. Hence proved.

***§ 8. Potential Energy and Potential Function.**

Definition. The potential energy of a particle is the amount of work that *would be* done by the forces (which must belong to conservative system) acting on the particle *if it be allowed to move* from its present position to some standard position.

Thus if we denote the potential energy by V, the resultant force acting on the particle by $\mathbf{F}$, the position vectors of the present and standard positions A and E of the particle by $\mathbf{r}$ and $\mathbf{r}_0$, then

$$V=\int_{\mathbf{r}}^{\mathbf{r}_0}\mathbf{F}\cdot d\mathbf{r} \qquad ...(i)$$

If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $d\mathbf{r}=\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz$

and let $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$.

$\therefore\ \mathbf{F}\cdot d\mathbf{r}=(F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz)$

$=F_1\,dx+F_2\,dy+F_3\,dz$. ...(ii)

Now let us assume that there exists a function $V(x,y,z)$

such that $\quad F_1=-\dfrac{\partial V}{\partial x},\ F_2=-\dfrac{\partial V}{\partial y},\ F_3=-\dfrac{\partial V}{\partial z}$, ...(iii)

then from (ii), we get $\mathbf{F}\cdot d\mathbf{r}=-\frac{\partial V}{\partial x}dx-\frac{\partial V}{\partial y}dy-\frac{\partial V}{\partial z}dz$
$=-dV.$

$\therefore$ From (i), the work done by the force **F** in moving from A to $B=\int_A^B \mathbf{F}\cdot d\mathbf{r}=-\int_A^B dV=\int_B^A dV=V_A-V_B.$ ...(iv)

$\therefore$ Under assumptions (iii), the work done depends upon the positions A and B only and not on the path followed by the particle and hence force **F** is conservative.

Conversely, if **F** is conservative, then there exists a function $V(x, y, z)$ such that

$$F_1=-\frac{\partial V}{\partial x},\ F_2=-\frac{\partial V}{\partial y},\ F_3=-\frac{\partial V}{\partial z},\ \text{where } \mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}.$$

Such a function $V(x, y, z)$ is called the **potential function** of the conservative force F.

Note 1. If m be the mass of a particle placed at a point lying in conservative field of force, then the potential energy of the particle in that position $=mV$, where V is the potential function at that point.

Note 2. Generally V denotes the potential energy of the particle of mass m. Here we suppose that m is absorbed in the potential function.

****§ 9. Principle of conservation of Energy.**

(*Bhopal 85; Lucknow 87*)

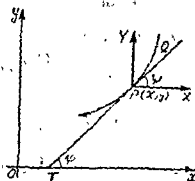
If a particle is acted on by a conservative system of force and be in motion then the sum of the kinetic and potential energies of the particle remains constant.

Let P be the position of the particle at time t and let v be its velocity at that time. Let X, Y be the components of the forces acting on the particle parallel to the coordinate axes.

Considering motion along the tangent at P, we have the equation of motion

$$mv\frac{dv}{ds}=X\cos\psi+Y\sin\psi,$$

where ψ is the inclination of the tangent at P.



(Fig. 10)

or $mv\frac{dv}{ds}=X\frac{dx}{ds}+Y\frac{dy}{ds}$, since $\cos\psi=\frac{dx}{ds}$ and $\sin\psi=\frac{dy}{ds}$

or $$mv\,dv=X\,dx+Y\,dy$$

Integrating, $m\dfrac{v^2}{2}=\int (X\,dx+Y\,dy)+C.$ ...(i)

If the particle moves from $P(x, y)$ to $Q(x+\delta x, y+\delta y)$, then the work done by the forces X and Y in moving the particle from P to Q is $X\delta x+Y\delta y$, hence $\int (X\,dx+Y\,dy)$ is the total work done by the forces X and Y in moving the particle from its initial position to P.

Also suppose $(X dx+Y dy)$ is the complete differential of some function $f(x, y)$, then from (i) we get $\frac{1}{2}mv^2=f(x, y)+C$

i.e. K.E.$=f(x, y)+C$ ...(ii)

Also potential energy of the particle in the position $P(x, y)$ is equal to the amount of work that would be done if the particle be allowed to move from its position $P(x, y)$ to some standard position (x_1, y_1) say.

$$\therefore\ \text{P. E. at } P=\int_{(x,y)}^{(x_1, y_1)} (X\,dx+Y\,dy)=\int_{(x,y)}^{(x_1, y_1)} df(x, y)$$

$$=\Big[f(x, y)\Big]_{(x,y)}^{(x_1, y_1)}$$

$$=f(x_1, y_1)-f(x, y).\qquad \text{...(iii)}$$

Adding (ii) and (iii) we have at P,

K.E.+P.E.$=f(x_1, y_1)+C=$constant, since x_1, y_1 are constants as (x_1, y_1) is the standard position of the particle *i.e.* a fixed point. Hence proved.

In vector notations :

[illegible]

Also we know change in kinetic energy=work done (See § 7 Page 19 of this chapter).

$\therefore\ T_B-T_A=$ work done by **F** as the particle moves from A to B.

$$=\int_A^B \mathbf{F}\cdot d\mathbf{r}=\int_A^B (F_1\,dx+F_2\,dy+F_3\,dz),$$

as in § 8 (ii) Page 20

$$=\int_A^B \left[\left(-\frac{\partial V}{\partial x}\right)dx+\left(-\frac{\partial V}{\partial y}\right)dy+\left(-\frac{\partial V}{\partial z}\right)dz\right],$$

taking $F_1=-\partial V/\partial x$ etc.

$$=\int_A^B(-dV)=\Big[-V\Big]_A^B=V_A-V_B$$

or $T_B+V_B=T_A+V_A$

i.e. sum of kinetic and potential energies of the particle remains constant. Hence proved.

Solved Examples on Energy.

Ex. 1. A bullet loses $\frac{1}{15}$ the of its velocity in passing through a plank. *Find with the help of principle of work and energy, how many such uniform planks will be required to bring the bullet to rest.*

Solution. Let the bullet strike the plank with a velocity u. The velocity of the bullet is reduced to $\left(1-\frac{1}{15}\right)u$ *i.e.* $\frac{14}{15}u$ in passing through one plank of thickness x, say.

Let F be the resistance offered by the plank.

We know 'change in K.E.=work done'

$$\therefore \tfrac{1}{2}mu^2-\tfrac{1}{2}m\left(\frac{14}{15}u\right)^2=F.x$$

or $$F.x=\tfrac{1}{2}mu^2\left[1-\left(\frac{14}{15}\right)^2\right]=\tfrac{1}{2}mu^2\times\frac{29}{225} \quad ...(i)$$

Let us suppose that n planks will be required to bring the bullet to rest. It means that the velocity of the bullet will be reduced to zero from u in passing through a thickness $n\times x$.

$\therefore$ From 'change in K.E.=work done', we have

$$\tfrac{1}{2}mu^2-\tfrac{1}{2}m(0)^2=F(n\times x)$$

or $$\tfrac{1}{2}mu^2=nFx=n\left(\tfrac{1}{2}mu^2\times\frac{29}{225}\right), \text{ from (i)}$$

or $$n=\frac{225}{29}=7\cdot76 \text{ nearly } i.e. \ 8.$$ **Ans.**

Ex. 2. A shot of mass m fired horizontally penetrates a thickness s of a fixed plate of mass M, prove if that M is free to move, thickness penetrated is $Ms/(M+m)$.

Solution. Let the shot strike the plate with a velocity u. Let F be the resistance offered by the plate.

When the plate is fixed, the velocity of the shot is reduced from u to zero after it has penetrated a distance s.

Also we know change in K.E.=work done.

$$\therefore \tfrac{1}{2}mu^2-\tfrac{1}{2}m.0=F.s \text{ or } F.s=\tfrac{1}{2}mu^2, \quad ...(i)$$

When the plate is free to move, let v be velocity after impact of the plate with the shot. Then as

total momenta before impact=total momenta after impact

so $$mu+M.0=(m+M)v \quad \text{or} \quad v=mu/(m+M) \quad ...(ii)$$

If s_1 be the distance penetrated in this case when the plate is free to move from the principle of energy *viz.*

change in K.E.=work done,

we have $\frac{1}{2}mu^2-\frac{1}{2}(m+M)v^2=F.s_1$.

or $F.s_1=\frac{1}{2}mu^2-\frac{1}{2}(m+M).\{mu/(m+M)\}^2$, from (ii)

$$=\frac{1}{2}mu^2-\frac{1}{2}\frac{(m^2u^2)}{(m+M)}=\frac{1}{2}mu^2\left(1-\frac{m}{m+M}\right)$$

or $F.s_1=\frac{1}{2}m\,Mu^2/(m+M)$. ...(iii)

Dividing (iii) by (i), we get

$$\frac{s_1}{s}=\frac{M}{(m+M)} \text{ or } s_1=\frac{Ms}{(m+M)} \text{ or } \frac{s}{1+(m/M)}$$

Hence proved.

*Ex. 3. A bullet of mass m, moving, with velocity v strikes a block of mass M, which is free to move in the direction of the bullet and is embedded in it. Show that a portion $M/(m+M)$ of the kinetic energy is lost. (*Lucknow* 87 ; *Rohilkhand* 86)

If the block is afterwards struck by an equal bullet moving in the same direction with the same velocity, show that there is a further loss of K.E. equal to $mM^2v^2/2\,(M+m)\,(M+2m)$.

Solution. Let V be the velocity of the block embedded with the first bullet when it belongs to move.

$\therefore$ From principle of momentum viz. total momenta before impact=total momenta after impact, we have

$$mv=(m+M)\,V \quad \text{or} \quad V=mv/(m+M) \qquad ...(i)$$

$\therefore$ Loss of K.E.=K.E. before impact−K.E. after impact

$$=\frac{1}{2}mv^2-\frac{1}{2}(m+M)V^2$$

$$=\frac{1}{2}mv^2-\frac{1}{2}(m+M)\{m^2v^2/(m+M)^2\}, \text{ from (i)}$$

$$=\frac{1}{2}m\,v^2\left[1-\frac{m}{m+M}\right]=\frac{M}{m+M}\left(\frac{1}{2}mv^2\right).$$

Hence proved.

Let V' be the velocity of the block after the second bullet strikes it, then from Principle of momentum, we get

$$mv+(m+M)\,V=(2m+M)\,V',$$

the first term on the left hand side is due to second bullet.

or $mv+mv=(2m+M)\,V'$ from (i)

or $V'=2mv/(2m+M)$. ...(ii)

$\therefore$ Loss of K.E. due to this impact.

=K.E. before impact−K.E. after impact

$$=[\tfrac{1}{2}mv^2+\tfrac{1}{2}(m+M)V^2]-[\tfrac{1}{2}(2m+M).V'^2]$$

$$=\frac{1}{2}mv^2+\frac{1}{2}(m+M).\frac{m^2v^2}{(m+M)^2}-\frac{1}{2}(2m+M).\frac{4m^2v^2}{(2m+M)^2}$$

$$=\frac{1}{2}mv^2\left[1+\frac{m}{(m+M)}-\frac{4}{(2m+M)}\right]=\frac{1}{2}\frac{mM^2v}{(m+M)(2m+M)}$$

Hence proved.

Ex. 4. A shot of mass m is fired from a gun of mass M with velocity u relative to the gun ; show that the actual velocities of the shot and the gun are $Mu/(m+M)$ and $mu/(m+M)$ respectively, and that their kinetic energies are inversely proportional to their masses. *(Rohilkhand 85)*

Solution. Let v and V be the velocities of the shot and the gun respectively, the directions of these velocities being opposite to each other.

Since u is velocity of the shot relative to the gun, therefore

$$u=v-(-V) \quad \text{or} \quad u=v+V. \qquad \ldots\text{(i)}$$

Also as momentum of the shot = momentum of the gun.

$$\therefore \quad mv=MV. \qquad \ldots\text{(ii)}$$

From (i) and (ii) we get $u=\dfrac{MV}{m}+V=\dfrac{(m+M)\,V}{m}$

or $\quad V=\dfrac{mu}{m+M}$ and $v=\dfrac{MV}{m}=\dfrac{Mu}{m+M}$. $\qquad \ldots$(iii)

$$\therefore \frac{\text{K.E. of the shot}}{\text{K.E. of the gun}}=\frac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2}=\frac{m}{M}\frac{v^2}{V^2}$$

$$=\frac{m}{M}\cdot\frac{M^2}{m^2}, \text{ from (ii)}$$

$$=\frac{M}{m}=\left(\frac{1}{m}\right)\Big/\left(\frac{1}{M}\right).$$

Hence K. Energies of the shot and the gun are inversely proportional to their masses.

****Ex. 5. A gun of mass M fires a shell of mass m horizontally and the energy of the explosion is such as would be sufficient to project that shot vertically to a height h. Show that the velocity of recoil of the gun is $[2m^2gh/M\ (m+M)]^{1/2}$.**

Solution. Let E be the energy of the explosion. As this energy is just sufficient to raise the mass m to height h so we have

$$E=mgh. \qquad \ldots\text{(i)}$$

Let v and V be the velocity of shell and the recoil of gun respectively.

Then as momentum of the shell = momentum of the gun so we have $\quad mv=MV. \qquad$ .. (ii)

Also energy $E=\frac{1}{2}mv^2+\frac{1}{2}MV^2$. $\qquad \ldots$(iii)

$\therefore$ from (i) and (iii), we get $mgh=\frac{1}{2}mv^2+\frac{1}{2}MV^2$

$$=\tfrac{1}{2}m\left(\frac{MV}{m}\right)^2+\tfrac{1}{2}MV^2, \text{ from (ii)}$$

$$=\tfrac{1}{2}MV^2\,(M+m)/m$$

or $\quad V^2=2m^2gh/M\,(m+M)$

or $\quad V=[2m^2gh/M\,(m+M)]^{1/2}$. $\qquad$ Hence proved.

Ex. 6. Assuming that in a cannon the force on the ball depends only on the volume of the gas generated by the gun powder, show that the ratio of the final velocity when the gun is free to recoil to its velocity when the gun is fired is $\sqrt{\{M/(m+M)\}}$, where M and m are masses of the gun and the ball respectively.

Solution. Let V be final velocity of the ball when the gun is fired and E be the kinetic energy released by explosion.

$$\therefore \quad E=\tfrac{1}{2}mV^2 \qquad \text{...(i)}$$

When the gun is free to recoil, let u and v be the velocity of the ball and recoil of the gun respectively.

Since momentum of the ball = momentum of gun,

therefore $\quad mu=Mv \qquad$ ...(ii)

Also kinetic energy $E=\frac{1}{2}\,mu^2+\frac{1}{2}\,Mv^2 \qquad$ ...(iii)

From (i) and (iii), we get $\frac{1}{2}mV^2=\frac{1}{2}mu^2+\frac{1}{2}Mv^2$

or $\quad mV^2=mu^2+Mv^2=Mu^2+M\,(mu/M)^2$, from (ii)

$$=u^2\left[m+\frac{m^2}{M}\right]=\frac{m}{M}\,u^2\,(M+m)$$

or $\quad \dfrac{u^2}{V^2}=\dfrac{M}{(m+M)}$ or $\dfrac{u}{V}=\sqrt{\left(\dfrac{M}{m+M}\right)}$. Hence proved.

[illegible] body of mass 400 gms is at rest at a height 3 metres [illegible]

and potential energies when it has fallen through a distance of 2 metres? Find answers in joules, (given that g = 9·8 m/sec²).

Solution. Mass of the body = 400 gms. = $\frac{2}{5}$ kg.

Height above the ground = 3 metres.

$\therefore$ Potential energy at this height = 'mgh'

$= \frac{2}{5}\times 9{\cdot}8\times 3 = 11{\cdot}76$ joules. **Ans.**

[illegible] be the velocity of the body when it has fallen [illegible] tial position *i.e.* at a [illegible]

[illegible] $2\times 9{\cdot}8\times 2$ or $v^2=39{\cdot}2$.

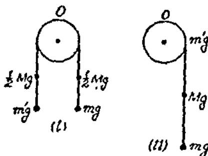
$\therefore$ K. Energy at this height = "$\frac{1}{2}mv^2$" = $\frac{1}{2}\times\frac{2}{5}\times 39{\cdot}2$

$= 7{\cdot}84$ joule. **Ans.**

and Potential energy at this height (*i.e.* 1 m. above the ground)

= "mgh" = $\frac{2}{5}\times 9{\cdot}8\times 1 = \frac{1}{5}\,(19{\cdot}6) = 3{\cdot}92$ joule. **Ans.**

[illegible] uniform string of mass M and length $2a$ is placed [illegible]



(Fig. 11)

Solution. Let O be the peg. Initially when the string of mass M and length $2a$ is placed symmetrically over the peg and the masses m and m' are at the two ends of the string, then the depth of the centre of gravity of the system from the peg O [see fig. (i)]

$$=\frac{mg\,a+m'g\,a+\frac{1}{2}Mg.\frac{1}{2}a+\frac{1}{2}Mg\,\frac{1}{2}a}{mg+m'g+\frac{1}{2}Mg+\frac{1}{2}Mg}$$

$$=\frac{1}{2}a.\,\frac{M+2\,(m+m')}{M+m+m'} \quad \text{...(i)}$$

Secondly when the string runs off the peg [see fig. (ii)], the depth of the centre of the system from the peg O is

$$=\frac{mg\,2a+m'g.0+Mga}{mg+m'g+Mg}=a\,\frac{M+2m}{M+m+m'} \quad \text{.. (ii)}$$

The initial velocity of the system was zero and the velocity of the system when the string runs off the peg is v (say), the

The change in K.E. $=\frac{1}{2}\,(M+m+m')\,v^2-\frac{1}{2}\,(M+m+m')\,0.$

$=\frac{1}{2}\,(M+m+m')\,v^2.$ ...(iii)

Also work done (by the weight of the system)

$$=(M+m+m')\,g.\left[a.\,\frac{M+2m}{M+m+m'}-\frac{1}{2}a\,\frac{M+2m+2m'}{M+m+m'}\right],$$

from (i) and (ii)

$=\frac{1}{2}ag\,\{M+2\,(m-m')\}.$

$\therefore$ From the change in K.E. = work done, we have

$\frac{1}{2}\,(M+m+m')\,v^2=\frac{1}{2}ag\,\{M+2\,(m-m')\}$

or $$v=\sqrt{\left[\frac{M+2\,(m-m')}{M+m+m'}.ag\right]}.$$ Hence proved.

Ex. 8 (b). A flexible but inextensible chain of length l and weight wl is held on a smooth table with the length $l-a$ on the table and the length a over hanging. Apply the principle of energy to obtain the velocity with which the chain will leave the table if released.

Solution. Let AB be the chain of length l. Initially the length $(l-a)$ is on the table and the length a is overhanging. [See fig. (i)].

$$=\frac{w\,(l-a)\times 0+wa\times\frac{1}{2}a}{w\,(l-a)+wa},$$ where w is the weight per unit length (given)

$$=\frac{a^2}{2l} \quad \text{...(i)}$$

Secondly when the chain is released and it leaves the table, the depth of C.G. of the chain below the table $=\frac{1}{2}l$ [see fig. (ii)]. ... (ii)

∴ From the instant of releasing the chain to the instant when it leaves the table, the work done by the weight of chain.

$$=wl\times\left[\frac{l}{2}-\frac{a^2}{2l}\right]=\frac{w(l^2-a^2)}{2}. \quad \text{...(iii)}$$

Also initially the velocity of the chain was zero and its velocity when it leaves the table is v (say), then

the change in K.E. $=\frac{1}{2}\cdot\frac{wl}{g}\cdot v^2-\frac{1}{2}\cdot\frac{wl}{g}\cdot 0=\frac{1}{2}\cdot\frac{wl}{g}v^2$. ...(iv)

∴ From change in K.E. = work done, we have

$$\frac{1}{2}\cdot\frac{wl}{g}v^2=\frac{w(l^2-a^2)}{2}, \text{ from (iii) and (iv)}$$

or $v=\sqrt{[g(l^2-a^2)/l]}$. Hence the result.

*Ex. 9. A shell of mass m fired from a gun of mass M which can recoil freely on a horizontal base, and the elevation of the gun is α. Prove that the inclination on the path of the shell to the horizon at the time of projection is $\tan^{-1}\left\{\left(1+\frac{m}{M}\right)\tan\alpha\right\}$. (*Rohilkhand* 86)

Prove also that the energy of the shell on leaving the gun is to that of the gun as $[M^2+(m+M)^2\tan^2\alpha] : Mm$, assuming that none of the energy of the explosion is lost.

Solution. Let the velocity and elevation to the horizontal of the shell on leaving the gun be u and θ respectively. Let v be the velocity of the gun when recoiling horizontally.

The horizontal and vertical components of the velocity of the shell relative to the gun are $(u\cos\theta+v)$ and $u\sin\theta$ respectively.

Since the elevation of the gun is given to be α.

∴ $\tan\alpha=u\sin\theta/(u\cos\theta+v)$. .. (i)

Also by the Principle of conservation of momentum, we have

$Mv=mu\cos\theta$ ...(ii)

∴ From (i), $\tan\alpha=\frac{u\sin\theta}{u\cos\theta+v}=\frac{u\sin\theta}{u\cos\theta+\left(\frac{mu\cos\theta}{M}\right)}$

$$=\frac{Mu\sin\theta}{(M+m)\cos\theta}=\frac{M\tan\theta}{(M+m)}$$

or $\qquad \tan\theta=\left(\dfrac{M+m}{M}\right)\tan\alpha=\left(1+\dfrac{m}{M}\right)\tan\alpha \qquad$ ...(iii)

or $\qquad \theta=\tan^{-1}\left\{\left(1+\dfrac{m}{M}\right)\tan\alpha\right\}.$ $\qquad$ Hence proved.

Also $\dfrac{\text{energy of the shell}}{\text{energy of the gun}}=\dfrac{\frac{1}{2}mu^2}{\frac{1}{2}Mv^2}=\dfrac{m}{M}\left(\dfrac{u}{v}\right)^2$

$$=\frac{m}{M}\cdot\left(\frac{M}{m\cos\theta}\right)^2, \quad \because \text{ from (ii) } \frac{u}{v}=\frac{M}{m\cos\theta}$$

$$=\frac{M}{m}\sec^2\theta=\frac{M}{m}(1+\tan^2\theta)$$

$$=\frac{M}{m}\left[1+\left(1+\frac{m}{M}\right)^2\tan^2\alpha\right], \text{ from (iii)}$$

$$=\frac{M}{m}\left[\frac{M^2+(M+m)^2\tan^2\alpha}{M^2}\right]=\frac{M^2+(M+m)^2\tan^2\alpha}{mM}.$$

Hence proved.

Ex. 10. A shell of mass m is projected from a gun of mass M by an explosion which generates kinetic energy E. Prove that the initial velocity of the shell is $\sqrt{\{2EM/m(m+M)\}}$, it being assumed that at the instant of explosion the gun is free to recoil.

Solution. Let u and v be the velocities of the shell and recoil of gun respectively. Then as momentum of the shell = momentum of the gun, so we get $\qquad mu=Mv.$ $\qquad$ ...(i)

Also energy $\qquad E=\frac{1}{2}mu^2+\frac{1}{2}Mv^2.$ $\qquad$ ...(ii)

Eliminating v from (i) and (ii), we get

$$E=\tfrac{1}{2}mu^2+\tfrac{1}{2}M(mu/M)^2=\frac{m(m+M)u^2}{2M}$$

or $\qquad u=\sqrt{\{2EM/m(m+M)\}}.$ $\qquad$ Hence proved.

Ex. 11. A body of mass (m_1+m_2) moving in a straight line is split into two parts of masses m_1 and m_2 by an internal explosion [illegible] that if after the explosion [illegible] their relative speed is [illegible]

Solution. Let u be the velocity of the body before explosion. Let the velocities of the parts of masses m_1 and m_2 be v_1 and v_2 after explosion.

Then by principle of conservation of momentum, we have

$$(m_1+m_2)u=m_1v_1+m_2v_2. \qquad \text{...(i)}$$

Also total kinetic energy before explosion = kinetic energy after explosion.

i.e. $\qquad E+\frac{1}{2}(m_1+m_2)u^2=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$

or $\qquad 2E+(m_1+m_2)u^2=m_1v_1^2+m_2v_2^2.$ $\qquad$ ...(ii)

Squaring (i), we get $(m_1+m_2)^2u^2=(m_1v_1+m_2v_2)^2$ $\qquad$ ...(iii)

Multiplying (ii) by (m_1+m_2), we get

$$2E(m_1+m_2)+(m_1+m_2)^2u^2=(m_1+m_2)(m_1v_1^2+m_2v_2^2) \qquad \text{...(iv)}$$

Subtracting (iii) from (iv), we get

$$2E(m_1+m_2)=(m_1+m_2)(m_1v_1^2+m_2v_2^2)-(m_1v_1+m_2v_2)^2$$
$$=m_1m_2(v_1^2+v_2^2-2v_1v_2)=m_1m_2(v_1-v_2)^2$$

or $v_1-v_2=\sqrt{\{2E(m_1+m_2)/m_1m_2\}}$.

$\therefore$ Relative velocity of the two parts after explosion

$$=v_1-v_2=\sqrt{\{2E(m_1+m_2)/m_1m_2\}}.$$ Hence proved.

**Ex. 12. A shell is moving with velocity u in the line AB. An internal explosion which generates an energy E, breaks it into two fragments of masses m_1 and m_2 which moves in the line AB, show that velocities are

$$u+\sqrt{\{2Em_2/m_1(m_1+m_2)\}} \text{ and } u-\sqrt{\{2Em_1/m_2(m_1+m_2)\}}.$$

Solution. Let the velocities of the fragment of masses m_1 and m_2 be v_1 and v_2 after explosion.

Then by the principle of conservation of momentum, we have

$$(m_1+m_2)u=m_1v_1+m_2v_2. \quad ...(i)$$

Also total kinetic energy before explosion=kinetic energy after explosion

i.e., $\frac{1}{2}(m_1+m_2)u^2+E=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2$

or $(m_1+m_2)u^2+2E=m_1v_1^2+m_2v_2^2$. ...(ii)

Also by inspection we find that $v_1=u+x/m_1$ and $v_2=u-x/m_2$ satisfy (i) for all values of x.

If these values satisfy (ii) also then we have

$$(m_1+m_2)u^2+2E=m_1\left[u+\frac{x}{m_1}\right]^2+m_2\left[u-\frac{x}{m_2}\right]^2$$
$$=m_1\left(u^2+\frac{2ux}{m_1}+\frac{x^2}{m_1^2}\right)+m_2\left(u^2-\frac{2ux}{m_2}+\frac{x^2}{m_2^2}\right)$$
$$=(m_1+m_2)u^2+x^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$$

or $2E=x^2(m_1+m_2)/m_1m_2$

or $x^2=2Em_1m_2/(m_1+m_2)$ or $x=\sqrt{\{2Em_1m_2/(m_1+m_2)\}}$.

Hence $v_1=u+x/m_1=u+\sqrt{\{2Em_2/m_1(m_1+m_2)\}}$

and $v_2=u-x/m_2=u-\sqrt{\{2Em_1/m_2(m_1+m_2)\}}$.

**Ex. 13. A hammer of mass M gms. falls freely from a height h cm, on the top of an inelastic pile of mass m gms. which is driven into the ground to a distance a cm. Assuming that the resistance of ground is constant, show that the time during which the pile is in motion is given by $\frac{a(m+M)}{M}\left(\frac{2}{gh}\right)^{1/2}$. *(Kanpur 85)*

Find also the kinetic energy lost at the impact. What weight would have to be placed on the top of pile in order to drive it slowly into the ground? *(Rohilkhand 87)*

Solution. Let u cm/sec be the velocity with which the hammer strikes the pile, then $u^2=2gh$. ...(i)

Let velocity of the pile after being struck by the hammer be v

cm/sec. Then the pile being inelastic, the hammer will not rebound after striking but will move with the pile with velocity v of the pile.

$\therefore$ from principle of momentum, we have

$$Mu=(M+m)\,v \quad \text{or} \quad v=Mu/(m+m). \qquad ...(ii)$$

Let the force due to resistance of the ground be P dynes. Then the force acting on the system is $P-(m+M)\,g$, which brings the pile to rest after moving through a distance a cm.

Now change in kinetic energy = work done.

$$\therefore \tfrac{1}{2}(m+M)\,v^2-0=[P-(m+M)\,g]\,a$$

or

$$P=\tfrac{1}{2}(m+M)\,(v^2/a)+(m+M)\,g$$

$$=\frac{(m+M)}{2a}\cdot\frac{M^2u^2}{(m+M)^2}+(m+M)\,g, \text{ from (ii)}$$

$$=\frac{(m+M)}{2a}\cdot\frac{M^2 .2gh}{(m+M)^2}+(m+M)\,g, \text{ from (i)}$$

or

$$P=\left[\frac{M^2h}{a\,(m+M)}+(m+M)\right]g \text{ dynes} \qquad ..(iii)$$

Also loss of kinetic energy $=\tfrac{1}{2}Mu^2-\tfrac{1}{2}(m+M)\,v^2$

$$=\tfrac{1}{2}Mu^2-\tfrac{1}{2}(m+M)\cdot\frac{M^2u^2}{(m+M)^2}, \text{ from (ii)}$$

$$=\tfrac{1}{2}Mu^2\left[1-\frac{M}{m+M}\right]=\frac{Mmu^2}{2\,(m+M)}. \qquad \text{Ans.}$$

Let f cm/sec² be the retardation due to resistance of the ground, then from "$v^2=u^2+2fs$", we get $0=v^2-2fa$

or

$$f=v^2/2a. \qquad ...(iv)$$

Also if t be the required time, from "$v=u+ft$", we get

$$0=v-ft \quad \text{or} \quad t=\frac{v}{f}=\frac{v.2a}{v^2}, \text{ from (iv)}$$

or

$$t=\frac{2a}{v}=\frac{2a\,(m+M)}{mu}, \text{ from (ii)}$$

$$=\frac{2a\,(m+M)}{m.\sqrt{(2gh)}}=\frac{a\,(m+M)}{m}\cdot\sqrt{\left(\frac{2}{gh}\right)}. \qquad \text{Hence proved.}$$

Now if W be the weight placed on the top of the pile in order to drive it slowly into the ground, then

W + weight of the pile = resistance of the ground

or

$$W+mg=P$$

or

$$W=P-mg=\left[\frac{M^2hg}{a\,(m+M)}+(m+M)\,g\right]-mg, \text{ from (iii)}$$

$$=M\left[\frac{Mh}{a\,(m+M)}+1\right]g \text{ dynes}. \qquad \text{Ans.}$$

Ex. 14. Prove that if a hammer weighing W lbs. striking a nail weighing w lbs. with velocity V feet per second, drives it a feet into a fixed block of wood, the average resistance of the wood in

pounds to the penetration of the nail is $\frac{W^2V^2}{2ga\,(W+w)}$.

If however the block is free to recoil and weighs M lbs. the resistance obtained would be $\frac{MW^2V^2}{2ag\,(M+W+w)\,(W+w)}$, motion in the case of a nail being driven is in the horizontal direction.

Solution. Case I. *If the block be fixed.* Let the hammer and the nail move with a common velocity v after the nail is being struck by the hammer.

Then as total momenta before impact = total momenta after impact,

$$\text{so} \quad WV=(W+w)\,v \quad \text{or} \quad v=\frac{WV}{(w+W)} \qquad \ldots(i)$$

If R be the average resistance, by the Principle of Energy, we have

$$\tfrac{1}{2}\,(W+w)\,v^2=Rag$$

$$\text{or} \quad R=(W+w)\,v^2/2ag=(W+w).W^2V^2/2ag\,(W+w)^2, \text{ from (i)}$$

$$=W^2V^2/2ag\,(W+w). \qquad \text{Hence proved.}$$

Case II. *If the block be free to recoil.*

Let the hammer, nail and the block move with a common velocity v_1 after the penetration ceases, then we have

$$(m+W+w)\,v_1=WV \quad \text{or} \quad v_1=\frac{WV}{(M+W+w)} \qquad \ldots(ii)$$

If R_1 be the average resistance in this case, then by Principle of Energy, we have

$$\tfrac{1}{2}\,(W+x)\,v^2-\tfrac{1}{2}\,(M+W+w)\,v_1^2=R_1ga$$

$$\text{or} \quad R_1=\frac{1}{2ag}\,[(W+w)\,v^2-(M+W+w)\,v_1^2]$$

$$=\frac{1}{2ag}\left[(W+w)\frac{W^2V^2}{(W+w)^2}-(M+W+w).\frac{W^2V^2}{(M+W+w)^2}\right], \text{ from (i) and (ii)}$$

$$=\frac{W^2V^2}{2ag}\left[\frac{1}{W+w}-\frac{1}{W+M+w}\right]$$

$$=MW^2V^2/[2ag\,(W+w)\,(M+W+w)]. \qquad \text{Hence proved.}$$

Ex. 15. A hammer of 1 kilogram strikes a nail of 100 gms. with a velocity of 100 cm./sec. Find the loss in kinetic energy due to striking.

S[illegible] combined body

[illegible]

[illegible] 100 gm.

$$\text{or} \quad 1100\,v=1000\times100 \quad \text{or} \quad v=(1000/11) \text{ cms./sec.}$$

$\therefore$ *Total K.E. before impact* $=\tfrac{1}{2}\times100\times0^2+\tfrac{1}{2}\times1000\times(100)^2$

$=500\times(100)^2$ cm.-gm.

and total K.E. after impact $=\frac{1}{2}(100+1000)\times\left(\frac{1000}{11}\right)^2$

$=550\times(1000/11)^2$ cm. gm.

$\therefore$ Required loss in K.E.

$=[500\times(100)^2-550\times(1000/11)^2]$ cm.-gm.

$=500\times(100)^2\left[1-\frac{11\times10}{121}\right]=500\times(100)^2\times\frac{1}{11}$

$=(50/11)\times10^6$ cm.-gm. $=(4\cdot555)\times10^5$ ergs. Ans.

***Ex. 16. In starting a train the pull of the engine on the rails is at first constant and equals P, after the speed attains a certain value u the engine works at a constant rate $R=Pu$. Prove that when the engine has attained a speed $v>u$, the time t from the start is given by $t=M(v^2+u^2)/2R$, where M is the mass of the engine and the train.**

Solution. Let t_1 be the time taken in the first part of the journey when pull is constant and equal to P and let t_2 be the time taken in the second part of the journey when the engine works at a constant rate $R=Pu$.

In the first part the velocity of the train changes from O to u and in second part it changes from u to v. Let x_1 and x_2 be the distances moved in these two parts of journey.

Since change in K. Energy = work done in the interval.

$\therefore$ $\frac{1}{2}Mu^2=Px_1$, for the first part of the journey

or $$x_1=\frac{Mu^2}{2P}. \quad ...(i)$$

And for the second part of the journey,

$$\frac{1}{2}M(v^2-u^2)=\int_{x=x_1}^{x_2}F\cdot dx, \quad ..(ii)$$

where F is the variable force in this part.

Let ω be the velocity of the particle at any point.

Then $$F.\omega=R$$

or $$F=R/\omega. \quad .(iii)$$

$\therefore$ From (ii), $\frac{1}{2}M(v^2-u^2)=\int_{x_1}^{x_2}\frac{R}{\omega}dx$

$=\int_{x_1}^{x_2}\frac{R}{(dx/dt)}dx=\int_{t=0}^{t_2}R\,dt=R.t_2$

or $$t_2=\frac{M(v^2-u^2)}{2R}. \quad ...(iv)$$

Also change in momentum = Impulse of the force.

$\therefore$ $M(u-0)=Pt_1$, for the first part of journey

or $t_1=Mu/P=Mu^2/R$, $\because$ $R=Pu$

$\therefore$ Required total time $=t_1+t_2=\frac{Mu^2}{R}+\frac{M(v^2-u^2)}{2R}$

$=M(v^2+u^2)/2R$. Hence proved.

Exercises on Energy

**Ex. 1. A shell lying in a straight smooth horizontal tube suddenly breaks into two portions of masses m_1 and m_2. If s is the distance apart, in the tube, of the masses after a time t, show that the work done by the explosion is $\frac{1}{2}\frac{m_1 m_2}{(m_1+m_2)} \cdot \frac{s^2}{t^2}$.

(*Rohilkhand* 85)

[Hint : As in Ex. 11 Page 29 of this chapter we can show that

$$v_1 - v_2 = \sqrt{\left[\frac{2E(m_1+m_2)}{m_1 m_2}\right]}.$$

Also here s=relative distance of the portions after explosion, at time t

or $s=(v_1-v_2)\,t$. Find s^2/t^2].

Ex. 2. A bullet is fired at a speed of 600 metres per second on a target, of mass 12 kg and free to move. If the target embedded with the bullet, after it penetrates the target, moved with a velocity of 1·5 metres per second then find the mass of the bullet and also the percentage of loss of kinetic energy.

Ans. $\frac{4}{1.3}$ kg and 99·7%

Ex. 3. A stone is projected vertically upwards with velocity 980 cm./sec. Find the height at which its kinetic and potential energies are equal. (Take g=980 cm./sec²). Ans. 245 cm.

Ex. 4. A shell of mass (m_1+m_2) is fired with a given velocity in a given direction. At the highest point of its path, the shell explodes into two fragments of masses m_1 and m_2. The explosion produces an additional kinetic energy E and the fragments strike the ground at a distance which is equal to $\frac{V}{g}\sqrt{\left[2E\left(\frac{1}{m_1}\right)+\left(\frac{1}{m_2}\right)\right]}$, where V is the vertical component of the velocity of projection.

**Ex. 5. A shell of mass M is moving with velocity V. An internal explosion generates an amount of energy E and breakes the shell into two portions whose masses are in the ratio $m_1 : m_2$. The fragments continue to move in the original line of motion of the shell. Show that their velocities are

$$V+\sqrt{\left(\frac{2m_2E}{m_1M}\right)} \quad \text{and} \quad V-\sqrt{\left(\frac{2m_1E}{m_2M}\right)}$$

[Hint : See Ex. 12 Page 30].

§ 10. Impulse of a force. (*Calcutta* 85; *Kanpur* 85)

Case I. *When the force is constant.* If a force F remains constant (in magnitude and direction) for an interval of time t

seconds, then *Ft i.e.* the *product of the force F and time t is* called the impulse of the force F during the time t.

Case II. *When the force is variable.* At time t, let the magnitude of the force be F. In the interval of time δt the force practically remains constant, so the impulse of the force F in the interval of time $\delta t = F\,\delta t$.

$\therefore$ In an interval of time $t_2 - t_1$,

the impulse of the force $= \int_{t_1}^{t_2} F\,dt$.

In vector notations :—

Definition. If a force **F** acting on a particle remains constant (both in magnitude and direction) for an interval of time t, then the vector $\mathbf{I} = t\,\mathbf{F}$ is called the impulse of the force **F** during the time t.

If however, if a variable force $\mathbf{F}(t)$ is acting on the particle then during an interval of time $t_2 - t_1$, the vector

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\,dt \qquad \ldots\text{(i)}$$

is known as the impulse of the force during the interval (t_1, t_2).

If F is the resultant of the number of concurrent forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$, then as

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = \int_{t_1}^{t_2} \sum_{i=1}^{n} \mathbf{F}_i\,dt = \sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{F}_i\,dt$$

$\therefore$ Impulse of $\mathbf{F} = \mathbf{I} = \sum_{i=1}^{n} \int_{t_1}^{t_2} \mathbf{F}_i\,dt$, during the interval (t_1, t_2)

i.e. the impulse of the resultant of a number of concurrent forces is equal to the sum of the impulses of the concurrent forces taken separately.

§ 11. Impulse-momentum Principle.

Statement : *The change in momentum of a particle during an interval of time is equal to the impulse of the force acting on the particle during that interval.*

Case II. *When the force is constant.* In this case acceleration f is also constant, since force F (say) is constant.

If u and v are the velocities of the mass m (which is being acted upon by the force F) at the beginning and end of the interval t, then we have $v = u + ft$ or $ft = v - u$. ...(i)

Also $F.t = mf.t$, since $F = mf$

$= m(v - u)$, from (i)

or $F.t = mv - mu$.

$\therefore$ Impulse of the force $=$ change in the momentum.

Case II. *When the force is variable.* If the force is variable, the acceleration is also variable.

$\therefore$ acceleration $= dw/dt$, where w is the velocity at time t.

$\therefore$ Force $F = m.dw/dt$

or $F\,dt = m\,dw$.

$\therefore$ In the interval $(t_2 - t_1)$, we have

$$\int_{t_1}^{t_2} F\,dt = m \int_u^v dw = m \left(w \right)_u^v = m\,(v-u)$$

or impulse in the interval $(t_2 - t_1) = mv - mu =$ change in momentum

Note. If u and v are finite and interval $(t_2 - t_1)$ is very small then the force must be very large and such a force is called impulsive force.

In vector notations :—

Let F be the force (resultant external force) acting on the particle of mass m during an interval of time (t_1, t_2). Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocity vectors of the particle at the instants t_1 and t_2 respectively and let $\mathbf{v}$ be the velocity vector of the particle at any instant during this interval (t_1, t_2).

Then the equation of motion of the particle is

$$m\,(d\mathbf{v}/dt) = \mathbf{F}. \qquad \text{.. (i)}$$

$\therefore$ If I be the impulse vector of the force F during the interval (t_1, t_2) then we have

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{F}\,dt = \int_{t_1}^{t_2} m\,\frac{d\mathbf{v}}{dt}\,dt, \text{ from (i)}$$

$$= \int_{\mathbf{v}_1}^{\mathbf{v}_2} d\mathbf{v} = m\,(\mathbf{v}_2 - \mathbf{v}_1).$$

Hence impulse of the force F during the interval $(t_1, t_2) =$ change in momentum of the particle on which F is acting. Hence proved.

§ 12. Units of Impulse.

The units of impulse are those used for momentum, since from § 11 above we know impulse of a force = change in momentum.

Thus the absolute units of impulse are :—

In F. P. S. system lb.ft./sec ; in M. K. S. system kg.-m/sec. and in C. G. S. system gm.-cm./sec.

§ 13. Principle of conservation of linear momentum for a particle. (*Ranchi 85*)

Statement. *If the resolved part in a given direction of the resultant force acting on a moving particle is zero, then the resolved part of the momentum in that direction remains constant.*

Proof. Let F be the resultant force acting on a particle of mass m at the instant when its velocity is v. Then from Newton's II law we have $m\,\frac{dv}{dt} = F$ or $\frac{d}{dt}\,(mv) = F$. ...(i)

Now let $F \cos \alpha$ be the resolved part of force F in a given direction and let it be zero *i.e.* $F \cos \alpha = 0$, where α is constant.

Then we have

$$\frac{d}{dt}(mv\cos\alpha)=\cos\alpha.\ \frac{d}{dt}(mv)$$

$$=(\cos\alpha)\ F,\text{ from (i)}$$

$$=0,\ \because\ F\cos\alpha=0$$

$\therefore\ mv\cos\alpha=\text{constant}$

i.e. the resolved part of the momentum in the given direction is constant. Hence proved.

In vector notations :

Let **F** be the resultant force acting on a particle of mass m at the instant when its velocity vector is **v**. Then from Newton's II law, we have $m\dot{\mathbf{v}}=\mathbf{F}$ or $\frac{d}{dt}(m\mathbf{v})=\mathbf{F}$ ...(ii)

Let **e** be the unit vector in the direction in which the resolved part of **F** vanishes.

i.e. $\mathbf{F}\cdot\mathbf{e}=0$, where **e** is a constant vector being in a given direction.

$$\therefore\ \frac{d}{dt}(\mathbf{e}\cdot m\mathbf{v})=\mathbf{e}\cdot\frac{d}{dt}(m\mathbf{v}),\ \because\ \mathbf{e}\text{ is constant}$$

$$=\mathbf{e}\cdot\mathbf{F},\text{ from (ii)}$$

$$=0,\ \because\ \mathbf{F}\cdot\mathbf{e}=0.$$

$\therefore\ \mathbf{e}\cdot m\mathbf{v}=\text{constant}$

i.e. the resolved part of the momentum vector $m\mathbf{v}$ in the given direction is zero. Hence proved.

§ 14. Principle of conservation of linear momentum for a system particles.

Statement. *If the sum of the resolved parts in a given direction of the external force acting on a system of particles is zero, then the resolved part of the total momentum of the system in that direction remains constant.*

Proof : In vector notations :—Let us consider a system of two particles of masses m_1 and m_2 only, let their velocity vectors at any instant be $\mathbf{v}_1$ and $\mathbf{v}_2$ and let the resultant external forces acting on them be $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively.

Let **R** be the internal force acting on the particle of mass m_1 due to that of mass m_2.

Then from Newton's II law of Motion, we have

$$\frac{d}{dt}(m_1\mathbf{v}_1)=\mathbf{F}_1+\mathbf{R}\text{ and }\frac{d}{dt}(m_2\mathbf{v}_2)=\mathbf{F}_2-\mathbf{R}.\quad\text{(Note)}$$

Adding these, we get $\frac{d}{dt}(m_1\mathbf{v}_1+m_2\mathbf{v}_2)=\mathbf{F}_1+\mathbf{F}_2$ ...(i)

Now if the sum of the resolved parts of F_1 and F_2 in a given direction vanishes and if e be the unit vector in that direction, then

$(F_1+F_2)\cdot e=0$, where e is a constant vector

$$\therefore \frac{d}{dt}[(m_1v_1+m_2v_2)\cdot e]=e\cdot\frac{d}{dt}(m_1v_1+m_2v_2), \because \text{ e is constant}$$
$$=e\cdot(F_1+F_2), \text{ from (i)}$$
$$=0$$

or $(m_1v_1+m_2v_2)\cdot e=$ constant

i.e. the resolved part of the total momentum of the system in the given direction is constant. Hence proved.

Solved Examples on Impulse

Ex. 1. A force acting on a body of mass 5 lbs. changes its velocity from 30 miles/hr to 45 miles/hr. Find the impulse of the force.

Sol. We know (from § 11 Page 35) that

Impulse of force=change in momentum ...(i)

$$\text{Here final momentum}=5\times\frac{45\times1760\times3}{60\times60}$$
$$=5\times66 \text{ lbs. ft/sec}$$

$$\text{and initial momentum}=5\times\frac{30\times1760\times3}{60\times60}$$
$$=5\times44 \text{ lbs. ft/sec}$$

From (i), the required impulse

$=(5\times66)-(5\times44)=110$ lbs. ft/sec. Ans.

Ex. 2. A mass of 5 lbs attached to a string is at rest. If a sudden jerk of the string produces in the mass a velocity of 12 ft/sec, find the impulse of the jerk.

Solution. Initial momentum $=5\times0=0$ lbs. ft/sec.

Final momentum $=5\times12=60$ lbs. ft/sec.

$\therefore$ The required impulse of the jerk

=change in momentum

$=(60-0)=60$ lbs. ft/sec. Ans.

Exercise on Impulse

Ex. Correct the following sentence :—

"Impulsive force is a...force acting for .... time."

Ans. Impulsive force is a *very large* force acting for a *very short time*.

MISCELLANEOUS SOLVED EXAMPLES

Ex. 1. A jet of water issues vertically at a speed of 30 ft per second from a nozzle of 0·1 square inch section. A ball weighing 1 lb. is balanced in the air by the impact of water on its inner side.

Show that the height of the ball above the level of the jet is 4·6 feet approximately. (1 cu. ft. of water weighs 62·5 lbs).

Solution. The section of the nozzle = 0·1 sq. inch

$= 1/1440$ sq. ft.

The speed of the jet of water = 30 ft./sec.

$\therefore$ Mass m of water issued from the jet per second

$= (1/1440) \times 30 \times 62 \cdot 5$ lbs. ($\because$ 1 cu. ft. of water = 62·5 lbs.)

$= \dfrac{3 \times 625}{1440}$ lbs.

Also if h be the height of the ball above the level of the jet and v the velocity of the water at the time of striking the ball, then from "$v^2 = u^2 + 2fs$" we get $v^2 = (30)^2 - 2gh$ or $v = \sqrt{(900 - 2gh)}$. ...(ii)

Now the mass given by (i) strikes [illegible]he ball with velocity v [given by (ii)] and reduces to rest and the force in the vertical direction is the weight of the ball which is 1 lb. or g-poundals (given).

Also by Newton's second law, rate of change of momentum in the vertical direction = the force in that direction.

$\therefore \quad m(v - 0) = g$

or $\quad \dfrac{3 \times 625}{1440} \times \sqrt{(900 - 2gh)} = g$, from (i) and (ii)

or $\quad \sqrt{(900 - 2gh)} = 1440g/(3 \times 625) = 96g/125$

or $\quad 900 - 2gh = (96/125)^2 . g^2$, where $g = 32$ ft/sec^2

or $\quad h = \dfrac{900}{2g} - \left(\dfrac{96}{125}\right)^2 \left(\dfrac{g}{2}\right) = \dfrac{900}{64} - \dfrac{96 \times 96 \times 16}{125 \times 125} = 4 \cdot 6$ approx.

Hence proved.

Ex. 2. A cylindrical cork of length l and [illegible] extracted from the [illegible]

[illegible]

Solution. Let a length h of cork be in the contact with the neck of the bottle. Then the area of the surface of the cork in contact with the neck of the bottle is $2\pi rh$.

Since P is the [illegible] it of area between [illegible] k at any time. so we [illegible] and [illegible]

H[illegible] done in extracting a further length δh of the cork $= \mu 2\pi rh \, P . \delta h$.

$\therefore$ Required work done $= 2\pi\mu rP \int_0^l h \, dh = 2\pi\mu rP \left[\tfrac{1}{2} h^2 \right]_0^l$

$= \pi\mu rP . l^2$. Ans.

Ex. 3. *Water issuing from a nozzle of radius r cms. with a velocity v cm/sec., impinges on a vertical wall, the jet being at right angles to the wall. If there is no splash, find the pressure exerted on the wall.*

Solution. When the jet strikes the wall, the wall exerts a force R dynes per square centimetre (say) on the jet which destroys its momentum along a normal to the wall.

Then the mass of water that reaches the wall in one second

$$=(\pi r^2)\, v \text{ gms.}$$

∴ Change in momentum $=\pi r^2 v\,[0-(-v)]$, Ans.

in the direction of R which is opposite to the direction of the jet.

i.e. change in momentum $=\pi r^2 v^2$ gm-cm/sec. ...(i)

Also the impulse of the force R in one second

$=(\pi r^2 R).1$ gm-cm/sec. (Note) ...(ii)

∵ Change in momentum $=$ impulse of the force.

∴ $\pi r^2 v^2 = \pi r^2 R$, from (i) and (ii)

or $R = v^2$ dynes/cm².

∴ From Newton's III Law, the pressure exerted on the wall

$=R=v^2$ dynes/cm². Ans.

Ex. 4. *A uniform chain is called in a heap on a horizontal plane, a man takes hold of one end raises it uniformly with a velocity v. Show that when his hand is at a height h above the plane, the pressure on his hand is equal to the weight of a length $(h+v^2/g)$ of the chain.*

Solution. Let m be the mass per unit length of the chain. Let at time t the length of the chain raised above the plane be x.

The weight of the length x of the chain $=xmg$ and its momentum $=xm.v$, where v is its velocity (given)

∴ *Total force on this mass at time $t=P-mgx$,*

where P is the uniform pull of the man.

Also by Newton's second law of motion, we konw rate of change of momentum $=$ impressed force.

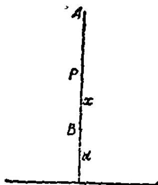
$$\therefore \frac{d}{dt}(xm.v)=P-mgx \quad \text{or} \quad mv.\frac{dx}{dt}=P-mgx$$

or $mv^2=P-mgx$, ∵ $dx/dt=v$

or $P=mgx+mv^2$.

∴ When his hand is at a height h above the plane, the pressure on his hand $=mgh+mv^2=mg\,(h+v^2/g)$. Hence proved.

... so ... the ... stant



(Fig. 13)

Solution. The chain AB of length l is held by its upper end A so that the lower end B is at a height a above the horizontal plane. Let m be the mass per unit length of the chain.

Take a point P of the chain, such that $BP=x$. Also the height of P above the horizontal plane$=(d+x)$. $\therefore$ The velocity of P on reaching the horizontal plane$=\{2g(d+x)\}$, when the upper end is let go and the string moves vertically downward and under gravity.

$\therefore$ In the time δt a length $\sqrt{\{2g(d+x)\}}.\delta t$ of chain will strike the horizontal plane with a velocity $\sqrt{\{2g(d+x)\}}$ which will remain the same for each point of this length as δt is small.

$\therefore$ Change in momentum in time δt

$$=m\sqrt{\{2g(d+x)\}}.\delta t.\sqrt{\{2g(d+x)\}}=2mg(d+x).\delta t$$

$$\therefore \text{ Rate of change of momentum}=\frac{2mg(d+x)\,\delta t}{\delta t}=2mg(d+x)$$

Now we know that rate of change of momentum is equal to the impressed force.

$\therefore$ When the point P strikes the horizontal plane, the force due to striking$=2m(d+x)$.

Hence at the instant when a length x of the chain coiled on upon the plane, total pressure on the plane

=Pressure due to striking+weight of the chain coiled on the plane$=2mg(d+x)+mxg=3mxg+2mgd$. **Ans.**

Ex. 6. **A block of mass M rests on a smooth horizontal table and a bullet of mass m is fired into it. The penetration of the bullet is opposed by a constant resisting force. If the experiment is repeated with block firmly fixed, show that the depth of penetration of the bullet and the time which elapses before bullet is at rest relatively to the block are each increased in the ratio**

$$[1+(m/M)]:1.$$

Solution. When the block is fixed. Let the bullet strike the block with a velocity u. Let x be the depth of penetration and t the time of this penetration. Let P be the constant resisting force.

$\therefore$ From "change in momentum=impulse of the force" we have

$$mu=P.t \quad \text{...(i)}$$

Also from "change in kinetic energy=work done," we have

$$\tfrac{1}{2}mu^2=P.x. \quad \text{.. (ii)}$$

When the block is free to move. In this case let x' be the depth of penetration and t' the time of this penetration. Let the bullet and the block move with velocity v after this penetration ceases.

Then, from "change in momentum = impulse of the force," we have
$$mv = P.t'. \quad \text{(iii)}$$
And from "change in kinetic energy = work done," we get
$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}(m+M)v^2 = P.x' \quad \text{(Note)} \quad ...(iv)$$
Also from principle of conservation of momentum, we have
$$mu = (m+M)v. \quad ...(v)$$
Dividing (i) by (iii), we have
$$\frac{t}{t'} = \frac{u}{v} = \frac{u}{mu/(m+M)}, \text{ from (v)}$$
$$= \frac{m+M}{m} = \left(1 + \frac{M}{m}\right)$$
or
$$t : t' = [1 + (M/m)] : 1. \quad \text{Hence proved.}$$
Also from (ii) and (iv), we get
$$\frac{x}{x'} = \frac{\frac{1}{2}mu^2}{\frac{1}{2}mu^2 - \frac{1}{2}(m+M)v^2} = \frac{u^2}{u^2 - [(m+M)/m]v^2}$$
$$= \frac{(m+M)^2 v^2/m^2}{\frac{(m+M)^2 v^2}{m^2} - \left(\frac{m+M}{m}\right)v^2}, \text{ from (v)}$$
$$= \frac{1}{1 - [m/(m+M)]} = \frac{m+M}{m} = 1 + \frac{m}{M}$$
or
$$x : x' = [1 + (m/M)] : 1. \quad \text{Hence proved.}$$

***Ex. 7.** **Two particles of masses $3m$ and m are attached to the ends A, B respectively of a light rod 3 m. long which is freely hinged at a fixed point O in the rod where BO = 2 m. and the rod is constrained to rotate in a vertical plane. Initially A is above O, show that if the rod is just disturbed A will pass through the lowest position with velocity $\frac{1}{3}\sqrt{(28g)}$.**

Solution. Given $AB = 3$ metres, $BO = 2$ m. and $OA = 1$ m.

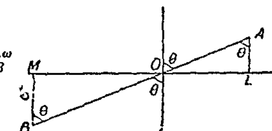
If the rod is slightly displaced, A and B will describible circles with common centre O but radius AO and BO respectively. At A and B, there are two forces acting *viz.* tension and weight. Tension will act at A in the sense AO and at B in the sense BO. But there being obstacle at O, the tension in two parts will be different. Also these tensions will do no work.

The angular velocity of the particle at A is the same as that of the particle at B (angles described being equal)

Let the angular velocity be $\omega = d\theta/dt$.

$\therefore$ The linear velocity of A = "$a d\theta/dt$" = $OA.\omega = 1.\omega$ and the linear velocity of B = $OB.\omega = 2\omega$.

Kinetic energy
$= \frac{1}{2}(3m)\,\omega^2 + \frac{1}{2}(m)(2\omega)^2$
$= (7/2)m\omega^2$ and



potential energy $= 3mg.AL - mg.BM$ (Fig. 14)

$= 3mg\ OA \cos\theta - mg\ OB \cos\theta$

$= 3mg \cos\theta - 2mg \cos\theta$. $\because$ $OA = 1$ m, $OB = 2$ m

$= mg \cos\theta$.

$\because$ Sum total of two energies is constant.

$(7/2)\ m\omega^2 + mg \cos\theta = \text{constant} = k$ (say). ...(i)

Initially *i.e.* when A is above O, $\theta = 0$, so $\omega = 0$, $k = mg$ ...(ii)

Finally when A is the lowe point and AB is vertical

$\theta = \pi$ and let $\omega = \omega_0$.

$\therefore$ From (i), $(7/2)\ m\omega_0^2 - mg = k = mg$, from (ii) or $\omega_0^2 = \frac{4}{7}g$.

$\therefore$ velocity of A in its lowest position = "$a\ d\theta/dt$"

$= OA\ \omega_0 = 1.\sqrt{(\frac{4}{7}g)} = \frac{1}{7}\sqrt{(28g)}$. Hence proved.

Ex. 8. Two equal weights P and P are supported by a string passing over smooth pegs A and B in the same horizontal line, a weight Q=2P/√3 is attached to the middle point of the string between A and B. Prove that Q will descend until QAB forms an equilateral triangle.

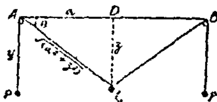
Solution. Let $AB = 2a$

The pegs being smooth, the tension throughout the string (even before attachment) will remain the same. When Q is attached, the tension will be different in two parts, but due to perfect symmetry these tensions will be equal. Let the string be of length $2l$. Then initially hanging portions are each equal to $(l-a)$. Also no work is done by these tensions of the strings.

Let in any portion, $\angle QAB = \theta$. Let at any instant $QD = z$ and the hanging portions $= y$.

We know sum total of two energies = const.

$\therefore$ For the whole system we have



(Fig. 15)

$[2\cdot\frac{1}{2}P\dot{y}^2+\frac{1}{2}Q\dot{z}^2]+[-2Pgy-Qgz]=\text{const.}=k$ (say) (Note)

Initially $y=(l-a)$, $z=0$, $\dot{y}=0$ and $\dot{z}=0$.

$\therefore\ 0+0-2Pg\,(l-a)-0=k$ or $k=-2Pg\,(l-a)$

$\therefore\ P\dot{y}^2+\frac{1}{2}Q\dot{z}^2-2Pgy-Qgz=-2Pg\,(l-a)$. ...(i)

Also from the figure it is evident that

$$OA+AP=\tfrac{1}{2}\text{ (length of string)}$$

or $$\sqrt{(a^2+z^2)}+y=l.$$ ...(ii)

Differentiating with respect to t, we get

$$\frac{z}{\sqrt{(a^2+z^2)}}\,\dot{z}+\dot{y}=0.$$ ...(iii)

When motion stops, then Q stops or $\dot{z}=0$ and $\dot{y}=0$ at the same time.

Let Q descend upto $z=z_1$ and $y=y_1$.

$\therefore$ When motion stops from (i), we have

$$0+0-2Pgy_1-Qgz_1=-2Pg\,(l-a)$$

or $$2Py_1+2Pz_1/\sqrt{3}=2P\,(l-a),\quad\because\ Q=2P/\sqrt{3}$$

or $$y_1+(z_1/\sqrt{3})=l-a.$$ ...(iv)

And when the motion stops, from (ii), we have

$$\sqrt{(a^2+z_1^2)}+y_1=l\quad\text{or}\quad y_1=l-\sqrt{(a^2+z_1^2)}.$$

Substituting this value of y_1 in (iv), we have

$$l-\sqrt{(a^2+z_1^2)}+(z_1/\sqrt{3})=l-a$$

or $$\sqrt{(a^2+z_1^2)}=a+(z_1/\sqrt{3}).$$

Squaring, $a^2+z_1^2=a^2+(z_1^2/3)+(2az_1/\sqrt{3})$

or $$(2z_1^2/3)=(2az_1/\sqrt{3})\quad\text{or}\quad z_1=a\sqrt{3}\ \text{or}\ 0.$$

But $z_1=0$ gives the original level. Hence $z_1=a\sqrt{3}$.

$\therefore$ In $\triangle ABQ$, $\cot\theta=\dfrac{AD}{DQ}=\dfrac{a}{z_1}=\dfrac{a}{a\sqrt{3}}=\dfrac{1}{\sqrt{3}}$

or $\tan\theta=\sqrt{3}$ or $\theta=60°$ or $\triangle ABQ$ is an equilateral triangle.

... inelastic plat- ...y. One of the ...ntre of gravity men leaping ... t if he ... e same energy through a height ... e of g ... ise a height from the platfor... being $\frac{(M+2m)\,h}{2\,(M+m)}$, initiall... ...ion Let th... ...e ... locit...

and V be the common velocity of moving system which comprises of mass m on one side and mass $(m+M)$ on the other.

$\therefore$ From principle of momentum, we have

$$Mv=\{m+(m+M)\}\ V \qquad ...(i)$$

If due to leaping, the C.G. of the man rises to a height h_1, then from "$v^2=u^2+2fs$" we get $0=v^2-2gh_1$ or $v^2=2gh_1$ ...(ii)

Also energy of the jump $=Mgh$.

$\therefore$ By the principle of conservation of energy, we have

$$\tfrac{1}{2}Mv^2+\tfrac{1}{2}(2m+M)\ V^2=Mgh$$

or $\frac{1}{2}M(2gh_1)+\frac{1}{2}(2m+M).\{Mv/(2m+M)\}^2=Mgh$, from (i) and (ii)

or $Mgh_1+\dfrac{M^2v^2}{2(2m+M)}=Mgh$ or $gh_1+\dfrac{M(2gh_1)}{2(2m+M)}=gh$, from (ii)

or $h_1\left[1+\dfrac{M}{2m+M}\right]=h$ or $h_1=\dfrac{(2m+M)\ h}{2(m+M)}$. Hence proved.

Ex. 10. A platform of mass m and a counterpoise of mass $(m+M)$ are connected by a light cord which passes over a small smooth pulley. A man of mass M is standing on the platform which is at rest. If the man leaps vertically with velocity, u, find the distance through which the platform will descend and show that when the man meets the platform again both are in their original positions.

Solution. Let the man leap from the platform with velocity v and V be the common velocity of the moving system which comprises of mass m on one side and mass $(m+M)$ on the other.

$\therefore$ From principle of momentum, we have

$$Mv=\{m+(m+M)\}\ V \quad \text{or} \quad V=Mv/(M+2m). \qquad ...(i)$$

If f be the retardation to the motion of the system, then

$$f=\text{“}\left(\frac{m_1-m_2}{m_1+m_2}\right)g\text{”}=\frac{(M+m)-m}{(M+m)+m}g=\frac{Mg}{(M+2m)}. \qquad ...(ii)$$

If t be the time taken by the platform in coming to its original, then we have

$$t=2t_1,\ \text{where}\ 0=V-ft_1,\ \text{from}\ ‘v=u+ft’$$

$$=\frac{2V}{f}=2.\frac{Mv}{(M+2m)}.\frac{(M+2m)}{Mg}=\frac{2v}{g}$$

$=$time taken by the man in coming to his original position.

Hence when the man meets the platform again both are in their original position.

Also from "$v^2=u^2+2fs$" we have the distance moved by the system before coming to rest, then

$$0=V^2-2fs \text{ or } s=\frac{V^2}{2f}=\frac{M^2v^2}{2(M+2m)^2}\cdot\frac{(M+2m)}{Mg}, \text{ from (i) and (ii)}$$

or $s=Mv^2/[2g(M+2m)]$.

***Ex. 11. A mass m is attached to one end of an elastic string of length a, the other end being fixed to a peg. Initially m is held near the peg and projected with a velocity v vertically downwards. If the particle moves all along in a straight line, the depth h below the peg of m when it is first at rest is given by the equation $v^2+2gh=\lambda(h-a)^2/ma$, where λ is the modulus of elasticity of the string.**

Solution. O is the fixed peg and OA is the natural length of the string.

As the mass m falls from O to A, there will be no tension in the string and only the weight of the particle (of mass m) will do work in moving downwards. Let u be the velocity of this mass at A.



(Fig. 16)

Then from "change in K. Energy = work done", we get

$$\tfrac{1}{2}m(u^2-v^2)=mg\,a. \quad ...(i)$$

After the natural length is covered, the weight will act downwards and the tension of the elastic string will act upwards.

∴ The work will be done against tension.

∴ From "change in kinetic energy = work done", we get

$$\tfrac{1}{2}m(0-u^2)=mg(h-a)-\frac{(h-a)}{2}\left[\frac{\lambda(h-a)}{a}+0\right]$$

or $$-\tfrac{1}{2}mu^2=mg(h-a)-\frac{\lambda(h-a)^2}{2a}. \quad ...(ii)$$

Adding (i) and (ii), we get $-\tfrac{1}{2}mv^2=mgh-\dfrac{\lambda(h-a)^2}{2a}$

or $$mv^2+2mgh=\frac{\lambda}{a}(h-a)^2 \text{ or } v^2+2gh=\frac{\lambda}{am}(h-a)^2.$$

Hence proved.

****Ex. 12. If a particle falls under gravity, prove that the sum of its kinetic and potential energies is constant at any instant.**

(*Kanpur* 85)

Solution. Refer Fig. 16 Page 46.

Let the particle fall from the point O which is at a height h above the ground to B on the ground. Let A be a position of the particle at a depth a below O.

At O, the point of start, we have

Kinetic energy = "$\frac{1}{2}mv^2$" = 0, $\because$ $v=0$ at O

Potential energy = work done if the particle be allowed to fall from O to B

$= mgh$, where m is the mass of the particle.

$\therefore$ Sum of the energies at $O = 0 + mgh = mgh$. ...(i)

At A, if v be the velocity of the particle, then from "$v^2=u^2+2fs$" we have $v^2 = 0 + 2ga$.

$\therefore$ Kinetic energy at A = "$\frac{1}{2}mv^2$" $=\frac{1}{2}m\,(2ga)=mga$.

Potential energy at A = work done if the particle be allowed to fall from A to B

$= mg\,(h-a)$.

$\therefore$ Sum of energies at $A = mga + mg\,(h-a) = mgh$ ...(ii)

At B if, V be the velocity of the particle, the from "$v^2=u^2+2fs$" we have $V^2=0+2gh$.

$\therefore$ Kinetic energy at B = "$\frac{1}{2}mv^2$" $=\frac{1}{2}m\,(2gh)=mgh$.

Potential energy at $B=0$, since the particle has already reached its standard position (See § 5 Pages 17-18 of this Chapter).

$\therefore$ Sum of energies at $B = mgh + 0 = mgh$. ...(iii)

From (i), (ii) and (iii), we observe that the sum of energies is constant at any instant.

****Ex. 13. Prove that the mean kinetic energy of a particle of mass m moving under a constant acceleration, in any interval fo time, is $\frac{1}{6}m\,(u_1^2+u_1u_2+u_2^2)$, where u_1 and u_2 are the initial and final velocities.**

Solution. Let after time t, the velocity of the particle be v. Let f be the constant acceleration and T be the total time taken.

Then from "$v=u+ft$", we have

$v=u_1+ft$ .. (i) and $u_2=u_1+fT$ ...(ii)

Now mean K. Energy $=\dfrac{1}{T}\displaystyle\int_{t=0}^{T}(\tfrac{1}{2}mv^2)\,dt$ (Note)

$$=\frac{m}{2T}\int_0^T (u_1+ft)^2\,dt, \text{ from (i)}$$

$$=\frac{m}{2T}\left[\frac{(u_1+ft)^3}{3f}\right]_0^T=\frac{m}{6fT}[(u_1+fT)^3-u_1^3]$$

$$=\frac{m}{6(u_2-u_1)}[u_2^3-u_1^3], \text{ from (ii)}$$

and putting $fT=u_2-u_1$

$$=\tfrac{1}{6}m(u_2^2+u_2u_1+u_1^2). \quad \text{Hence proved.}$$

EXERCISES ON IMPUISE, WORK, ENERGY

Ex. 1. Find the energy of an elastic extended string.

Ex. 2. A heavy chain of length l is held by its upper end, so that its lower end is at a height l above a horizontal plane, if the upper end is let go, prove that, at the instant when half the chain is coiled upon the plane, the pressure on the plane is to the weight of the chain in the ratio 7 : 2.

[**Hint.** See Ex. 5 Page 40].

§ **Relative Motion.** Everything is in motion or in rest only *relative to the earth.* But earth itself is in motion relative to the sun. Hence we can find only the relative motion of any object and not its absolute motion.

When the velocity of a moving body relative to another moving body is obtained, it is said to be relative velocity.

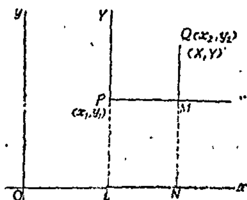
§ 2. **Relative velocity and acceleration.**

Consider the relative velocity of two points P and Q moving over a plane.

With reference to the axes of coordinates Ox and Oy, let the coordinates of P and Q be (x_1, y_1) and (x_2, y_2) respectively.

Referred to these axes the component velocities of P and Q are $\dot{x}_1$, $\dot{y}_1$ and $\dot{x}_2$, $\dot{y}_2$ respectively.

Pass through P two straight lines PX and PY parallel to Ox and Oy respectively.



(Fig. 1)

With reference to P (as if it is fixed) the component velocities of Q are dX/dt and dY/dt.

Let us suppose Ox and Oy are fixed but P is a moving point. Hence Px and Py are not fixed. From P and Q draw PL and QN perpendiculars to Ox.

Then $OL = x_1$, $PL = y_1$, $PM = X = x_2 - x_1$ and $QM = Y = y_2 - y_1$

Differentiating $(dX/dt) = \dot{x}_2 - \dot{x}_1$, and $(dY/dt) = \dot{y}_2 - \dot{y}_1$

(dX/dt) and (dY/dt) are relative component velocities of Q referred to point P, $\dot{x}_2$, $\dot{y}_2$ are the absolute velocities of the point Q referred to the point O as Ox and Oy are taken to be fixed.

Similarly $\dot{x}_1$ and $\dot{y}_1$ are the absolute velocities of the point P referred to the point O.

From above we find $\dfrac{dX}{dt} = \dot{x}_2 - \dot{x}_1$ and $\dfrac{dY}{dt} = \dot{y}_2 - \dot{y}_1$

which show that the velocity components of the point Q relative to P are obtained by adding to the actual velocity components of Q the actual velocity components of P which are equal but opposite to

those of Q.

$\therefore$ **velocity of Q relative to P is the resultant of the absolute velocity of Q and reversed velocity of P.**

Also $\frac{d^2X}{dt^2}=\ddot{x}_2-\ddot{x}_1$ and $\frac{d^2Y}{dt^2}=\ddot{y}_2-\ddot{y}_1$

Here $\frac{d^2X}{dt^2}$ and $\frac{d^2Y}{dt^2}$ are the accelerations of Q, relative to P and as before we conclude that :—

Acceleration of Q relative to P is the resultant of the *absolute* velocity of Q and reversed velocity of P.

§ 3. **Relative Velocity** (*Vector approach*).

In the previous chapters we have so far considered the motion of a particle referred to some origin which we supposed as fixed in space. Now it may so happen that this origin is not fixed but moving in space.

Let **r** be the position vector of a moving point P at time t referred to O as origin.

Let **R** be the position vector of the point O' at time t referred to O as origin.

Now if **r'** be the position vector of P at time t referred to O' as origin, then we have (see fig. 2) $\mathbf{r}=\mathbf{R}+\mathbf{r}'$ ...(i)

If **V** be the velocity vector of O' and **v** that of P relative to O, then $\mathbf{V}=\frac{d\mathbf{R}}{dt}$ and $\mathbf{v}=\frac{d\mathbf{r}}{dt}$ ...(ii)

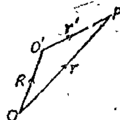


Fig. 2)

Let $\mathbf{v}'=\frac{d\mathbf{r}'}{dt}$, then **v'** is called the velocity vector of P relative to O'.

Now $\mathbf{v}'=\frac{d\mathbf{r}'}{dt}=\frac{d}{dt}(\mathbf{r}-\mathbf{R})$, from (i)

$=\frac{d\mathbf{r}}{dt}-\frac{d\mathbf{R}}{dt}=\mathbf{v}-\mathbf{V}$, from (ii)

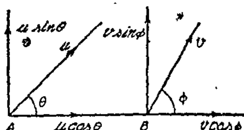
i.e. $\mathbf{v}'=\mathbf{v}-\mathbf{V}$ ...(iii)

which shows that the velocity of P relative to O' is obtained by adding the *reversed* velocity of O' to the *absolute* velocity of P. Hence by absolute velocity of P is meant the velocity of P referred to some fixed origin.

Hence we conclude that if **u** and **v** be the velocity vectors of two moving points P and Q, then the velocity vector of P relative to Q is $\mathbf{u}-\mathbf{v}$ and that of Q relative to P is $\mathbf{v}-\mathbf{u}$.

§ 4. Relative angular velocity.

At any time t, let A and B be the positions of two moving points.



(Fig. 3)

Let their velocities u and v make angles θ and ϕ with the line AB.

Reduce A to rest by applying a velocity $-u$ to each of the points A and B *i.e.* by applying velocities $-u\cos\theta$ and $-u\sin\theta$ along and perpendicular to AB to each of the points A and B.

Then the components of velocity of B will be $(v\cos\phi-u\cos\theta)$ and $(v\sin\phi-u\sin\theta)$ in the sense of AB and perpendicular to AB.

$\therefore$ **Angular velocity of B relative to A** $=\dfrac{v\sin\phi-u\sin\theta}{AB}$,

$(v\cos\phi-u\cos\theta)$ having no effect in rotating AB about A.

Solved Examples on Relative Motion.

***Ex. 1. Wind is blowing in the direction of the railway track. Two trains are moving with the same speed in opposite directions, the steam track of one of the trains is twice that of the other. Show that the speed of each train is three times that of the wind.**

Sol. Let u and v be the velocities of the train and wind respectively.

The relative velocity of the train moving in the direction of the wind $=u-v$.

And the relative velocity of the other train which is moving in the direction opposite to that of the wind $=u+v$.

$\therefore$ The lengths of the steam tracks of the above trains in unit time are $(u-v).1$ and $(u+v).1$ respectively.

$\therefore$ According to the given problem, we have

$$\frac{(u-v).1}{(u+v).1}=\tfrac{1}{2}$$

or $\quad 2(u-v)=u+v$ or $u=3v$

i.e. the speed of the train is three times that of wind.

***Ex. 2.** **Two trains are moving on parallel tracks in the same direction with velocities 40 km./hr. and 30 km./hr. Find the velocity of the second train relative to the first.**

Sol. The required velocity of the second train relative to the first=(the velocity of second)−(the velocity of first)

$=30-40=-10$ km./hr **Ans.**

Ex. 3. **A car driver is moving with a velocity of 36 km./hr. towards east and a cyclist is moving with a velocity of 4 km./hr. towards west. Find the velocity with which the cyclist appears to move to the car driver.**

Sol. Velocity of cyclist in the western direction=4 km./hr.

velocity of car-driver in the western direction=−36 km./hr. (Note)

$\therefore$ Velocity of cyclist relative to the car-driver

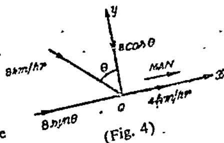
=(velocity of cyclist in the western direction)

−(velocity of car-driver in the same *i.e.* western direction)

$=4-(-36)=40$ km./hr.

$\therefore$ The cyclist appears to move with a velocity of 40 km./hr. to the car-driver.

***Ex. 4.** **To a man walking at the rate of 4 km./hr. rain appears to fall vertically. If its actual velocity is 8 km./hr. find the actual direction (by vector method).**



(Fig. 4)

Sol. Let the man be walking along Ox and let the actual direction of rain make an angle θ with yO, where yO is perpendicular to Ox. Let **i** and **j** be the unit vectors along Ox and Oy respectively.

Then the velocity of the man=4**i** and the velocity of the rain=$(8 \sin \theta)$ **i**−$(8 \cos \theta)$ **j**.

$\therefore$ Velocity of rain relative to the man

$=[(8 \sin \theta) \mathbf{i}-(8 \cos \theta) \mathbf{j}]-4\mathbf{i}$

$=(8 \sin \theta-4) \mathbf{i}-(8 \cos \theta) \mathbf{j}$

But it is given that the rain appears to the man to fall verti cally so we must have $8 \sin \theta-4=0$ or $\sin \theta=\frac{1}{2}$ or $\theta=30°$. **Ans.**

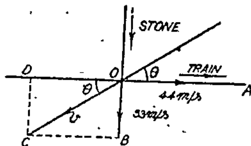
***Ex. 5.** **A train is moving at a speed of 44 metres per second. A stone strikes it at right angles with a speed of 33 metres/second. Find the magnitude and direction of the velocity with which it would appear to strike to the passengers sitting in the train.** **Ans.**

Sol. Let the velocities of the train and the stone in magnitude and direction be represented by OA and OB respectively then we have $OA=44$, $OB=33$.

Produce AO to D such that $OD=OA=44$.

$\therefore$ OD represents the velocity of the train in the reversed direction.

Complete the parallelogram (actually it is rectangle) $OBCD$. Then the velocity of the stone relative to the train will be represented in magnitude and direction by the diagonal OC.



(Fig. 5)

i. e. $v=OC=\sqrt{(OB^2+OD^2)}=\sqrt{[(33)^2+(44)^2]}=11\sqrt{(3^2+4^2)}=55$

Also if $\angle COD=\theta$, then $\tan\theta=\dfrac{CD}{OD}=\dfrac{OB}{OD}=\dfrac{33}{44}=\dfrac{3}{4}$

$\therefore$ $\tan\angle AOC=\tan(180°-\theta)$, see figure above

$=-\tan\theta=-(3/4)$

or $\angle AOC=\tan^{-1}(-3/4)$

$\therefore$ The stone will appear, to the passengers sitting in the train, to move with a velocity 55 m/second in a direction making an angle $\tan^{-1}(-3/4)$ with the direction of motion of the train.

Ans.

Ex. 6. The drops of water falling from the ceiling of the channel appears to make an angle $\tan^{-1}(1/2)$ with the horizodtol clear the window. of the train and the velocity of the drops is 24 decimetres per second. Find the velocity of the train, neglecting air-resistance.

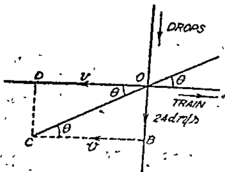
Sol. Let the velocity of the train be v decimetres/second.

Let the actual velocity of the drops of water and the reversed velocity of the train be represented in magnitude and direction by OB and OD, then we have $OB=24$, $OD=v$.

Complete the parallelogram $ODGB$, then the velocity of the drops relative to the train will be represented in magnitude and direction by the diagonal OC.

If the apparent velocity of the drops makes an angle θ with the horizontal, then $\tan\theta=\frac{1}{2}$ (given) or $\cot\theta=2$

Now in $\triangle OBC$, we find that



$$\frac{BC}{\sin(90°-\theta)}=\frac{OB}{\sin\theta}$$ (see figure above)

(Fig. 6)

or $\frac{v}{\cos\theta}=\frac{24}{\sin\theta}$ or $v=24\cot\theta=24\ (2)=48$.

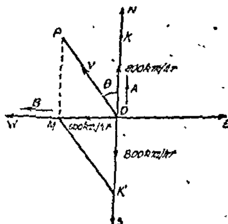
$\therefore$ The required velocity of the train$=48$ decimetres/second. **Ans.**

Ex. 7. To an observer on the ground, the aeroplane A appears to fly at a constant height towards north with a speed 800 km./hr. To the pilot of this plane another plane B appears to fly at the same height towards west with a speed 600 km./hr. To the observer on ground find in what direction and with what speed the plane B will appear to fly ? *(Roorkee Entrance)*

Sol. Let O be the position of the observer on the ground. Let to the pilot of the plane A, when it is flying exactly over observer O towards north with a speed 800 km./hr., the second plane B appears to fly towards west with a speed of 600 km./hr.

The velocity 600 km./hr. of the plane B is relative to th plane A, so it is the resultant of the *reversed velocity of A* and the *absolute velocity of B.*

Let OK' and OM represent in magnitude and direction the reversed velocity of the plane A and the absolute velocity of B.



(Fig. 7)

Let V be the absolute velocity of the plane B, which is represented in magnitude and direction by OP (see Fig. 7 above).

Join MP and MK' to complete the parallelogram $PMK'O$.

Let $\angle POK=\theta$, then $\angle POM=90°-\theta$

$\therefore$ In $\triangle POM$, $\tan(90°-\theta)=\frac{PM}{OM}=\frac{800}{600}=\frac{4}{3}$ (Note)

or $\cot\theta=4/3$ or $\tan\theta=3/4$ or $\theta=\tan^{-1}(3/4)$. ...(i)

Again in $\triangle POM$, we find $\sin(90°-\theta)=\frac{PM}{OP}$ or $\cos\theta=\frac{800}{V}$

or $V=800\sec\theta$, where $\tan\theta=3/4$ which gives $\sec\theta=5/4$

or $V=800\ (5/4)=1000$ km./hr.

Hence to the observer at O on the ground, the plane B will *appear to fly at a speed of 1000 km./hr. in a direction making an* angle $\tan^{-1}(3/4)$ towards west with the north. **Ans.**

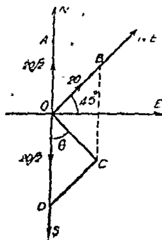
Ex. 8. From a light house an observer observes two ships A and B, the ship A proceeding towards east at a speed of 20 $\sqrt{2}$ km./hr. and the ship B proceeding towards north-east at a speed of 20 km./hr.

Find in which direction and with what speed the ship B would appear to move to a man standing on the deck of the ship A.

(Roorkee Entrance)

Sol. Let O be the position of the observer in the light-house.

Let the velocities of the ships A and B be represented in magnitude and direction by OA and OB [illegible] $\sqrt{2}$, $OB=20$. [illegible] $=OA=20\sqrt{2}$. [illegible]rsed velocity of the ship A. Complete the parallelogram $ODCB$. Then the diagonal OC represents in magnitude and direction the velocity V of B relative to A. Also $\angle BOD=45°+90°$ or $\angle BOD=135°$.



(Fig. 8)

Then V=resultant of velocities represented by OB and OD

$$=\sqrt{[(OB)^2+(OD)^2+2(OB)\ (OD)\cos 135°]} \quad \text{(Note)}$$

$$=\sqrt{[(20)^2+(20\sqrt{2})^2+2(20)\ (20\sqrt{2})\ (-1/\sqrt{2})]}$$

$$=(20)\sqrt{[1+2-2]}=20$$

Also if $\angle COD=\theta$, then $\tan\theta=\dfrac{20\sin 135°}{20\sqrt{2}+20\cos 135°}$ (Note)

or $\tan\theta=\dfrac{20\ (1/\sqrt{2})}{20\sqrt{2}+20(-1/\sqrt{2})}=\dfrac{20}{40-20}=1=\tan 45°$

or $\theta=45°$ *i.e.* OC is in the south-east direction.

Hence to the man standing on the deck of A, the ship B would appear to move with velocity $V=20$ km./hr. in the south-east direction. Ans.

Ex. 9 (a). A ship is sailing at a speed of 20 km./hr. in the north-east direction and to the man sitting in the ship the wind appears to blow at a speed of 32 km./hr. from the north. Find the absolute velocity (upto one decimal place) of the wind and also the angle (in degrees) which this absolute velocity makes with the north.

(Roorkee Entrance)

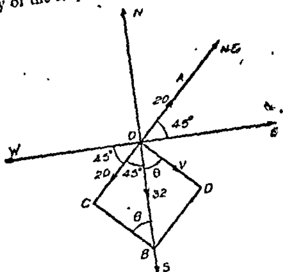
Sol. Let OA and OB represent in magnitude and direction the velocities of the ship moving in the north-east direction and the relative velocity of the wind from the north relative to the man O, on the ship.

The velocity of the ship is given to be 20 km./hr. and the relative velocity of the wind to be 32 km./hr.

Now the relative velocity of the wind is the resultant of the reversed velocity of the ship and the absolute velocity of the wind.

Let OC and OD represent in magnitude and direction the

reversed velocity of the ship and the absolute velocity V of the wind respectively.



(Fig. 9)

Then $OC=OA=20$; $OD=V$. Let OD make an angle θ with OB. Complete the parallelogram $BDAC$. Then $CB=OD=V$ and $DB=OC=20$.

Now in $\triangle OBC$, $\cos 45° = \dfrac{OB^2+OC^2-BC^2}{2\,OB.OC} = \dfrac{(32)^2+(20)^2-V^2}{2\,(32)(20)}$

or $\dfrac{1}{\sqrt{2}} = \dfrac{1024+400-V^2}{1280}$ or $\dfrac{1280}{\sqrt{2}}=1424-V^2$

or $V^2=1424-640\sqrt{2}$

or $V^2=1424-640\,(1{\cdot}414)=1424-904{\cdot}96=519{\cdot}04$ or $V=22{\cdot}8$

Also $\dfrac{OC}{\sin\theta}=\dfrac{BC}{\sin 45°}$, in $\triangle OBC$

or $\dfrac{20}{\sin\theta}=\dfrac{V}{(1/\sqrt{2})}$

or $\sin\theta=\dfrac{20}{V\sqrt{2}}=\dfrac{20}{(22{\cdot}8)\sqrt{2}}=\dfrac{20\sqrt{2}}{(22{\cdot}8)\times 2}=\dfrac{5\sqrt{2}}{11{\cdot}4}$

or $\sin\theta=0{\cdot}6201=\sin 38{\cdot}5°$ or $\theta=38{\cdot}5°$ approx.

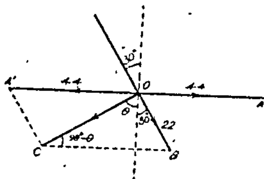
Hence the absolute velocity of the wind is 22·8 km./hr. making an angle of 38·5° with the north towards the west. **Ans.**

Ex. 9 (b). A ship is sailing at a speed of 10 miles per hour in the north-east direction and to the man sitting in the ship the wind appears to blow at a speed of $10\sqrt{2}$ miles per hour from the north. Find the absolute velocity of the wind both in magnitude and direction.

Sol. Do as Ex. 9 (a) above. **Ans.** 10 miles/hr. making an angle of 45° with the north towards the west.

Ex. 10. A train is moving on horizontal rails at speed of 45 m./sec. and due to the wind blowing in the direction of the train, the rain is falling at a speed of 22 m/sec. making an angle of 30° with the vertical. What is the apparent direction of the rain to any man sitting in the train ?

Sol. Let O be the position of the observer in the train. Let OA and OB represent in magnitude and direction the velocities of the train and the rain.



(Fig. 10)

Produce AO to A', such that $OA'=OA=44$

Then OA' represents the reversed velocity of the train in magnitude and direction. Complete the parallelogram $OACB$.

Then the diagonal OC represents the resultant of reversed velocity of the train and absolute velocity of the rain *i.e.* OL represents the apparent velocity of the rain.

In $\triangle BOC$, we have

$$\frac{BC}{\sin \angle BOC}=\frac{OB}{\sin \angle BCO} \quad \text{or} \quad \frac{44}{\sin(\theta+30°)}=\frac{22}{\sin(90°-\theta)}$$

(see Fig. 10 above)

or $2\cos\theta=\sin(\theta+30°)=\sin\theta\cos 30°+\cos\theta\sin 30°$

or $2\cos\theta=(\sin\theta)(\sqrt{3}/2)+(\cos\theta)(1/2)$

or $3\cos\theta=\sqrt{3}\sin\theta$ or $\tan\theta=\sqrt{3}=\tan 60°$

or $\theta=60°$ which gives $\angle BOC=\theta+30°=60°+30°=90°$

Hence the apparent direction of the rain is at right angles to the direction of its absolute velocity.

Ex. 11. A fighter plane is flying towards east over a city A at a speed of 100 km/hr. After six minutes another fighter plane starts flying in the north-east direction from a point B, situated at 40 km. south of A. With what speed the second plane should fly so that it may catch the first plane ?

Sol. Let the second plane start from B with a velocity V

km./hr. in the north-east direction and let it catch the first plane at *C*.

Then $\angle ABC = 45°$, as *BC* is in the north east direction and *AC* in the east.

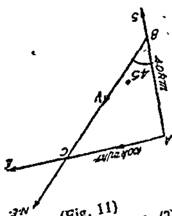
$\therefore \angle ACB = 45°$, also.

So in ΔABC we have $AC = AB = 40$ km.

and so $BC = \sqrt{(AB^2 + CA^2)} = \sqrt{[(40)^2 + (40)^2]} = 40\sqrt{2}$ km.;

Also time taken by the first plane in moving from *A* to *C*

$$= \frac{\text{distance}}{\text{speed}} = \frac{AC}{100} = \frac{40}{100} = \frac{2}{5} \text{ hr}.$$



(Fig. 11)

Again it is given that the second plane starts six minutes later, so the time taken by the second plane in moving from *B* to $C = (2/5)$ hr. $-$ 6 min

$$= \left(\frac{2}{5} - \frac{6}{60}\right) \text{hr.} = \frac{3}{10} \text{hr}.$$

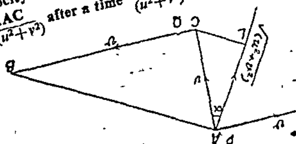
$\therefore$ The distance *BC* moved by the second plane with a speed *V* km/hr. is given by $BC = V \times (3/10)$

or $40\sqrt{2} = V\,(3/10)$

or $V = \dfrac{40\sqrt{2} \times 10}{3} = \dfrac{400}{3}\sqrt{2}$ km/hr.

Hence the required speed of the second plane is $(400\sqrt{2})/3$ km./hr.

Ex. 12. ABC is a triangle right-angled at C, a particle P starts from A and moves along AD with uniform velocity u, a second particle Q starts from C at the same instant and moves along CB with uniform velocity v; show that the shortest distance between P and Q will be $\dfrac{v.AC}{\sqrt{(u^2+v^2)}}$ after a time $\dfrac{u.AC}{(u^2+v^2)}$.



(Fin. 12)

Sol. Reduce *Q* to rest by applying a velocity *v* at *A* in sense opposite to *BC* as shown in the figure.

Resultant velocity $\sqrt{(u^2+v^2)}$ of P will be in the direction AL making an angle α with AC.

$\therefore$ $\tan\alpha=(v/u)$ whence $\sin\alpha=v/\sqrt{(u^2+v^2)}$, $\cos\alpha=u/\sqrt{(u^2+v^2)}$.

From C draw straight line CL perpendicular to the resultant velocity. This distance CL would be the required shortest distance.

$\therefore$ $CL=AC\sin\alpha=AC\, v/\sqrt{(u^2+v^2)}$, from (i)

and $AL=AC.\cos\alpha=AC.u/\sqrt{(u^2+v^2)}$, from (i)

$\therefore$ Time taken to cover the distance AL with resulting velocity $\sqrt{(u^2+v^2)}$

$$=\frac{AL}{\sqrt{(u^2+v^2)}}=\frac{ACu/\sqrt{(u^2+v^2)}}{\sqrt{(u^2+v^2)}}=\frac{u\,AC}{(u^2+v^2)}.$$ Hence proved.

Ex. 13. ***Two particles A and B are moving in a plane in such a manner that the line joining A and B is of constant length a and the velocities of A and B are in directions which make angles α and β respectively with AB. Prove that angular velocity of AB is***

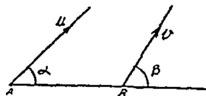
$\{u(\sin(\alpha-\beta))/(a\cos\beta)$, u being the velocity of A.

Sol. Let v be the velocity of B

$\therefore$ The angular velocity of AB.

i.e. the angular velocity of A relative to B.

$$=\frac{u\sin\alpha-v\sin\beta}{AB}\quad\ldots(i)$$



(See § 4 Page 2 of this chapter) (Fig. 13)

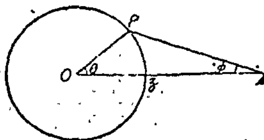
Also as the line joining A and B is of constant length throughout the motion, so in the direction AB the velocity of A relative to B is zero *i.e.* $u\cos\alpha-v\cos\beta=0$ or $v=(u\cos\alpha)/\cos\beta$.

Substituting this value of v in (i), we have the required angular velocity of AB

$$=\frac{u\sin\alpha-(u\cos\alpha/\cos\beta)\sin\beta}{AB}=\frac{u\sin(\alpha-\beta)}{a\cos\beta},$$

since $AB=a$ (given). Hence proved.

Ex. 14. **A crank OP rotates about a fixed centre O while the**



(Fig. 14)

end A of the connecting rod PA is constrained to move along OA. If the angles POA and PAO are θ and ϕ, prove that when the angular velocity of P is ω, the linear velocity of A is

$$\frac{-OP \sin(\theta+\phi)}{\cos\phi} \cdot \omega$$

Sol. Angular velocity of P about O

$= d\theta/dt = \omega$ (given) ...(i)

Also from ▲ OPN we have $\frac{OP}{\sin\phi} = \frac{PA}{\sin\theta}$

or $OP.\sin\theta = PA\sin\phi$ ...(ii)

Differentiating, $OP\cos\theta\frac{d\theta}{dt} = PA.\cos\phi.\frac{d\phi}{dt}$ ...(iii)

Let $OA = z$, then from the figure we find that

$z = OP.\cos\theta + PA.\cos\phi$

∴ Linear velocity of A

$= \frac{dz}{dt} = -OP.\sin\theta\frac{d\theta}{dt} - PA.\sin\phi\frac{d\phi}{dt}$, on differentiating (iii)

$= -OP.\sin\theta\frac{d\theta}{dt} - OP\cos\theta\frac{d\theta}{dt}.\frac{\sin\phi}{\cos\phi}$, from (ii)

$= -OP.\frac{(\sin\theta\cos\phi + \cos\theta\sin\phi)}{\cos\phi}.\frac{d\theta}{dt}$

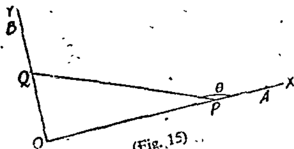
$= -OP\frac{\sin(\theta+\phi)}{\cos\phi}\omega$, from (i)

Hence proved.

Ex. 15. Two points are moving with uniform velocity u, v in perpendicular lines OX and OY, the motions being towards O. If initially, their distances from the origin are a and b respectively, calculate the angular velocity of the line joining them at the end of t seconds, and show that it is greatest when $t = (au+bv)/(u^2+v^2)$.

Sol. Let A and B be the initial positions of the points moving with uniform velocities u and v. Given $OA = a$ and $OB = b$.

Let after time t the particles be at P and Q, then we hav[e]



(Fig. 15)

$$AP = u.t \text{ and } BQ = v.t \quad ...(i)$$

Let $\angle APQ = \theta$, then $\tan OPQ = \tan(\pi - \theta) = OQ/OP$

or $$-\tan\theta = \frac{OB - BQ}{OA - AP} = \frac{b - vt}{a - ut}, \text{ from (i)}$$

or $$\theta = \tan^{-1}\{(vt - b)/(a - ut)\} \quad ...(ii)$$

$\therefore$ Angular velocity of $PQ = d\theta/dt$

$$= \frac{1}{1 + [(vt - b)/(a - ut)]^2} \cdot \frac{(a - ut)\, v - (vt - b)\,(-u)}{(a - ut)^2}, \text{ from (ii)}$$

$$= \frac{av - bu}{(a - ut)^2 + (vt - b)^2}.$$

This angular velocity will be greatest when the denominator $(a - ut)^2 + (vt - b)^2$ is least

Let $(a + ut)^2 + (vt - b)^2 = x$.

Then $dx/dt = -2u\,(a - ut) + 2v\,(vt - b) = 0$
gives $t = (au + bv)/(u^2 + v^2)$. Also $d^2x/dt^2 = 2u^2 + 2v^2 =$ positive.

$\therefore$ x is minimum *i.e.* the angular velocity of PQ is maximum when $t = (au + bv)/(u^2 + v^2)$.

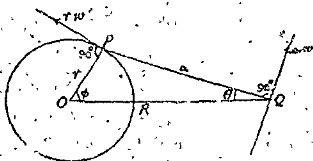
****Ex. 16. A point P describes a circle of radius r with angular velocity w' about its centre O. Another point Q moves so that PQ is of constant length a and is turning with angular velocity w in the plane of the first circle. Prove that angular velocity of OQ, when its length is R, is $[w\,(R^2 + a^2 - r^2) + w'\,(R^2 + r^2 - a^2)]/(2R^2)$,**

Sol. The linear velocity of P and Q are rw' and aw acting in the directions as shown in the figure

$\therefore$ Angular velocity of Q about O

$$= \frac{\text{velocity of } Q \text{ perpendicular to } OQ}{OQ}$$ ...See § 4 Page 3

$= (1/OQ)$ (resolved part of rw' perpendicular to OQ + resolved part of aw perpendicular to OQ).



(Fig. 16)

$= \dfrac{aw \sin (90°-\theta)+rw' \sin (90°-\phi)}{OQ}$ ...See figure 16 P. 13

$=(aw \cos \theta + rw' \cos \phi)/R$

$= \dfrac{1}{R}\left[aw\left(\dfrac{a^2+R^2-r^2}{2aR}\right)+rw'\left(\dfrac{r^2+R^2-a^2}{2rR}\right)\right],$

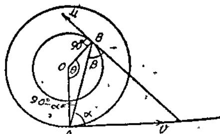
from "$\cos A=(b^2+c^2-a^2)/2bc$"

$=(1/2R^2)\,[w\,(a^2+R^2-r^2)+w'\,(r^2+R^2-a^2)]$. Hence proved.

Ex. 17. Two planets A and B revolve round the sun in circular orbits of radii a and b. If the angular velocity in a planetary orbits of radius r, is given by $\omega^2 r^3=\mu$; show that when the planets appear stationary one with respect to another, the line joining them subtends at the sun an angle given by $\cos^{-1}[\sqrt{(ab)}/\{a-\sqrt{(ab)}+b\}]$.

Sol. Let the velocities of the planets A and B be v and u making angles α and β with the line AB.

Directions of v and u are at right angles to OA and OB respectively, where O is the centre of the circle.



(Fig. 17)

Let angle $AOB=\theta$, where O is the sun. Then from $\triangle OBA$, it is evident that $\theta=\alpha+\beta$ ...(i)

Also from $\triangle AOB$, we find that $\dfrac{OA}{\sin \angle ABO}=\dfrac{OB}{\sin \angle OAB}$.

or $\dfrac{a}{\sin (90°-\beta)}=\dfrac{b}{\sin (90°-\alpha)}$ or $\dfrac{a}{\cos \beta}=\dfrac{b}{\cos \alpha}$

or $a \cos \alpha = b \cos \beta = b \cos (\theta-\alpha)$, from (i)

$=b\,(\cos \theta \cos \alpha + \sin \theta \sin \alpha)$

or $(a-b \cos \theta) \cos \alpha = b \sin \theta \sin \alpha$ ...(ii)

Also we know for a particle moving in a circle of radius r, the linear velocity v and angular velocity ω are connected by the relation $v=r\omega$

But $\omega^2 r^3=\mu$ (given) or $\omega^2 r^2=\mu/r=v^2$, $\therefore\ v=\sqrt{(\mu/r)}$ ...(iii)

Hence for the planet A and B we have

$v=\sqrt{(\mu/a)}$ and $u=\sqrt{(\mu/b)}$

If the planet A appears stationary with respect to the planet B, then the angular velocity of AB i.e. velocity of A relative to B is zero

i.e. $\dfrac{v \sin \alpha-(-u \sin \beta)}{AB}=0$, (Note) See § 4 Page 3 of this chapter the directions of $v \sin \alpha$ and $u \sin \beta$ being opposite.

or $v \sin \alpha + u \sin \beta = 0$ or $\sqrt{(\mu/a)} \sin \alpha + \sqrt{(\mu/b)} \sin \beta = 0$, from (iii)

or $\sqrt{b} \sin \alpha + \sqrt{a} \sin (\theta-\alpha)=0$, from (i)

or $\sqrt{b}\sin\alpha+\sqrt{a}(\sin\theta\cos\alpha-\cos\theta\sin\alpha)=0$

or $$\sqrt{a}\sin\theta\cos\alpha=(\sqrt{a}\cos\theta-\sqrt{b})\sin\alpha \quad \text{...(iv)}$$

Dividing (ii) by (iv) we get $\dfrac{a-b\cos\theta}{\sqrt{a}\sin\theta}=\dfrac{b\sin\theta}{(\sqrt{a}\cos\theta-\sqrt{b})}$

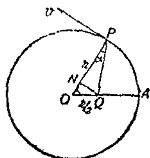
or $a\sqrt{a}\cos\theta-b\sqrt{a}\cos^2\theta-a\sqrt{b}+b\sqrt{b}\cos\theta=b\sqrt{a}\sin^2\theta$

or $(a\sqrt{a}+b\sqrt{b})\cos\theta=b\sqrt{a}(\cos^2\theta+\sin^2\theta)+a\sqrt{b}=b\sqrt{a}+a\sqrt{b}$
$=\sqrt{(ab)}(\sqrt{b}+\sqrt{a})$

or $\cos\theta=\dfrac{\sqrt{(ab)}(\sqrt{b}+\sqrt{a})}{(a\sqrt{a}+b\sqrt{b})}=\dfrac{\sqrt{(ab)}(\sqrt{b}+\sqrt{a})}{(\sqrt{a}+\sqrt{b})\{a-\sqrt{(ab)}+b\}}$ ...(Note)

or $\theta=\cos^{-1}\left[\dfrac{\sqrt{(ab)}}{a-\sqrt{(ab)}+b}\right]$. Hence proved.

Ex. 18. A point P describes a circle of radius r with uniform angular velocity ω. Show that the angular velocity of P about a point Q distant $\frac{1}{2}r$ from the centre O fluctuates between 2ω and $\frac{2}{3}\omega$.



(Fig. 18)

Sol. Let v be the linear velocity of P, which describes a circle of radius r with uniform angular velocity ω.

Then $v=r\omega$. ...(i)

Join OQ and produce it meeting the circle in A.

Let $\angle POQ=\theta$ and $\angle OPQ=\alpha$.

Now angular velocity at P about Q

$=\dfrac{\text{velocity of } P \text{ perpendicular to } PQ}{PQ}$

...See § 4 Pages 3 of this chapter

$=\dfrac{v\sin(90°-\alpha)}{PQ}=\dfrac{r\omega\cos\alpha}{PQ}$

Also from $\triangle POQ$, $\cos\theta=\dfrac{r^2+(\frac{1}{2}r)^2-PQ^2}{2r(\frac{1}{2}r)}$

or $r^2\cos\theta=(5/4)r^2-PQ^2$ or $PQ^2=(5/4)r^2-r^2\cos\theta$ ...(iii)

From Q draw QN perpendicular to OP.

Then $PN=OP-ON$

or $PQ.\cos\alpha=r-(\frac{1}{2}r)\cos\theta$ or $\cos\alpha=(2-\cos\theta)r/(2.PO)$ ...(iv)

∴ From (ii) angular velocity of P about R

$=\dfrac{r\omega(2-\cos\theta)r}{2PQ^2}$, from (iv)

$=\dfrac{(2-\cos\theta)\omega r^2}{2[(5/4)r^2-r^2\cos\theta]}$, from (iii)

$=\dfrac{2\omega(2-\cos\theta)}{(5-4\cos\theta)}=\dfrac{\omega}{2}\left[1+\dfrac{3}{5-4\cos\theta}\right]$.

This angular velocity is greatest when $(5-4\cos\theta)$ is least i.e. when $\cos\theta$ is greatest i.e. when $\theta=0$.

$\therefore$ Maximum angular velocity of P about $Q = \frac{1}{2}\omega\,(1+3) = 2\omega$

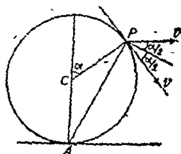
Also angular velocity of P about Q is least when $(5-4\cos\theta)$ is greatest *i.e.* when $\cos\theta$ is least *i.e.* when $\theta = \pi$.

$\therefore$ Minimum angular velocity of P about Q

$$= \tfrac{1}{2}\omega\,[1+(3/9)] = (2/3)\,\omega.$$

$\therefore$ Angular velocity of P about Q fluctuates between 2ω and $(2/3)\,\omega$.

Ex. 19. A wheel rolls a ... v. Show that the actual vel... point of contact of the wheel ...



(Fig. 9)

Sol. Actual velocity of P is the resultant of two component velocities (i) a velocity v along the tangent at P and (ii) a velocity v parallel to the road.

If α be the angle between these two equal components, then the velocity of P = resultant of these components

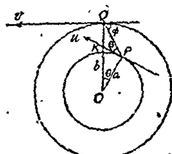
$$= 2v\cos\tfrac{1}{2}\alpha = v.\ \frac{2a\cos\frac{1}{2}\alpha}{a}$$

$$= v\,\frac{AP}{CP},\quad \because\ AP = 2a\cos\tfrac{1}{2}\alpha \text{ and } a \text{ is the radius of the wheel}$$

$\because$ These two components are equal, $\therefore$ their resultant ...

... scribing coplanar concen... in the same ... to P when ... ther relative to the other is zero when $\cos\theta = \dfrac{au+bv}{av+bu}$.

Sol. Let the directions of motion of P and Q make angles β and ϕ with PQ.



(Fig. 20)

Then from $\triangle POQ$, we find

$b = a\cos\theta + PQ\cos(90°-\phi)$

and $a = b\cos\theta - PQ\cos(90°-\beta)$

i.e. $b = a\cos\theta + PQ\sin\phi$ ...(i)

and $a = b\cos\theta - PQ\sin\beta$ ...(ii)

Eliminating PQ from (i) and (ii) we get $b\sin\beta + a\sin\phi$

$$= (a\sin\beta + b\sin\phi)\cos\theta$$

or $\cos\theta = \dfrac{a\sin\phi + b\sin\beta}{a\sin\beta + b\sin\phi}$

Angular velocity of Q relative to P

$= \dfrac{v \sin\phi - u \sin\beta}{PQ}$, See § 4 Page 3 of this chapter

If this angular velocity is zero, then

$v \sin\phi - u \sin\beta = 0$ or $\sin\phi = (u \sin\beta)/v$

Substituting this value of $\sin\phi$ in (iii), we get

$\cos\theta = \dfrac{a\{(u \sin\beta)/v\} + b \sin\beta}{a \sin\beta + b\{(u \sin\beta)/v\}} = \dfrac{au + bv}{av + bu}$ Hence proved.

Also from the figure we can prove from $\triangle PKQ$ that

$90° - \theta = \beta + (90° - \phi)$ or $\theta = \phi - \beta$ ...(iv)

Now velocity of Q relative to P

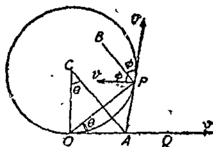
$= \sqrt{[(v \cos\phi - u \cos\beta)^2 + (v \sin\phi - u \sin\beta)^2]}$, See § 4 Page 3

$= \sqrt{[v^2 + u^2 - 2uv(\cos\phi \cos\beta + \sin\phi \sin\beta)]}$

$= \sqrt{[u^2 + v^2 - 2uv \cos(\phi - \beta)]} = \sqrt{(u^2 + v^2 - 2uv \cos\theta)}$, from (iv)

Ex. 21. Two points P and Q start simultaneously from the same point O, move uniformly in a circle and in a straight line which touches the circle, respectively, each with a speed v. Find the magnitude and direction of relative velocity of P with respect to Q at any instant, and also its respective path.

Sol. OA is a line touching the circle with centre C at O. Let the positions of the particle be at P and Q at time t, where P is a point on the circle and Q is a point on the line OA.



(Fig. 21)

Velocity of P relative to Q is the resultant of two velocities (i) absolute velocity v of P acting along the tangent PA at P and (ii) a velocity acting at P in the direction opposite to that in which the velocity v of Q is acting.

These two components being equal, their resultant will be acting along the bisector of the angle between their directions and if 2ϕ be the angle between these components then their resultant $= 2v \cos\phi$, acting along PB (see figure) and it can be proved easily that PB is parallel to CA. Also CA is perpendicular to OP. Let $\angle POA = \theta$, then $\angle OCA = \theta$ or $\angle OCP = 2\theta$. Hence arc $OP = a.\ 2\theta$, where a is the radius of the circle and $OP = 2.OC \sin\theta = 2a \sin\theta$.

Since velocities of P and Q are the same, so $OQ =$ arc $OP = 2a\theta$.

$\therefore$ Referred to Q as origin and OA and OC as coordinate axes we have the coordinates of Q as $(2a\theta, 0)$.

And the coordinates of P are $(OP \cos\theta, OP \sin\theta)$

or $(a \sin 2\theta, 2a \sin^2\theta)$, $\because$ $OP = 2a \sin\theta$

$\therefore$ If the coordinates of Q with respect to P are (x, y)

then $x = 2a\,\theta - a \sin 2\theta = a\,(2\theta - \sin 2\theta)$ ...(i)

and $y = 2a \sin^2 \theta = a\,(1 - \cos 2\theta)$ ...(ii)

Equations (i) and (ii) show that the path of Q relative to P is a cycloid.

Exercises on Relative Motion

Ex. 1. To a man walking at a rate of 2 km/hr. the rain appears to fall vertically. When he increases his speed to 4 kms/hr., it appears to meet him at an angle of 45°, find the real direction and speed of the rain.

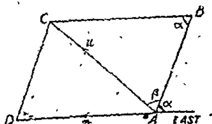
Ex. 2. To a man walking at 6 km/hr. along a road running ... hile to a cyclist ... s to come from ... of the wind.

Ex. 3. On board a ship steaming due east with velocity $2v$, the wind appears to blow from the sout-east, while on board ship steaming due north with velocity v it appears to blow from the west. Find the actual velocity and direction of the wind.

Ex. 4. A stream, of width (1/2) km., flows with a velocity of $1\frac{1}{2}$ km/hr. Find in what direction a man, who swims with a velocity of $2\frac{1}{2}$ km./hr., should start in order to cross the strerm perpendicularly.

MISCELLANEOUS SOLVED EXAMPLES

*Ex. 1. **An aeroplane flies in a horizontal line with velocity u relative to the air which has a velocity v due east. If the machine is moving in a direction making an angle α with east, find the direction in which it is pointed.**



Sol. Let A be the aeroplane's position. It is moving in the direction AB making angle α with the east. The air has a velocity due east. Reduce the air to rest by introducing a velocity v at A in the direction AD i.e. opposite, to that of velocity of air.

Let the velocity of the machine be represented by AB and the velocity equal and opposite to th... by AD.

Complete the paralle... D. Then ... will actually move in the directi... velocity u.

Let $\angle BAC = \beta$.

Then from ΔABC the ...

$$\frac{CB}{...} = \frac{AC}{...\alpha}$$

or $\sin \beta = ...$

∴ The direction in which the aeroplane actually moves makes with east an angle $=\alpha+\beta=\alpha+\sin^{-1}\{(v \sin \alpha)/u\}$.

**Ex. 2 Two particles start simultaneously from the same point and moves along two straight lines, one with uniform velocity u and other from rest with uniform acceleration f. Show that their relative velocity is least after a time $(u \cos \alpha)/f$ and that least relative velocity is $u \sin \alpha$, where α is the angle between the two lines.

Also show that their relative path is

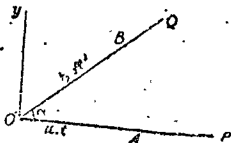
$$(x \sin \alpha-y \cos \alpha)^2=[(2u^2 \sin \alpha)/f]\, y$$

Sol. Let the two straight lines be OP and OQ inclined at an angle α to each other. The first particle moves along OP with a uniform velocity u and the other moves along OQ with uniform acceleration f and starting from rest.

Take OP and OY, through O perpendicular to OP, as coordinate axes.

Then if after time t, the positions of the particles be at A and B, then $OA=u.t$ and $OB=\frac{1}{2}ft^2$.

∴ The coordinates of A and B are $(ut, 0)$ and $(\frac{1}{2}ft^2 \cos \alpha,\ \frac{1}{2}ft^2 \sin \alpha)$ respectively.



Let the coordinates of second relative to first particle be (x, y),

Then $x=\frac{1}{2}ft^2 \cos \alpha-ut$ ...(i) ; $y=\frac{1}{2}ft^2 \sin \alpha$ ...(ii)

Also differentiating these we get

$\dot{x}=ft \cos \alpha-u$ ...(iii) ; $\dot{y}=ft \sin \alpha$ ...(iv)

∴ Velocity of second relative to first particle

$=\sqrt{(\dot{x}^2+\dot{y}^2)}=\sqrt{[(ft \cos \alpha-u)^2+(ft \sin \alpha)^2]}$, from (iii), (iv)

$=\sqrt{(u^2-2uft \cos \alpha+f^2 t^2)}=\sqrt{\{(u \cos \alpha-ft)^2+u^2 \sin^2 \alpha\}}$

The velocity is least when $u \cos \alpha-ft=0$

or $t=(u \cos \alpha)/f$ and then its value $=u \sin \alpha$.

Also eliminating t between (i) and (ii) we can get the equation of their relative path.

Multiplying (i) by $\sin \alpha$ and (ii) by $\cos \alpha$ and subtracting we get

$$x \sin \alpha-y \cos \alpha=-ut \sin \alpha$$

or $$(x \sin \alpha-y \cos \alpha)^2=(u^2 \sin^2 \alpha)\, t^2$$

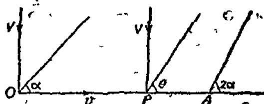
$$=(u^2 \sin^2 \alpha)\, [2y/f \sin \alpha] \text{ from (ii)}$$

or $$(x \sin \alpha-y \cos \alpha)^2=[(2u^2 \sin \alpha)/f]\, y.$$ Hence proved.

*Ex. 3. A man runs a race starting with velocity ... tion of the wind appears to him ... It then appears to veer ... about at right angles to his race. If the wind blows

course. and if T be the time of race ; show that the length of course is
$\{(vT \tan \alpha)/\alpha\} \log (2 \cos \alpha)$.

Sol. Let O be the point of start and A that of finish of the race. Actually the wind blows at right angles to the course OA but



(Fig. 24)

appears to the man making angles α and 2α with OA at O and A respectively. Let actual velocity of the wind be V which is uniform and always acts at right angles to OA.

At time t, let the man be at P, such that $OP=x$, say and let the wind appear to blow in a direction making an angle θ with OA at P.

Since the direction of the wind veers round uniformly

$\therefore\ d\theta/dt=\text{constant}=\omega$ (say) ...(i)

$\therefore$ Also in time T, *i.e.* in moving from O to A, change in the angle $=2\alpha-\alpha=\alpha$

$\therefore\ \omega=\alpha/T$ or from (i) $d\theta/dt=\alpha/T$ ...(ii)

Also at start, $\tan \alpha=V/v$ and at P, $\tan \theta=V/(dx/dt)$

or $\dfrac{dx}{dt}=V\cot\theta$ or $\dfrac{dx}{dt}\cdot\dfrac{d\theta}{dt}=V\cot\theta$ or $\dfrac{dx}{dt}\cdot\dfrac{\alpha}{T}=V\cot\theta$, from (ii)

or $\dfrac{dx}{d\theta}=\dfrac{TV\cot\theta}{\alpha}=\dfrac{T\cot\theta}{\alpha}\cdot v\tan\alpha$, from $\tan\alpha=\dfrac{V}{v}$

Integrating, $OA=\dfrac{Tv\tan\alpha}{\alpha}\displaystyle\int_{\theta=\alpha}^{2\alpha}\cot\theta\,d\theta=\dfrac{Tv\tan\alpha}{\alpha}\Big[\log\sin\theta\Big]_{\alpha}^{2\alpha}$

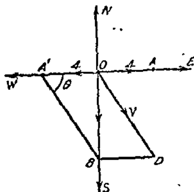
$$=\frac{Tv\tan\alpha}{\alpha}[\log\sin 2\alpha-\log\sin\alpha]$$

$$=\frac{Tv\tan\alpha}{\alpha}\log\left(\frac{\sin 2\alpha}{\sin\alpha}\right)=\frac{Tv\tan\alpha}{\alpha}\log(2\cos\alpha)$$

***Ex. 4. To the man walking towards east with a speed of 4 kms/hr. the wind appears to blow directly from the north. It appears to blow from north east when he doubles his speed. Find the absolute velocity and direction of the wind.** *(Roorkee Entrance)*

Sol. Let V km./hr. be the absolute velocity of the wind.

Case I. When the man is walking at a speed of 4 km./hr.



(Fig, 25)

Let OA and OB represent in magnitude and direction the velocities of the man in eastern direction and the relative velocity of the wind blowing in the northern direction respectively.

Let OA' and OD represent in magnitude and direction the reversed velocity of the man and the absolute velocity V of the wind. Complete the parallelogram $OA'BD$.

Let $\angle AOD=\theta$,

then $\angle OA'B=\theta$,

$\angle OBA'=90°-\theta$

$\therefore$ In $\triangle\ OBA'$, we have

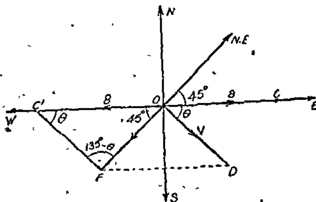
$$\frac{OA'}{\sin(90°-\theta)}=\frac{A'B}{\sin 90°} \text{ or } \frac{4}{\cos\theta}=\frac{V}{1}$$

or $V\cos\theta=4$ ...(i)

Case. II. When the man doubles his speed.

In this case the speed of the man is 8 km./hr.

Let OC, OC', OD and OF represent in magnitude and direction the velocity of the man, his reversed velocity, the absolute velocity of the wind and the relative velocity of the wind respectively. Then $\angle FOC'=45°$, as the wind appears to blow from north-east.



(Fig. 26)

Then from $\triangle\ OFC'$ we have

$$\frac{OC'}{\sin[180°-(45°+\theta)]}=\frac{C'F}{\sin 45°} \text{ or } \frac{8}{\sin(135°-\theta)}=\frac{V}{\sin 45°},$$

Let the bullet strike the coach at A with velocity V km./hr. at an angle α, where $\sin \alpha = 3/5$ (given).

Since the bullet moves in the direction of the diagonal AC within the coach, so this is the direction of the velocity of the bullet relative to the train.

Let AE represent in magnitude and direction the reversed velocity of the train and AF the absolute velocity V of the bullet.

Complete the parallelogram $EAFC$, then AC will represent in magnitude and direction the velocity of the bullet relative to the train.

Let $\angle BAC = \theta$, then $\tan \theta = \frac{24}{18} = \frac{4}{3}$

Now in $\triangle ACF$ we have $\frac{AF}{\sin \angle ACF} = \frac{CF}{\sin \angle CAF}$

i.e. $\frac{V}{\sin (180° - \theta)} = \frac{28}{\sin (\theta - \alpha)}$ or $\frac{V}{\sin \theta} = \frac{28}{\sin (\theta - \alpha)}$

or $V (\sin \theta \cos \alpha - \cos \theta \sin \alpha) = 28 \sin \theta$, where $\sin \alpha = 3/5$ (given) and $\tan \theta = 4/3$ (Proved above)

or $V [(4/5)(4/5) - (3/5)(3/5)] = 28$ or $V (7/25) = 28 (4/5)$

or $V = \frac{28 \times 4 \times 25}{7 \times 5} = 80$ km./hr. Hence proved.

Again if V' km./hr. be the relative velocity of the bullet, then AC represents V' and from $\triangle ACF$ we have

$$\frac{AC}{\sin \angle AFC} = \frac{AF}{\sin \angle ACF} \text{ or } \frac{V'}{\sin \alpha} = \frac{V}{\sin (180° - \theta)}$$

or $V' \sin \theta = V \sin \alpha$, or $V' (4/5) = 80 (3/5)$ or $V' = 60$ km./hr

Also in $\triangle ABC$, we have

$$AC = \sqrt{[(AB)^2 + (BC)^2]} = \sqrt{[(18)^2 + (24)^2]} = 30 \text{ dm.}$$

or $$AC = \frac{30}{10 \times 1000} \text{ km.} = \frac{3}{1000} \text{ km.}$$

$\therefore$ The time taken by the bullet in moving from A to C

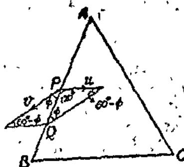
$$= \frac{AC}{V'} = \frac{3}{1000} \times \frac{1}{60} \text{ hr.} = \frac{3}{1000} \times \frac{1}{60} \times 60 \times 60 \text{ sec.}$$

$$= \frac{9}{50} \text{ second.}$$ Hence proved.

***Ex. 7. If an aviator flies round a triangular course, each side of which is c kms. long, while the wind blows at u kms. per hour parallel to one side, show that he takes $c\,[v^2 + \sqrt{(4v^2 - 3u^2)}]/[v^2 - u^2]$ hours to complete the circuit in either direction, v being his velocity relative to the air.**

Sol. ABC is the triangular course. The aviator flies along AB, BC and CA. Let the wind blow at u kms.-hour parallel to the side BC.

v is the velocity of the aviator relative to the air and is directed in such a way that the apparent velocity remains along the side AB. At time t, let the aviator be at B moving with a velocity along AB. Then this velocity V at P is the resultant of (i) a velocity u of the wind acting parallel to BC and (ii) a velocity v of the engine making an angle ϕ (say) with AB.



(Fig. 29)

Then we get $\dfrac{v}{\sin 120°}=\dfrac{V}{\sin (60°-\phi)}=\dfrac{u}{\sin \phi}$ ...(i)

From (i) we have $\dfrac{u}{\sin \phi}=\dfrac{v}{\sin 120}$ or $\sin \phi=\dfrac{u \sin 120°}{v}$

or $\sin \phi=\dfrac{u\sqrt{3}}{2v}$ $\therefore$ $\cos \phi=\sqrt{(1-\sin^2 \phi)}=\sqrt{(4v^2-3u^2)}/2v$

Also from (i) we get $\dfrac{V}{\sin (60°-\phi)}=\dfrac{v}{\sin 120°}$

or $V=\dfrac{2v}{\sqrt{3}} \sin (60°-\phi)=\dfrac{2v}{\sqrt{3}} [\sin 60° \cos \phi-\cos 60° \sin \phi]$

$=\dfrac{2v}{\sqrt{3}}\left[\dfrac{\sqrt{3}}{2}.\dfrac{\sqrt{(4v^2-3u^2)}}{2v}-\frac{1}{2}.\dfrac{u\sqrt{3}}{2v}\right]$

or $V=\frac{1}{2}\sqrt{(4v^2-3u^2)}-\frac{1}{2}u,$ ...(i))

which is the velocity with which the aviator moves along AB. This will also be the velocity with which the aviator will move along CA. Hence the time taking in moving from A to B or C to A is equal to c/V, since $AB=c=CA$.

Now when the aviator moves along BC, the wind is also blowing in the same direction and as such the velocity with which the aviator moves along BC is $(u+v)$.

$\therefore$ time taking in moving from B to $C=\dfrac{BC}{u+v}=\dfrac{c}{u+v}$

$\therefore$ The total time taken in completing the circuit

$=2\dfrac{c}{V}+\dfrac{c}{u+v}=\dfrac{4c}{[\sqrt{(4v^2-3u^2)}-u]}+\dfrac{c}{u+v}$; from (ii)

$=\dfrac{4c[\sqrt{(4v^2-3u^2)}+u]}{(4v^2-3u^2)-u^2}+\dfrac{c}{(u+v)}$, rationalising the denom.

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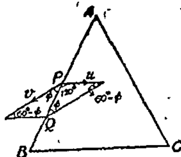
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(Fig. 29)

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or $\sin\phi=\dfrac{u\sqrt{3}}{2v}$ $\therefore$ $\cos\phi=\sqrt{(1-\sin^2\phi)}=\sqrt{(4v^2-3u^2)}/2v$

Also from (i) we get $\dfrac{V}{\sin(60°-\phi)}=\dfrac{v}{\sin 120°}$

or $V=\dfrac{2v}{\sqrt{3}}\sin(60°-\phi)=\dfrac{2v}{\sqrt{3}}[\sin 60°\cos\phi-\cos 60°\sin\phi]$

$=\dfrac{2v}{\sqrt{3}}\left[\dfrac{\sqrt{3}}{2}\,\dfrac{\sqrt{(4v^2-3u^2)}}{2v}-\dfrac{1}{2}.\dfrac{u\sqrt{3}}{2v}\right]$

or $V=\frac{1}{2}\sqrt{(4v^2-3u^2)}-\frac{1}{2}u$, ...(ii)

which is the velocity with which the aviator moves along AB. This will also be the velocity with which the aviator will move along CA. Hence the time taking in moving from A to B or C to A is equal to c/V, since $AB=c=CA$.

Now when the aviator moves along BC, the wind is also blowing in the same direction and as such the velocity with which the aviator moves along BC is $(u+v)$.

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$=2\dfrac{c}{V}+\dfrac{c}{u+v}=\dfrac{4c}{[\sqrt{(4v^2-3u^2)}-u]}+\dfrac{c}{u+v}$, from (ii)

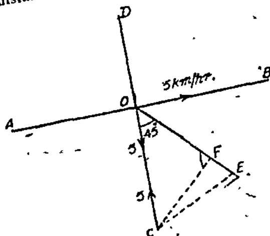
$=\dfrac{4c[\sqrt{(4v^2-3u^2)}+u]}{(4v^2-3u^2)-u^2}+\dfrac{c}{(u+v)}$, rationalising the denom.

$$=c\left[\frac{\sqrt{(4v^2-3u^2)}+u}{(v^2-u^2)}+\frac{1}{(u+v)}\right]$$

$$=c\left[\frac{\sqrt{(4v^2-3u^2)}+u+v-u)}{(v^2-u^2)}\right]=c\left[\frac{\sqrt{(4v^2-3u^2)}+v}{(v^2-u^2)}\right]$$

Hence proved.

Ex. 8. Two straight roads AOB and COD meet each other at right angles. A person walking at a speed of 5 km./hr, along AOB is at the crossing O at noon. Another person walking at the same speed along COD reaches the crossing O at 1-30 P. M. Find at what time the distance between them is least and what is its value ?



(Fig. 30)

Sol. Let C be the position of the man walking along COD at 12-00 noon. Then in 1 hour 30 min., *i.e.* in (3/2) hours he covers the distance CO with a speed of 5 km./hr.

$\therefore \quad CO=5\times(3/2)=(15/2)$ km.

Combining the reversed velocity of the second person with that of the first person, the velocify of the first person relative to the second=resultant of velocity 5 km./hr. of first person along OB and reversed velocity 5 km./hr, along OC of second person.

$=\sqrt{(5^2+5^2)}=5\sqrt{2}$ along OE, where OE is the bisector of $\angle COB$, since it is the line of action of the resultant of two equal velocities.

From C draw CF perpendicular to OE, then CF is the required least distance between the two persons. (Note

Now $CF = OC \sin 45°$, from $\triangle COF$

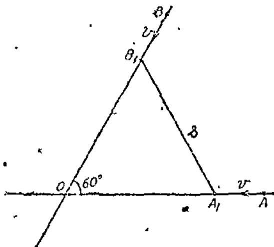
$= (15/2)(1/\sqrt{2}) = (15/4)\sqrt{2}$ km. **Ans.**

Also time taken in moving from O to F with velocity $5\sqrt{2}$ km./hr.

$$= \frac{CF}{5\sqrt{2}} = \frac{15\sqrt{2}}{4} \times \frac{1}{5\sqrt{2}} = \tfrac{3}{4} \text{ hr.} = 45 \text{ min.}$$

$\therefore$ Required time when the distance between the person will be least is 12—45 P. M. **Ans.**

Ex. 9. **Two straight roads meet at an angle of 60°. Initially one man is at a distance of 100 decimetres on one road from the crossing and the other at a distance of 200 decimetres on the other road from the crossing.** ***They start moving at the same instant with*** **same velocity towards the crossing. Find the distance of these men from the crossing, when the distance between them is least.**



(Fig. 31)

Sol. Let v dm./hr. be the speed of each man. After time t hr. let the positions of persons starting from A and B, where $OA = 100$ dm. and $OB = 200$ dm., be A_1 and B_1 respectively.

Then $AA_1 = vt = BB_1$

$\therefore$ $OA_1 = 100 - vt$ and $OB_1 = 200 - vt$

$\because$ In $\triangle OA_1B_1$, we have $\cos 60° = \dfrac{OA_1^2 + OB_1^2 - A_1B_1^2}{2 \cdot OA_1 \cdot OB_1}$

or $A_1B_1^2 = OA_1^2 + OB_1^2 - 2OA_1 \cdot OB_1 \cos 60°$

or $s^2 = (100 - vt)^2 + (200 - vt)^2 - 2(100 - vt)(200 - vt)(1/2)$,

where $A_1B_1 = s$ (say)

or $s^2=30000-300\,vt+v^2t^2$, on simplifying.

$\therefore \frac{d}{dt}(s^2)=-300\,v+2v^2t,\ \frac{d^2}{dt^2}(s^2)=2v^2=+ve$

$\therefore$ s^2 is minimum when $\frac{d\,(s^2)}{dt}=0$ *i.e.* $2v^2t-300\,v=0$

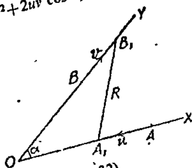
or $t=150/v$ or $vt=150$

$\therefore OA_1=OA-vt=100-vt=100-150=-50$ *i.e.* the person A moving along OA will be on the other side of the crossing at a distance 50 dm. from the crossing O. **Ans.**

And $OB_1=OB-vt=200-vt=200-150=50$ dm. **Ans.**

Ex. 10 (a). Two straight roads **OX** and **OY** are inclined to each other at an acute angle α; one car moves along **XO** with speed u, while a second car moves along **OY** with speed v, show that when the distance between them is least, the ratio to their distance from **O** is $(v+u\cos\alpha):(u+v\cos\alpha)$. *(Lucknow 91)*

(b) If the first car is a distance d from **O** when the second is at **O** show that the cars are at their least distance apart after time $d\,(u+v\cos\alpha)/(u^2+v^2+2uv\cos\alpha)$.



(Fig. 32)

Sol. (a). Let the first car start from A with velocity u and the second start from B with velocity v as shown in the figure. Let $OA=a$ and $OB=b$. After time t, let the cars be at A_1 and B_1 respectively. Then $AA_1=ut$ and $BB_1=vt$.

$\therefore OA_1=OA-AA_1=a-ut$,

$OB_1=OB+BB_1=b+vt$

$\therefore$ If the distance between the cars viz. $A_1B_1=R$, then from $\triangle OA_1B_1$ we have

$R^2=OA_1^2+OB_1^2-2\cdot OA_1\cdot OB_1\cos\alpha$

or $R^2=(a-ut)^2+(b+vt)^2-2\,(a-ut)\,(b+vt)\cos\alpha$

Differentiating, we have

$2R.\frac{dR}{dt}=-2u\,(a-ut)+2v\,(b+vt)-2\cos\alpha\,[(a-ut)\,v-u\,(b+vt)]$

When R is least $dR/dt=0$

$\therefore -2u\,(a-ut)+2v\,(b+vt)-2\cos\alpha\,[av-bu-2uvt]=0$

or $t\,[u^2+v^2+2uv\cos\alpha]=ua-bv+(av-bu)\cos\alpha$

or $$t=\frac{ua-bv+(av-bu)\cos\alpha}{u^2+v^2+2uv\cos\alpha} \quad ...(i)$$

We find from geometrical considerations that as time elapses the distance between the two cars increase, hence there is no possibility of getting a maximum value of the distance R.

∴ The value of R corresponding to the value of t, given by (i) is least.

∴ when R is least we have the required ratio

$$\frac{OA_1}{OB_1}=\frac{a-ut}{b+vt}=\frac{a-u\left[\dfrac{ua-bv+(av-bu)\cos\alpha}{u^2+v^2+2uv\cos\alpha}\right]}{b+v\left[\dfrac{au-bv+(av-bu)\cos\alpha}{u^2+v^2+2uv\cos\alpha}\right]}, \text{ from (i)}$$

$$=\left[\frac{v^2a+uav\cos\alpha+ubv+bu^2\cos\alpha}{bu^2+uva+buv\cos\alpha+av^2\cos\alpha}\right]$$

$$=\frac{va\,(v+u\cos\alpha)+bu\,(v+u\cos\alpha)}{bu\,(u+v\cos\alpha)+av\,(u+v\cos\alpha)}$$

$$=\frac{(v+u\cos\alpha)\,(av+bu)}{(u+v\cos\alpha)\,(bu+av)}=\frac{v+u\cos\alpha}{u+v\cos\alpha}$$

(b) In this case at start first car is at a distance d from O and second is at O.

∴ In part (a) above $OA=d$ and $OB=0$

i.e. put $a=d$ and $b=0$ in part (a) above.

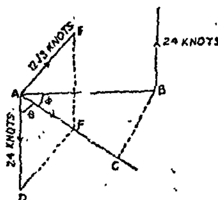
∴ From (i) of part (a), we have time t corresponding to the least value of R as $d\,(u+v\cos\alpha)/(u^2+v^2+2uv\cos\alpha)$. Hence proved.

Ex. 11. A battle-ship which can steam at $12\sqrt{3}$ knots sights a cruiser at a distance of 10 miles due to east of her. If the cruiser steam due north at 24 knots, find the course the battleship should steer to get as close to her as possible and find also the shortest distance between them.

Sol. Let A and B be the positions of the battle-ship and cruiser respectively. Let the battle-ship, to get as close as possible to the cruiser, start at an angle θ north of east.

Now suppose that the cruiser is at rest and the battle-ship is moving with relative velocity (which is the resultant of its own actual velocity $12\sqrt{3}$ knots and that of cruiser's reversed actual velocity 24 knots) and in the relative direction.

As the battle-ship is to get as close to the cruiser as possible its *velocity relative to the cruiser should make as small an angle as possible with AB.* (See fig. 33 below)



(Fig. 33)

i.e. angle ϕ is to be minimum or the angle FAD is to be the maximum. The angle FAD will be maximum when DF is perpendicular to AF and in that case

$$\sin \angle FAD = \frac{DF}{AD} \text{ or } \sin \theta = \frac{12\sqrt{3}}{24} = \frac{\sqrt{3}}{2} \text{ or } \theta = 60^\circ$$

Also the shortest distance between them

$= BC$ (*BC* is perpendicular distance from *B* upon *the relative direction of A*)

$= AB \sin \phi = 10 \sin (90^\circ - \theta)$, $\because$ $AB = 10$ (given)

$= 10 \cos \theta = 10 \cos 60^\circ = 5$ miles. Hence the result.

Ex. 12. Two cars A and B are running on mutually perpendicular roads at speed of 40 and 20 km/hr. A crosses the crossing of the roads when B has yet to cover 50 km. to reach the crossing. After how much time these cars will be nearest to each other and what would be least distance between them.

Sol. Let the car A move along XOX' and the car B along YOY'.

Let the car B be at K when the car A is at the crossing O. Then according to the problem $OK = 50$ km.

After time t hours let the car reach the point P from O and the car B reach the point Q from M.

Then according to the problem, we have

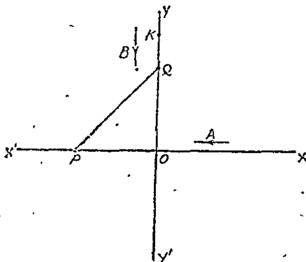
$OP=40\,t$ and $KQ=20\,t$

$\therefore\ OQ=OK-KQ=50-20\,t$

Now in $\triangle\ OPQ$, we have $PQ^2=OP^2+OQ^2$

or $(PQ)^2=(40\,t)^2+(50-20\,t)^2\equiv 1600t^2+2500+400t^2-2000\,t$

or $(PQ)^2=500\,(4t^2-4t+5)$ ...(i)



(Fig. 24)

Let $(PQ)^2=s$, then from (i) we get $s=500\,(4t^2-4t+5)$

Now PQ is minimum when s is minimum

i.e. when $\dfrac{ds}{dt}=0$ *i.e.* when $500\,(8t-4)=0$ *i.e.* when $t=\frac{1}{2}$

$\therefore$ The minimum value of t from (i) is given by

$(PQ)^2=500\,[4\times(1/4)-4\,(1/2)+5]=2000$

or $PQ=\sqrt{(2000)}=20\sqrt{5}$ km.

Hence the two cars A and B will be nearest after $\frac{1}{2}$ hour from the instant the car A crosses O and the least distance between them would be $20\sqrt{5}$ km. **Ans.**

EXERCISES ON RELATIVE MOTION

Ex. 1. The position of point P is defined by $x=t,\ y=2t$ and that of another Q by $x=3t,\ y=5t^2$.

Show that the path of Q relative to P is a parabola and find the velocity of Q relative to P when $t=3$.

(Hint. See Ex. 2 P. 19 of this chapter) Ans $2\sqrt{(197)}$ units/sec.

Ex. 2. Two straight railways converge to a level crossing at angle α; and two trains distant respectively a and b from this point

are moving towards it with speed u and v respectively. Find when and where they are nearest to each other and prove that their least distance apart is $\{(av-bu)\sin\alpha\}/\sqrt{(u^2+v^2-2uv\cos\alpha)}$.

(Hint. See Ex. 10 Page 28 of this chapter).

Ex. 3. Two points A and B move with uniform velocities u, v in two straight lines containing an angle α; prove that the time from the position in which AB is least to that in which it is double its least value is $\sqrt{(3)}\, cu \sin\alpha/(u^2+v^2-uv\cos\alpha)$, where c is the distance AB when A crosses the path of B.

Ex. 4. To a man walking towards north-east, the wind appears to blow from the north. But when he doubles his speed, then the wind appears to blow from north to east at an angle $\cot^{-1} 2$. Find the direction of the absolute velocity of the wind.

Ans. Wind blows from east.

Ex. 5. A ship A is sailing from the port X at a speed of 25 km/hr. towards west. Another ship B, which was stationary at a port Y situated at a distance of 30 km. from X in south east direction, starts sailing after half an hour at a speed of 20 km/hr. in the north-western direction. Find at what time, the ships will be nearest to each other, after A leaves the port X. *(Roorkee Entrance)*
